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Some algebraic properties of generalized rings

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Abstract In this paper we show that the identity function e of a generalized ring R is a generalized ring homomorphism if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$. Moreover, we show that if R is a generalized ring with an identity, then e is a generalized ring homomorphism. Properties of identities and identity mappings of generalized rings are considered. A method for constructing a new generalized ring with an identity via a given quotient generalized ring with an identity, is presented. Second isomorphism theorem and third isomorphism theorem for M-rings are proved.

Keywords generalized ring · M-ring · isomorphism theorem · quotient generalized ring

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1 Introduction

The notion of generalized rings has been studied first in [7]. Let us recall its definition. A generalized ring [7] is a non-empty set R with two different operations $(x, y) \mapsto x+y$ and $(x, y) \mapsto xy$ with the following axioms:
i) $x + (y + z) = (x + y) + z$, where $x, y, z \in R$;
ii) for all $x \in R$ there exists a unique $e(x) \in R$ such that $x + e(x) = e(x) + x = x$;
iii) for all $x \in R$ there exists $-x \in R$ such that $x + (-x) = (-x) + x = e(x)$;
iv) $x(yz) = (xy)z$, where $x, y, z \in R$;
v) for all $x, y, z \in R$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$. The properties (i), (ii), and (iii) mean that $(R, +)$ is a generalized group or completely simple semigroup (see [4, 5]). This notion has been applied in geometry (see [8, 9]) and dynamical systems (see [6]).

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Remark 1.1 Using (iii) and the associativity of $+$, one easily verifies $e(x) + e(x) = e(x)$ for every $x \in R$. Hence $e(e(x)) = e(x)$ follows by definitions and so $e^2 = e$ for the corresponding function $e : R \to R$.

A generalized ring with its operations is a ring if and only if e is a constant function.

Example 1.1 The two dimensional Euclidean space R^2 with the operations $(a_1, b_1) + (a_2, b_2) := (a_1, b_2)$ and $(a_1, b_1)(a_2, b_2) := (a_1a_2, b_1b_2)$ is a generalized ring.

A generalized ring R is called an M-ring if $e(xy) = e(x)e(y)$ and $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

In the next section we show that R is an M-ring if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$. In the other word the identity function e is a generalized ring homomorphism if $e(x + y) = e(x) + e(y)$, for all $x, y \in R$.

If there is $1 \in R$ such that $x \cdot 1 = 1 \cdot x = x$, for all $x \in R$, then R is called a generalized ring with an identity.

One can easily prove that the identity of a generalized ring is unique.

We show that if R is a generalized ring with an identity, then R is an M-ring. We present a method for constructing a generalized ring with an identity by using of a quotient generalized ring with an identity. We consider the properties of identity mappings in generalized rings with identities in the next section.

Rings have been considered recently from the new directions (see [1,2,10]). In section three we present the extensions of second isomorphism theorem and third isomorphism theorem of rings. In fact we prove the new version of them for M-rings.

2 The role of identity in generalized rings

In this section we assume that R is a generalized ring.

Theorem 2.1 *If R is a generalized ring, then $e(ab) = e(a)e(b)$, for all $a, b \in R$.*

Proof. Let $a, b \in R$ be given $ab+ae(b) = a(b+e(b)) = ab, ae(b)+ab = a(e(b)+b) = ab$. So $e(ab) = ae(b)$, $e(a)e(b) + ae(b) = (e(a) + a)e(b) = ae(b)$, $ae(b) + e(a)e(b) = (a + e(a))e(b) = ae(b)$. So $e(ae(b)) = e(a)e(b)$. Hence $e(e(ab)) = e(a)e(b)$. Thus $e(ab) = e(a)e(b)$, because $e^2 = e$. □

Corollary 2.2 *If R is a generalized ring, then $e(a)e(b) = ae(b) = e(a)b = e(ab)$, for all $a, b \in R$.*

Previous theorem implies that a generalized ring R is an M-ring if and only if $(R, +)$ is a normal generalized group (for definition of normal generalized group, see [5]).

