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Infinitely many solutions for a perturbed quasilinear two-point boundary value problem

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Abstract In this paper we deal with the existence of infinitely many weak solutions for a perturbed quasilinear two-point boundary value problem. More precisely the existence of an exactly determined open interval of positive parameters for which the problem admits infinitely many weak solutions is established. Our proofs are based on variational methods.

Keywords infinitely many solutions · perturbed Dirichlet problem · critical point · three solutions · variational methods

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1 Introduction

Consider the following perturbed quasilinear two - point boundary value problem on a bounded interval $[a, b]$ in $\mathbb{R}$ $(a < b)$

$$\begin{cases} -u'' = (\lambda f(x,u) + \mu p(x,u) + g(x,u))h(x,u') \quad \text{in } (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{1.1}$$

In the statement of the problem (1.1), $f, p : [a, b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists a constant $L > 0$ provided $|g(., t_1) - g(., t_2)| \le L|t_1 - t_2|$, for all $t_1, t_2 \in \mathbb{R}$ satisfying $g(., 0) = 0$,

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$h : [a, b] \times \mathbb{R} \to]0, +\infty[$ is a bounded and continuous function with $m := \inf h > 0$, and λ is a positive parameter and μ is a non-negative parameter. Denote $M := \sup h$ and suppose that the constant $L > 0$ satisfies $LM(b-a)^2 < 4$.

Employing a smooth version of Theorem 2.1 of [7] which is a more precise version of Ricceri's Variational Principle [30, Theorem 2.5] (see Theorem 1.2), under some appropriate hypotheses on the behavior of the potential of f, under some conditions on the potentials of p, g and h, at infinity, we establish the existence of a precise interval of parameters Λ such that, for each $\lambda \in \Lambda$, the problem (1.1) admits a sequence of weak solutions which are unbounded in the Sobolev space $W_0^{1,2}([a, b])$; this is done in Theorem 2.1. We also list some special cases of Theorem 2.1. Further, replacing the conditions at infinity of the potentials of f, p, g and h, by a similar one at zero, the same results hold and, in addition, the sequence of weak solutions uniformly converges to zero; this is done in Theorem 2.8.

To the best of our knowledge, no investigation has been devoted to establishing the existence of infinitely many solutions to such a problem as (1.1).

For a discussion about the existence of infinitely many solutions for differential equations, using Ricceri's Variational Principle [30] and its variants ([7] and [27]), we refer the reader to the papers [3, 4, 8–11, 13–15, 18, 19, 21, 25, 31]. We also refer the reader to the papers [26, 28, 29] where the existence of infinitely many solutions for some boundary value problems has been studied by using different approaches.

We mean by a (weak) solution of the problem (1.1), any $u \in W_0^{1,2}([a, b])$ such that:

$$\int_a^b \left[\left(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau\right) v'(x) - g(x, u(x))v(x) \right] dx - \lambda \int_a^b f(x, u(x))v(x)dx$$
$$- \mu \int_a^b p(x, u(x))v(x)dx = 0, \text{ for every } v \in W_0^{1,2}([a, b]).$$

A special case of our main result is the following theorem.

Theorem 1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and $h : \mathbb{R} \to]0, +\infty[$ be a bounded and continuous function, with $m := \inf h > 0$. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$ and $H(t) = \int_0^t \int_0^\tau \frac{1}{h(\delta)} d\delta d\tau$ for each $t \in \mathbb{R}$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0$$

and

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{H(4\xi) + H(-4\xi)} = +\infty.$$

Then, the problem

$$\begin{cases} -u'' = f(u)h(u'), & \text{in } (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

admits a sequence of pairwise distinct positive classical solutions.

Our main tool is the celebrated Ricceri's Variational Principle [30, Theorem 2.5] that we now recall as given by Bonanno and Molica Bisci in [7]:

Theorem 1.2 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(*a*) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

(*b_1*) *I_λ possesses a global minimum, or*

(*b_2*) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(*c*) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

(*c_1*) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(*c_2*) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

For other basic notations and definitions, and for a thorough account on the subject, we refer the reader to [1, 2, 5, 12, 16, 17, 20, 22–24, 26, 32].

