An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 1

Isophote curves on timelike surfaces in Minkowski 3-space

Fatih Doğan

Received: 21.I.2013 / Revised: 24.VII.2013 / Accepted: 3.IX.2013

Abstract Isophote curve comprises a locus of the surface points whose normal vectors make a constant angle with a fixed vector. In this paper, isophote curves are studied on timelike surfaces in Minkowski 3-space E_1^3. The axes of spacelike and timelike isophote curves are found via their Darboux frames. Subsequently, the relationship between isophotes and slant helices is shown on timelike surfaces.

Keywords isophote curve · spacelike curve · timelike curve · timelike surface · geodesic · slant helix

Mathematics Subject Classification (2010) 51B20 · 53A35

1 Introduction

Isophote is one of the characteristic curves on a surface such as parameter, geodesic and asymptotic curves or lines of curvature.

Isophote on a surface can be regarded as a nice consequence of Lambert's cosine law in optics branch of physics. Lambert's law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector N and the light vector d. According to this law the intensity is irrespective of the actual viewpoint, hence the illumination is the same when viewed from any direction (see [16]). In other words, isophotes of a surface are curves with the property that their points have the same light intensity from a given source (a curve of constant illumination intensity). When the source light is at infinity, we may consider that the light flow consists in parallel lines. Hence, we can give a geometric description of isophotes on surfaces, namely they are curves such that the surface normal vectors in points of the curve make a constant angle with a fixed direction (which represents the light direction). These curves are successfully used in computer

Fatih Doğan
Bartın University,
Faculty of Science,
Department of Mathematics,
74100, Bartın, Turkey
E-mail: fdogan@bartin.edu.tr

graphics but also it is interesting to study for geometry. Then, to find an isophote on a surface we use the formula

$$\frac{\langle N(u,v), d\rangle}{\|N(u,v)\|} = \cos\theta,\ 0 \le \theta \le \frac{\pi}{2},$$

where d is the light (fixed) vector and θ is the constant angle between the surface normal vector N and d.

Koenderink and van Doorn [8] studied the field of constant image brightness contours (isophotes). They showed that the spherical image (the Gauss map) of an isophote is a latitude circle on the unit sphere S^2 and the problem was reduced to that of obtaining the inverse Gauss map of these circles. By means of this they defined two kind singularities of the Gauss map: folds (curves) and simple cusps (apex, antapex points) and there are structural properties of the field of isophotes that bear an invariant relation to geometric features of the object.

Poeschl [13] used isophotes in car body construction via detecting irregularities along these curves on a free form surface. These irregularities emerge by differentiating of the equation $\langle N(u,v), l\rangle = \cos\theta = c$ (constant)

$$\begin{aligned} &\langle N_u, l\rangle\, du + \langle N_v, l\rangle\, dv = 0, \\ &\frac{dv}{du} = -\frac{\langle N_u, l\rangle}{\langle N_v, l\rangle},\ \ \langle N_v, l\rangle \neq 0, \end{aligned}$$

where l (d) is the light vector.

Sara [15] researched local shading of a surface through isophotes properties. By using fundamental theory of surfaces, he focused on accurate estimation of surface normal tilt and on qualitatively correct Gaussian curvature recovery.

Kim and Lee [7] parameterized isophotes for surface of rotation and canal surface. They utilized that both of these surfaces decompose into a set of circles where the surface normal vectors at points on each circle construct a cone. Again the vectors that make a constant angle with the fixed vector d construct another cone and thus tangential intersection of these cones gives the parametric range of the connected component isophote.

Dillen et al. [2] studied the constant angle surfaces in the product space $\mathbb{S}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the $\mathbb{R}$-direction. Then Dillen and Munteanu [3] investigated the same problem in $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is the hyperbolic plane. Again, Nistor [11] researched normal, binormal and tangent developable surfaces of the space curve from the viewpoint of constant angle surface. Recently, Munteanu and Nistor [10] gave an important characterization about constant angle surfaces and studied the constant angle surfaces taking with a fixed vector direction being the tangent direction to $\mathbb{R}$ in Euclidean 3-space. Thus, it can be said that all curves on a constant angle surface are isophote curves.