Theorem 2.3 *If R is a generalized ring with an identity, then R is an M-ring.*

Proof. By Corollary 2.2, $e(x) = e(x \cdot 1) = xe(1)$. So $e(x) + e(y) = xe(1) + ye(1) = (x + y)e(1) = e((x + y) \cdot 1) = e(x + y)$. Thus R is an M-ring. □

Lemma 2.4 *If R is a generalized ring with an identity, then $R_a = \{x \in R \mid e(x) = e(a)\}$ is an abelian group, for all $a \in R$.*

Proof. Theorem 2.3 implies that e is a homomorphism. So one can easily prove that R_a is a group. If $x, y \in R_a$, then $(x+y)(1+1) = x+x+y+y$, $(x+y)(1+1) = x+y+x+y$. So $x+x+y+y = x+y+x+y$. Hence $-x+x+x+y+y+(-y) = (-x)+x+y+x+y+(-y)$. Thus $e(x)+x+y+e(y) = e(x)+y+x+e(y)$. Since $e(x) = e(y)$, then $x+y = y+x$. So R_a is an abelian group. □

Theorem 2.5 *If R is a generalized ring, and if there is $x \in R$ such that $Rx = \{e(y) \mid y \in R\}$, then R is an M-ring.*

Proof. If $a, b \in R$, then there are $a_x \in R$ and $b_x \in R$ such that $e(a) = a_x x$ and $e(b) = b_x x$. So $e(a)+e(b) = (a_x+b_x)x$. Thus $e(a)+e(b)_= e(z)$ for some $z \in R$. Hence $e(e(a)+e(b)) = e(e(z)) = e(z) = e(a)+e(b)$. In Remark 2.3. of [3] we proved that $e(e(a)+e(b)) = e(a+b)$. So $e(a+b) = e(a)+e(b)$. Thus R is an M-ring. □

A subset I of an M-ring R is called a g-ideal (see [7]) if there exist a generalized ring D and a generalized ring homomorphism $f : R \to D$ such that $ker f = I$, where $ker f = \{r \in R \mid f(r) = f(e(a))$ for some $a \in R\}$. The set $R/I = \{x + ker f_r \mid x \in R_r$ and $f_r = f|_{R_r}\}$ with the operations $(x + ker f_r) + (y + ker f_k) = (x+y) + ker f_{r+k}$ and $(x + ker f_r)(y + ker f_k) = (xy) + ker f_{rk}$ is an M-ring (for the proof see Theorem 2.3 of [7]).

One can prove the following theorem by straightforward calculations.

Theorem 2.6 *Let R be an M-ring and I be a g-ideal of R. If R/I is a generalized ring with an identity, then $R^e = \{e(x) | x \in R\}$ is a generalized ring with an identity.*

Proof. It's clear that R^e is a generalized ring. Let $a + I_a$ be the identity of R/I. Then $(x+I_x)(a+I_a) = (a+I_a)(x+I_x) = x+I_x$, for all $x \in R$. Thus $xa + I_{xa} = ax + I_{ax} = x + I_x$, for all $x \in R$. Hence $e(xa) = e(ax) = e(x)$, for all $x \in R$. So $e(x)e(a) = e(a)e(x) = e(x)$ for all $x \in R$. Thus $e(a)$ is the identity of R^e. □

Theorem 2.7 *If R is a generalized ring with an identity, then:*
a) $(-x) = x(-1) = (-1)x$, for all $x \in R$.
b) $-(xy) = (-x)y = x(-y)$, for all $x, y \in R$.
c) $-(x+y) = (-x)+(-y)$, for all $x, y \in R$.

Definition 2.8 *If R, K are generalized rings, then a mapping $f : R \longrightarrow K$ is called an embedding if f is a monomorphism.*

Theorem 2.9 *If we can embed a generalized ring R to a generalized ring with an identity then:*
i) R is a M-ring.
ii) R_a is an abelian group, for all $a \in R$.

Proof. Let K be a generalized ring with an identity and $f : R \longrightarrow K$ be an embedding.

i) If $x, y \in R$, then $f(e(x)+e(y)) = f(e(x)) + f(e(y)) = e(f(x)) + e(f(y)) = e(f(x)+f(y)) = e(f(x+y)) = f(e(x+y))$. So $e(x)+e(y) = e(x+y)$. Thus R is an M-ring.

ii) If $x, y \in R_a$, then $f(x), f(y) \in K_{f(a)}$. Since K is a generalized ring with an identity, then by Lemma 2.4, $K_{f(a)}$ is an abelian group. So $f(x)+f(y) = f(y)+f(x)$. Thus $f(x+y) = f(y+x)$. Hence $x+y = y+x$. So R_a is an abelian group. □

Definition 2.10 *Let R be a generalized ring with an identity, an element $a \in R$ is called invertible if there is $x \in R$ such that $ax = xa = 1$. In this case x is called the inverse of a.*

If $a \in R$ is invertible then the inverse of a is unique and we denote it by a^{-1}.

The next theorem implies if R is a generalized ring with an identity then the role of e is similar to the role of zero in the ring theory.

Theorem 2.11 *If R is a generalized ring with an identity and $e(x) \in R$ is invertible for some $x \in R$, then $e(y) = y$ for all $y \in R$.*

Proof. Since $e(x)$ in invertible, then there is $z \in R$ such that $ze(x) = 1$. Corollary 2.2 implies $e(zx) = 1$. So $e(e(zx)) = e(1)$. Hence $e(zx) = e(1)$. Thus $e(1) = 1$. So $e(y) = ye(1) = y \cdot 1 = y$, for all $y \in R$. □

3 Isomorphism theorems

We begin this section by presenting equivalent conditions for the g-ideal concept.

Theorem 3.1 *If R is an M-ring and I is a subset of R, then I is a g-ideal if and only if:*

i) $x + (-y) \in R$, for all $x, y \in R$ i.e. I is a generalized subgroup of R;

ii) the map $g : K \times K \longrightarrow K$ defined by $g(x + I_a, y + I_b) = x + y + I_{a+b}$ is well defined, where $I_a = R_a \cap I$ and $R_a/I_a = \{x + I_a \mid x \in R_a\}$ and $K = \bigcup_{a \in R} R_a/I_a$;

iii) $xy \in I$ and $yx \in I$, for all $x \in I$ and $y \in R$.

Proof. If I is a g-ideal, then there exist an M-ring K and a generalized ring homomorphism $f : R \longrightarrow K$ such that $I = kerf$.

i) If $x, y \in kerf$, then Lemma 2.2 of [3] implies $f(x) = f(e(x))$, and $f(y) = f(e(y))$. So $f(-y) = -f(y) = -f(e(y)) = f(-e(y)) = f(e(y))$. Thus $f(x + (-y)) = f(e(x)) + f(e(y)) = f(e(x + y))$. So $x + (-y) \in kerf = I$.

ii) Since $I_a = R_a \cap I = kerf_a$, $K = \bigcup_{a \in R} R_a/kerf_a = R/I$, then Theorem 2.3 of [7] implies g is a well defined mapping.

iii) For $x \in I$ and $y \in R$, we have $f(xy) = f(x)f(y) = f(e(x))f(y) = f(e(x)y) = f(e(xy))$. So $xy \in I$. Similarly $yx \in I$.

Conversely, (i) implies I is a generalized subgroup of R. So I_a is a subgroup of R_a. We show that $K = \bigcup_{a \in R} R_a/I_a$, with the operations $(x + I_a) + (y + I_b) := (x + y) + I_{a+b}, (x + I_a) \cdot (y + I_b) := xy + I_{ab}$ is a generalized rings.