Izumiya and Takeuchi [6] defined a slant helix as a space curve that the principal normal lines make a constant angle with a fixed direction. They showed that a certain slant helix is also a geodesic on the tangent developable surface of a general helix.

Recently, Ali and Lopez [1] looked into slant helices in Minkowski 3-space E_1^3. They gave characterizations as to slant helix and its axis in E_1^3.

More recently, Doğan and Yayli [4] have investigated isophote curves in Euclidean 3-space E^3. Also, they [5] have studied isophote curves on spacelike surfaces in E_1^3. In both papers they viewed the close relation between isophote curves and special curves on the surfaces. For instance, an isophote can be generated by a curve which is both geodesic and slant helix.

This time we study isophote curves on timelike surfaces in E_1^3. The present paper is organized as follows. We give basic concepts concerning curve and surface theory in section 2. In section 3 and 4, we focus on finding the axis of spacelike and timelike isophote curves lying on timelike surfaces. Finally, we give main theorems for these curves in section 5.

2 Preliminaries

First of all, we begin to introduce Minkowski 3-space. Later, we mention some fundamental concepts of curves and surfaces in Minkowski 3-space E_1^3. The space R_1^3 is a three dimensional real vector space endowed with the inner product $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$. This space is called Minkowski 3-space and denoted by E_1^3. A vector in this space is said to be spacelike, timelike and lightlike (null) if $\langle x, x\rangle > 0$ or $x = 0$, $\langle x, x\rangle < 0$ and $\langle x, x\rangle = 0$ or $x \neq 0$, respectively. Again, a regular curve $\alpha : I \longrightarrow E_1^3$ is called spacelike, timelike and lightlike if the velocity vector α' is spacelike, timelike and lightlike, respectively [9]. The Lorentzian cross product of $x = (x_1, x_2, x_3)$ and

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2),$$

where δ_{ij} is Kronecker delta, $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = -e_2$.

Let $\{t, n, b\}$ be the moving Frenet frame along the curve α with arclenght parameter s. For a spacelike curve α, the Frenet-Serret equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t, t\rangle = 1$, $\langle n, n\rangle = \pm 1$, $\langle b, b\rangle = -\varepsilon$, $\langle t, n\rangle = \langle t, b\rangle = \langle n, b\rangle = 0$ and κ is the curvature and τ is the torsion of α. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal n and timelike binormal b. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal n and spacelike binormal b.

If α is a timelike curve, the Frenet-Serret equations are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t, t\rangle = -1$, $\langle n, n\rangle = \langle b, b\rangle = 1$, $\langle t, n\rangle = \langle t, b\rangle = \langle n, b\rangle = 0$.

Definition 2.1 ([14]) Let v and w be spacelike vectors.

(a) If v and w span a timelike vector subspace, then there is a unique non-negative real number $\theta \geq 0$ such that $\langle v, \omega\rangle = \|v\| \|w\| \cosh\theta$.

(b) If v and w span a spacelike vector subspace, then there is a unique non-negative real number $\theta \geq 0$ such that $\langle v, \omega\rangle = \|v\| \|w\| \cos\theta$.

Definition 2.2 ([14]) *Let v be a spacelike vector and w be a timelike vector in R_1^3. Then, there is a unique non-negative real number $\theta \geq 0$ such that $\langle v, w \rangle = \|v\| \|w\| \sinh \theta$.*

Definition 2.3 ([12]) *Let v and w be in the same timecone of R_1^3. Then, there is a unique real number $\theta \geq 0$, called the hyperbolic angle between v and w such that $\langle v, \omega \rangle = -\|v\| \|w\| \cosh \theta$.*

Lemma 2.4 *In Minkowski 3-space E_1^3, we have the following:*

(i) two timelike vectors cannot be orthogonal;
(ii) two null vectors are orthogonal if and only if they are linearly dependent;
(iii) a timelike vector cannot be orthogonal to a null (lightlike) vector.