According to (ii) the operation $+$ is a well defined operation. Now, we show that the operation $\cdot$ is also a well defined operation.

If $x_1 + I_a = x_2 + I_a$ and $y_1 + I_b = y_2 + I_b$, then $x_2 = x_1 + r$ and $y_2 = y_1 + s$, for some $r \in I_a$ and $s \in I_b$. So $x_2y_2 = x_1y_1 + x_1s + ry_1 + rs$. By (iii), $x_1s, ry_1 \in I$ and $e(x_1s) = e(ry_1) = e(ab)$, so $x_1s + ry_1 + rs \in I_{ab}$. Hence $x_1y_1 + I_{ab} = x_2y_2 + I_{ab}$.

Now, according to Theorem 2.3 of [7] it's clear that K is an M-ring.

If we define $f : R \longrightarrow K$ by $f(x) = x + I_x$, then f is a homomorphism and $kerf_a = \{x \in R_a \mid f(x) = f(e(a))\} = \{x \in R_a \mid x + I_x = e(a) + I_{e(a)}\} = \{x \in R_a \mid x + I_a = e(a) + I_a\} = \{x \in R_a \mid x \in I_a\} = I_a$, so $kerf = I$. Thus I is a g-ideal. □

Corollary 3.2 *If R is a generalized ring with an identity, I is a g-ideal and there exists $x \in I$ such that x is invertible, then $I = R$.*

Lemma 3.3 *If $I \lhd_g R$, $J \lhd_g R$, $I \subseteq J$, then $I \lhd_g J$ and $J/I \lhd_g R/I$.*

Proof. Since $I \lhd_g R$, then there are an M-ring K and a homomorphism $f : R \longrightarrow K$ such that $ker f = I$. Since $J \lhd_g R$, then $f|_J : J \longrightarrow K$ is a homomorphism and $ker(f|_J) = I$. So $I \lhd_g J$.

Since $J \lhd_g R$, then there are an M-ring K' and a homomorphism $g : R \longrightarrow K'$ such that $ker g = J$, if we define $\overline{g} : R/I \to K'$ by $\overline{g}(x + I_a) = g(x)$, then it is a homomorphism and $ker(\overline{g}) = J/I$. So $J/I \lhd_g R/I$. □

Theorem 3.4 *If $I \lhd_g R$ and $T \lhd_g R/I$, then there exists $J \lhd_g R$ such that $I \subseteq J$ and $T = J/I$.*

Proof. Define $J := \{x \in R \mid x + I_x \in T\}$.

If $x \in I$, then $x + I_x = e(x) + I_x = e(x) + I_{e(x)} \in T$. So $x \in J$. Hence $I \subseteq J$.

One can prove that the properties (i), (ii), and (iii) of Theorem 3.1 are valid for J. So it is a g-ideal of R. The definition of J implies that $T = J/I$. □

Theorem 3.5 [**Second Isomorphism Theorem**] *If R is an M-ring and $I \lhd_g R$, $J \lhd_g R$ and $I \subseteq J$, then $(R/I)/(J/I) \cong R/J$.*

Proof. $R/I = \bigcup_{a\in R} R_a/I_a$ and $R/J = \bigcup_{a\in R} R_a/J_a$. We define $\varphi : R/I \longrightarrow R/J$ by $\varphi(x + I_a) = x + J_a$. We show that φ is a homomorphism.