Let M be a regular timelike surface in E_1^3 and let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike curve. Then, Darboux frame $\{T,\, B = N \times T,\, N\}$ is well-defined and positively oriented along the curve α where T is the tangent of α and N is the unit normal of M. In this case, the Darboux equations are given by

$$T' = k_g B - k_n N, \;\; B' = k_g T + \tau_g N, \;\; N' = k_n T + \tau_g B, \tag{2.1}$$

where k_n, k_g and τ_g are the normal curvature, the geodesic curvature and the geodesic torsion of α, respectively and $\langle T, T \rangle = \langle N, N \rangle = \langle n, n \rangle = 1$, $\langle B, B \rangle = -1$. Then, by using Eq. (2.1) we get

$$\kappa^2 = k_n^2 - k_g^2,\; k_g = \kappa \sinh \phi,\; k_n = \kappa \cosh \phi,\; \tau_g = \tau + \phi', \tag{2.2}$$

where ϕ is the angle between the surface normal vector N and the principal normal n of α.

If $\alpha : I \subset \mathbb{R} \longrightarrow M$ is a unit speed timelike curve, then the Darboux equations are given by

$$T' = k_g B + k_n N,\; B' = k_g T - \tau_g N,\; N' = k_n T + \tau_g B, \tag{2.3}$$

where $\langle T, T \rangle = -1$, $\langle N, N \rangle = \langle B, B \rangle = \langle n, n \rangle = 1$. From Eq. (2.3) we get,

$$\kappa^2 = k_g^2 + k_n^2,\; k_g = \kappa \sin \phi,\; k_n = \kappa \cos \phi,\; \tau_g = \tau + \phi',$$

where ϕ is the angle between the surface normal vector N and the principal normal n of α.

Theorem 2.5 *Let α be a unit speed spacelike curve in E_1^3. If the normal vector of α is spacelike, then α is a slant helix if and only if one of the two functions*

$$\frac{\kappa^2}{(\tau^2 - \kappa^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' \quad \text{and} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'$$

is a constant everywhere $\tau^2 - \kappa^2$ does not vanish (see [1]).

Theorem 2.6 *Let α be a unit speed timelike curve in E_1^3. Then α is a slant helix if and only if one of the two functions*

$$\frac{\kappa^2}{(\tau^2 - \kappa^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' \quad \text{and} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'$$

is a constant everywhere $\tau^2 - \kappa^2$ does not vanish (see [1]).

3 The axis of a spacelike isophote curve on timelike surfaces

Now, we find the fixed vector (axis) of a spacelike isophote curve via its Darboux frame. Let M be a timelike surface and let α be a unit speed spacelike isophote curve on M. Then, there are two cases for the axis d of α.

Case (1). If the axis d is spacelike vector, then from Definition 2.1 (a) and (b) we have $\langle N, d\rangle = \cosh\theta$ or $\langle N, d\rangle = \cos\beta$, where θ and β are the constant angles between the surface normal vector N and d, respectively.

(a) Let $\langle N, d\rangle = \cosh\theta$. If we differentiate this equation with respect to s along the curve α, by Eq. (2.1) we get $\langle N', d\rangle = 0, \langle k_n T + \tau_g B, d\rangle = 0, k_n \langle T, d\rangle + \tau_g \langle B, d\rangle = 0, \langle T, d\rangle = -\frac{\tau_g}{k_n}\langle B, d\rangle$. If we take $\langle B, d\rangle = a$, the axis d can be written as $d = -\frac{\tau_g}{k_n} aT - aB + \cosh\theta N$, where $\langle T, T\rangle = \langle N, N\rangle = 1$ and $\langle B, B\rangle = -1$. Since d is spacelike, we obtain $\langle d, d\rangle = \frac{\tau_g^2}{k_n^2}a^2 - a^2 + \cosh^2\theta = 1, (1 - \frac{\tau_g^2}{k_n^2})a^2 = \sinh^2\theta, a = \mp\frac{k_n}{\sqrt{k_n^2-\tau_g^2}}\sinh\theta$. By substituting this in the expression of d, we get the axis as

$$d = \pm\frac{\tau_g}{\sqrt{k_n^2 - \tau_g^2}}\sinh\theta T \pm \frac{k_n}{\sqrt{k_n^2 - \tau_g^2}}\sinh\theta B + \cosh\theta N. \tag{3.1}$$

If we differentiate N' with respect to s and take inner product with d, we conclude

$$N'' = (k_n' + k_g\tau_g)T + (\tau_g' + k_n k_g)B - (k_n^2 - \tau_g^2)N \tag{3.2}$$

$$\langle N'', d\rangle = \mp\frac{(\tau_g' k_n - k_n'\tau_g) + k_g(k_n^2 - \tau_g^2)}{\sqrt{k_n^2 - \tau_g^2}}\sinh\theta - (k_n^2 - \tau_g^2)\cosh\theta = 0$$

$$\begin{aligned}\coth\theta &= \mp\left[\frac{\tau_g' k_n - k_n'\tau_g}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right]\\ &= \mp\left[\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right].\end{aligned} \tag{3.3}$$

Indeed, d is a constant vector. If we differentiate the vector d, from Eq. (2.1) we get

$$\begin{aligned} d' = &\pm\sinh\theta\left[\left(\frac{\tau_g}{\sqrt{k_n^2 - \tau_g^2}}\right)'T + \frac{\tau_g}{\sqrt{k_n^2 - \tau_g^2}}(k_g B - k_n N)\right]\\ &\pm\sinh\theta\left[\left(\frac{k_n}{\sqrt{k_n^2 - \tau_g^2}}\right)'B + \frac{k_n}{\sqrt{k_n^2 - \tau_g^2}}(k_g T + \tau_g B)\right] + \cosh\theta\left[k_n T + \tau_g B\right].\end{aligned}$$

By Eq. (3.3), we have

$$\cosh\theta = \pm\sinh\theta\left[\frac{k_n'\tau_g - \tau_g' k_n}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} - \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right].$$

The last equality is replaced in the statement of d', it follows that

$$d' = \pm \sinh\theta \left[\frac{\tau_g'(k_n^2 - \tau_g^2) - \tau_g(k_n k_n' - \tau_g \tau_g')}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} + \frac{k_n k_n' \tau_g - k_n^2 \tau_g'}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} \right] T$$
$$\pm \sinh\theta \left[\frac{k_n'(k_n^2 - \tau_g^2) - k_n(k_n k_n' - \tau_g \tau_g')}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} + \frac{k_n' \tau_g^2 - k_n \tau_g \tau_g'}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} \right] B.$$

As can be immediately seen above, the coefficients of T and B become zero. Then $d' = 0$ in other words d is a constant vector.

(b) Let $\langle N, d\rangle = \cos\beta$. In that case, by Eq. (2.1) it concludes that $\langle T, d\rangle = -\frac{\tau_g}{k_n}\langle B, d\rangle$. If we take $\langle B, d\rangle = a$, the axis d can be written as $d = -\frac{\tau_g}{k_n} aT - aB + \cos\beta N$, where $\langle T, T\rangle = \langle N, N\rangle = 1$ and $\langle B, B\rangle = -1$. Since d is spacelike, we obtain

$$\langle d, d\rangle = \frac{\tau_g^2}{k_n^2} a^2 - a^2 + \cos^2\beta = 1, \quad a = \mp \frac{k_n}{\sqrt{\tau_g^2 - k_n^2}} \sin\beta.$$

In this case, the axis d becomes

$$d = \pm \frac{\tau_g}{\sqrt{\tau_g^2 - k_n^2}} \sin\beta T \pm \frac{k_n}{\sqrt{\tau_g^2 - k_n^2}} \sin\beta B + \cos\beta N. \tag{3.4}$$

From Eq. (3.2) we have $N'' = (k_n' + k_g \tau_g)T + (\tau_g' + k_n k_g)B - (k_n^2 - \tau_g^2)N$. By taking inner product of N'' and d, we get

$$\langle N'', d\rangle = \mp \frac{(\tau_g' k_n - k_n' \tau_g) - k_g(\tau_g^2 - k_n^2)}{\sqrt{\tau_g^2 - k_n^2}} \sin\beta + (\tau_g^2 - k_n^2)\cos\beta = 0$$
$$\cot\beta = \pm \left[\frac{k_n^2}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}} \left(\frac{\tau_g}{k_n}\right)' - \frac{k_g}{(\tau_g^2 - k_n^2)^{\frac{1}{2}}} \right]. \tag{3.5}$$