If $x_1 + I_a = x_2 + I_a$, then $-x_1 + x_2 \in I_a$. So $-x_1 + x_2 \in J_a$. Hence $x_1 + J_a = x_2 + J_a$. Thus φ is well defined. It's clear that φ is a homomorphism, and $ker\varphi = J/I$. By the first isomorphism theorem for M-rings (see [7]) $(R/I)/(J/I) \cong \bigcup^o_{a\in R} \varphi((R/I)_{a+I_a}) = \bigcup^o_{a\in R} \varphi(R_a/I_a) = \bigcup^o_{a\in R} R_a/J_a = R/J$. So $(R/I)/(J/I) \cong R/J$. □

Lemma 3.6 *If R is an M-ring, $I \lhd_g R$, and $J \lhd_g R$, then $I + J = \bigcup_{a\in R}(I_a + J_a)$ and $(I + J)_a = I_a + J_a$.*

Proof. There are generalized rings K, K' and homomorphisms $f : R \longrightarrow K$ and $g : R \longrightarrow K'$ such that $I = ker f$ and $J = ker g$. It's clear that $\bigcup_{a\in R}(I_a + J_a) \subseteq I + J$.

If $x \in I + J$, then $x = \alpha + \beta$ for some $\alpha \in I_a$ and $\beta \in J_b$. Since $f_{a+b}(\alpha + \beta) = f_a(\alpha) + f_b(\beta) = f_a(e(a)) + f_b(\beta) = f_{a+b}(e(a) + \beta)$, so $\alpha + \beta + (-(e(a) + \beta)) \in I_{a+b}$. Since $g_{a+b}(e(a) + \beta) = g_a(e(a)) + g_b(\beta) = g_a(e(a)) + g_b(e(b)) = g_{a+b}(e(a) + e(b)) = g_{a+b}(e(a+b))$, then $e(a)+\beta \in J_{a+b}$. Thus $\alpha+\beta \in I_{a+b}+J_{a+b}$. So $I+J \subseteq \bigcup_{a\in R}(I_a+J_a)$. Hence $I + J = \bigcup_{a\in R}(I_a + J_a)$. Since $I + J = \bigcup_{a\in R}(I_a + J_a)$, then $(I + J)_a = I_a + J_a$.
□

If $I, J \lhd_g R$, then I_a and J_a are normal subgroups of R_a, so $I_a + J_a = J_a + I_a$, therefore $I + J = \bigcup_{a\in R}(I_a + J_a) = \bigcup_{a\in R}(J_a + I_a) = J + I$.

Corollary 3.7 *If R is an M-ring, $I \lhd_g R$ and $J \lhd_g R$, then $I + J$ is a g-ideal.*

Lemma 3.8 *Let $\{I_\gamma\}_{\gamma\in\Gamma}$ be a family of g-ideals of R, then $\bigcap_{\gamma\in\Gamma} I_\gamma$ is a g-ideal.*

Proof. It's easy to show that $(\bigcap_{\gamma\in\Gamma} I_\gamma)_a = \bigcap_{\gamma\in\Gamma}(I_\gamma)_a$, and by Theorem 3.1, it's clear that $\bigcap_{\gamma\in\Gamma} I_\gamma$ is a g-ideal. □

Theorem 3.9 [**Third Isomorphism Theorem**] *If R is an M-ring and $I \lhd_g R$, $J \lhd_g R$, then $(I + J)/I \cong J/(I \cap J)$.*

Proof. As the proof of the Lemma 3.8, we assume that $I = kerf$ and $J = kerg$. Since $I+J = \bigcup_{a\in R}(I_a+J_a)$, then we define $\varphi : I+J \longrightarrow J/(I \cap J)$ by $\varphi(\alpha+\beta) = \beta+(I+J)_a$, for all $a \in R$, $\alpha \in I_a$, and $\beta \in J_a$.

φ is a homomorphism, and $ker\varphi = I$. So the first isomorphism theorem implies $(I + J)/I \cong \bigcup_{a\in R} f((I + J)_a) = \bigcup_{a\in R} f(I_a + J_a) = \bigcup_{a\in R} J_a/(I \cap J)_a = J/(I \cap J)$. Thus $(I + J)/I \cong J/(I \cap J)$. □

Corollary 3.10 *If G is a normal completely simple semigroup* [5]*, and H, K are e-completely simple subsemigroups of G, then $(HK)/H \cong K/(H \cap K)$.*

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