If we differentiate Eq. (3.4) and then use Eq. (3.5), we obtain

$$d' = \pm \sin\beta \left[\frac{\tau_g'(\tau_g^2 - k_n^2) - \tau_g(\tau_g \tau_g' - k_n k_n')}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}} + \frac{k_n^2 \tau_g' - k_n k_n' \tau_g}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}} \right] T$$
$$\pm \sin\beta \left[\frac{k_n'(\tau_g^2 - k_n^2) - k_n(\tau_g \tau_g' - k_n k_n')}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} + \frac{k_n \tau_g \tau_g' - k_n' \tau_g^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} \right] B.$$

Since the coefficients of T and B are zero, $d' = 0$, i.e., d is a constant vector.

Case (2). If the axis d is a timelike vector, then from Definition 2.2 we have $\langle N, d\rangle = \sinh\gamma$, where γ is the constant angle between the surface normal vector N

and d. By doing computations similar to the case (1) we get

$$d = \pm \frac{\tau_g}{\sqrt{k_n^2 - \tau_g^2}} \cosh \gamma T \pm \frac{k_n}{\sqrt{k_n^2 - \tau_g^2}} \cosh \gamma B + \sinh \gamma N$$
$$\tanh \gamma = \mp \left[\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} \left(\frac{\tau_g}{k_n} \right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}} \right]$$

and again similar to proof of the case (1) it can be showed that d is a constant vector.

From now on, we will obtain the axis of timelike isophote curves on timelike surfaces.

4 The axis of a timelike isophote curve on timelike surfaces

In this section, we find the fixed vector (axis) of a timelike isophote curve via its Darboux frame. Let M be a timelike surface and let α be a unit speed timelike isophote curve on M. Then, there are two cases for the axis d of α.

Case (3). If the axis d is spacelike, then from Definition 2.1 (b) and 2.1 (a) we have $\langle N, d \rangle = \cos \delta$ or $\langle N, d \rangle = \cosh \xi$, where δ and ξ are the constant angles between the surface normal vector N and d, respectively.

(a) Let $\langle N, d \rangle = \cos \delta$. Then, from Eq. (2.3) it follows that $\langle T, d \rangle = -\frac{\tau_g}{k_n} \langle B, d \rangle$. By taking $\langle B, d \rangle = a$, the axis d can be written as $d = \frac{\tau_g}{k_n} aT + aB + \cos \delta N$, where $\langle T, T \rangle = -1$ and $\langle N, N \rangle = \langle B, B \rangle = 1$. Since d is spacelike, we obtain

$$a = \pm \frac{k_n}{\sqrt{k_n^2 - \tau_g^2}} \sin \delta.$$

Then we have

$$d = \pm \frac{\tau_g}{\sqrt{k_n^2 - \tau_g^2}} \sin \delta T \pm \frac{k_n}{\sqrt{k_n^2 - \tau_g^2}} \sin \delta B + \cos \delta N$$
$$\cot \delta = \mp \left[\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}} \left(\frac{\tau_g}{k_n} \right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}} \right].$$

Like preceding cases, it can be showed that d is a constant vector.

(b) Let $\langle N, d \rangle = \cosh \xi$. Then we can easily obtain

$$d = \pm \frac{\tau_g}{\sqrt{\tau_g^2 - k_n^2}} \sinh \xi T \pm \frac{k_n}{\sqrt{\tau_g^2 - k_n^2}} \sinh \xi B + \cosh \xi N$$
$$\coth \xi = \pm \left[\frac{k_n^2}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}} \left(\frac{\tau_g}{k_n} \right)' + \frac{k_g}{(\tau_g^2 - k_n^2)^{\frac{1}{2}}} \right].$$

Case (4). If the axis d is a timelike vector, then from Definition 2.2 we have $\langle N, d\rangle = \sinh\nu$, where ν is the constant angle between the surface normal vector N and d. In this situation, we get

$$d = \pm\frac{\tau_g}{\sqrt{\tau_g^2 - k_n^2}}\cosh\nu T \pm \frac{k_n}{\sqrt{\tau_g^2 - k_n^2}}\cosh\nu B + \sinh\nu N$$

$$\coth\nu = \pm\left[\frac{k_n^2}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' - \frac{k_g}{(\tau_g^2 - k_n^2)^{\frac{1}{2}}}\right].$$

5 Main theorems

In this section, we give main theorems that characterize isophotes on timelike surfaces. Moreover, we show the relationship between isophote curves and slant helices on timelike surfaces.

Theorem 5.1 *A unit speed spacelike curve on a timelike surface is an isophote curve if and only if one of the following three functions*

$$(1)\quad \coth\theta = \eta(s) = \mp\left(\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right)(s)$$

$$(2)\quad \cot\beta = \mu(s) = \pm\left(\frac{k_n^2}{(\tau_g^2 - k_n^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' - \frac{k_g}{(\tau_g^2 - k_n^2)^{\frac{1}{2}}}\right)(s)$$

$$(3)\quad \tanh\gamma = \psi(s) = \mp\left(\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right)(s)$$

is a constant function (*the case* $1(a)$, *the case* $1(b)$ *and the case* 2, *respectively*).

Proof. (1) Since α is an isophote, the Gauss map along the curve α is a circle on the Lorentzian unit sphere S_1^2. Hence, if we compute the Gauss map $N_{|_\alpha} : I \longrightarrow S_1^2$ along the curve α, the geodesic curvature of $N_{|_\alpha}$ becomes $\eta(s)$ as shown below.

$$N'_{|_\alpha} = k_n T + \tau_g B$$

$$N''_{|_\alpha} = (k'_n + k_g\tau_g)T + (k_n k_g + \tau'_g)B - (k_n^2 - \tau_g^2)N$$

$$N'_{|_\alpha} \times N''_{|_\alpha} = -\tau_g(k_n^2 - \tau_g^2)T + k_n(k_n^2 - \tau_g^2)B + (k_g(k_n^2 - \tau_g^2) + k_n^2(\frac{\tau_g}{k_n})')N,$$

where $T \times B = N$, $B \times N = T$ and $N \times T = B$. Therefore, we obtain

$$\kappa = \frac{\sqrt{\langle N'_{|_\alpha} \times N''_{|_\alpha}, N'_{|_\alpha} \times N''_{|_\alpha}\rangle}}{\|N'_{|_\alpha}\|^3} = \frac{\sqrt{-(k_n^2 - \tau_g^2)^3 + (k_g(k_n^2 - \tau_g^2) + k_n^2(\frac{\tau_g}{k_n})')^2}}{\sqrt{(k_n^2 - \tau_g^2)^3}}$$

$$= \sqrt{-1 + \frac{(k_g(k_n^2 - \tau_g^2) + k_n^2(\frac{\tau_g}{k_n})')^2}{(k_n^2 - \tau_g^2)^3}}.$$

Let $\overline{k}_g$ and $\overline{k}_n$ be the geodesic curvature and the normal curvature of the Gauss map $N_{|\alpha}$ on S_1^2, respectively. Since the normal curvature $\overline{k}_n = 1$, if we substitute $\overline{k}_n$ and κ in the following equation, we obtain the geodesic curvature $\overline{k}_g$ as follows

$$\kappa^2 = (\overline{k}_g)^2 - (\overline{k}_n)^2$$

$$\overline{k}_g(s) = \eta(s) = \coth\theta = \mp\left(\frac{k_n^2}{(k_n^2 - \tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2 - \tau_g^2)^{\frac{1}{2}}}\right)(s),$$

where θ is the constant angle between the surface normal vector N and d. In that case, the spherical images (Gauss maps) of isophotes are circles if and only if one of the three functions $\eta(s)$, $\mu(s)$ and $\psi(s)$ is a constant. The proofs for (2) and (3) can be done in the same way. □

Theorem 5.2 *Let α be a unit speed spacelike isophote curve on a timelike surface (the case 1(a), the case 1(b) and the case 2, respectively). Then:*

(a) α is a geodesic on the timelike surface if and only if α is a slant helix with the spacelike axis

$$d = \pm\frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta T \pm \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta B + \cosh\theta N.$$

(b) α is a geodesic on the timelike surface if and only if α is a slant helix with the spacelike axis

$$d = \pm\frac{\tau}{\sqrt{\tau^2 - \kappa^2}}\sin\beta T \pm \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}\sin\beta B + \cos\beta N.$$

(c) α is a geodesic on the timelike surface if and only if α is a slant helix with the timelike axis

$$d = \pm\frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\cosh\gamma T \pm \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}\cosh\gamma B + \sinh\gamma N.$$

Proof. (a) Since α is a geodesic, we have $k_g = 0$. From Eq. (2.2) it follows that $k_n = \kappa$ and $\tau_g = \tau$. By substituting k_g and k_n in the expression of $\eta(s)$ we get

$$\eta(s) = \mp\left(\frac{\kappa^2}{(\kappa^2 - \tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$$

is a constant function. Then, from Theorem 2.5 α is a slant helix. Because $k_n = \kappa$ and $\tau_g = \tau$, using Eq. (3.1) we obtain the spacelike axis of slant helix as

$$d = \pm\frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta T \pm \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta B + \cosh\theta N.$$

Conversely, let α be a slant helix with the spacelike axis

$$d = \pm\frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta T \pm \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}\sinh\theta B + \cosh\theta N.$$

Then from Eq. (3.1) we get $k_n = \kappa$ and $\tau_g = \tau$. This means that $k_g = 0$, i.e., α is a geodesic on the timelike surface.

The proof of (b) and (c) can be done similar to the proof of (a). □

Theorem 5.3 *A unit speed timelike curve on a timelike surface is an isophote curve if and only if one of the following three functions*

$$(1)\quad \cot\delta=\sigma(s)=\mp\left(\frac{k_n^2}{(k_n^2-\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)'+\frac{k_g}{(k_n^2-\tau_g^2)^{\frac{1}{2}}}\right)(s)$$

$$(2)\quad \coth\xi=\rho(s)=\pm\left(\frac{k_n^2}{(\tau_g^2-k_n^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)'+\frac{k_g}{(\tau_g^2-k_n^2)^{\frac{1}{2}}}\right)(s)$$

$$(3)\quad \coth\nu=\omega(s)=\pm\left(\frac{k_n^2}{(\tau_g^2-k_n^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)'-\frac{k_g}{(\tau_g^2-k_n^2)^{\frac{1}{2}}}\right)(s)$$

is a constant function (*the case* 3(*a*)*, the case* 3(*b*) *and the case* 4*, respectively*)*.*

The proof of Theorem 5.3 is similar to Theorem 5.1.

Theorem 5.4 *Let α be a unit speed timelike isophote curve on a timelike surface* (*the case* 3(*a*)*, the case* 3(*b*) *and the case* 4*, respectively*)*. Then:*

(*a*) *α is a geodesic on the timelike surface if and only if α is a slant helix with the spacelike axis*

$$d=\pm\frac{\tau}{\sqrt{\kappa^2-\tau^2}}\sin\delta T\pm\frac{\kappa}{\sqrt{\kappa^2-\tau^2}}\sin\delta B+\cos\delta N.$$

(*b*) *α is a geodesic on the timelike surface if and only if α is a slant helix with the spacelike axis*

$$d=\pm\frac{\tau}{\sqrt{\tau^2-\kappa^2}}\sinh\xi T\pm\frac{\kappa}{\sqrt{\tau^2-\kappa^2}}\sinh\xi B+\cosh\xi N.$$

(*c*) *α is a geodesic on the timelike surface if and only if α is a slant helix with the timelike axis*

$$d=\pm\frac{\tau}{\sqrt{\tau^2-\kappa^2}}\cosh\nu T\pm\frac{\kappa}{\sqrt{\tau^2-\kappa^2}}\cosh\nu B+\sinh\nu N.$$

The proof of Theorem 5.4 can be done similar to Theorem 5.2.

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