We recall that a function $f : [a, b]\times\mathbb{R} \to \mathbb{R}$ is said to be L^1-Carathéodory if:

(κ_1) $x \to f(x, t)$ is measurable for every $t \in \mathbb{R}$;

(κ_2) $t \to f(x, t)$ is continuous for almost every $x \in [a, b]$;

(κ_3) for every $\varrho > 0$ there exists a function $l_\varrho \in L^1([a, b])$ such that:

$$\sup_{|t|\le\varrho} |f(x, t)| \le l_\varrho(x)$$

for almost every $x \in [a, b]$.

2 Main results

Let $f, p : [a, b] \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions, $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists a constant $L > 0$ provided

$$|g(., t_1) - g(., t_2)| \le L|t_1 - t_2|, \tag{2.1}$$

for all $t_1, t_2 \in \mathbb{R}$ satisfying $g(., 0) = 0$. Let $h : [a, b] \times \mathbb{R} \to]0, +\infty[$ be a bounded and continuous function with $m := \inf h > 0$. Denote $M := \sup h$ and suppose that the constant $L > 0$ satisfies $LM(b - a)^2 < 4$.

We introduce the functions $F:[a,b]\times\mathbb{R}\to\mathbb{R}$, $P:[a,b]\times\mathbb{R}\to\mathbb{R}$, $H:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $G:[a,b]\times\mathbb{R}\to\mathbb{R}$, respectively, as follows

$$F(x,t)=\int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t)\in[a,b]\times\mathbb{R},$$
$$P(x,t)=\int_0^t p(x,\xi)d\xi, \quad \text{for all } (x,t)\in[a,b]\times\mathbb{R},$$
$$H(x,t)=\int_0^t\int_0^\tau \frac{1}{h(x,\delta)}d\delta d\tau, \quad \text{for all } (x,t)\in[a,b]\times\mathbb{R}$$

and

$$G(x,t)=-\int_0^t g(x,\xi)d\xi, \quad \text{for all } (x,t)\in[a,b]\times\mathbb{R}.$$

Now, we state our main result.

Theorem 2.1 *Assume that there exist two positive constants μ and ν with $\mu+\nu<b-a$ such that:*
(A1) $F(x,t)\geq 0$ *for each* $(x,t)\in([a,a+\mu]\cup[b-\nu,b])\times\mathbb{R}$;
(A2) $\liminf_{\xi\to+\infty}\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}F(x,t)dx}{\xi^2}$
$<\frac{2}{M(b-a)}\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{K_1(\xi)}$ *where* $K_1(\xi):=\int_a^{a+\mu}[G(x,\frac{\xi}{\mu}(x-a))+H(x,\frac{\xi}{\mu})]dx+\int_{a+\mu}^{b-\nu}G(x,\xi)dx+\int_{b-\nu}^{b}[G(x,\frac{\xi}{\nu}(b-x))+H(x,-\frac{\xi}{\nu})]dx.$

Then, for each $\lambda\in]\lambda_1,\lambda_2,[$ *where* $\lambda_1:=\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{K_1(\xi)}}$ *and*

$$\lambda_2:=\frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty}\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}F(x,t)dx}{\xi^2}},$$

for every non-negative L^1-Carathéodory function $p:[a,b]\times\mathbb{R}\to\mathbb{R}$ such that:

$$p_\infty:=lim_{\xi\to+\infty}\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}<+\infty \tag{2.2}$$

and for every $\mu\in[0,\mu_{p,\lambda}[$ *where*

$$\mu_{p,\lambda}:=\frac{2}{M(b-a)p_\infty}\cdot$$
$$\cdot\left(1-\lambda\frac{M(b-a)}{2}\liminf_{\xi\to+\infty}\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}\right),$$

the problem (1.1) *has an unbounded sequence of weak solutions in* $W_0^{1,2}([a,b])$.

Proof. In order to apply Theorem 1.2 to our problem, we take $X = W_0^{1,2}([a,b])$ equipped with the norm

$$\|u\| = \left(\int_a^b |u'(x)|^2 dx\right)^{1/2}.$$

Fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let p be a non-negative L^1-Carathéodory function satisfying the condition (2.2). Arguing as in [6], we follow the proof in the case $\mu > 0$. Since, $\overline{\lambda} < \lambda_2$, one has

$$\mu_{p,\overline{\lambda}} := \frac{2}{M(b-a)p_\infty} \cdot \left(1 - \overline{\lambda}\frac{M(b-a)}{2} \liminf_{\xi \to +\infty} \frac{\int_a^b \sup_{t \in [-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}, \sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} P(x,t)dx}{\xi^2}\right) > 0.$$

Fix $\overline{\mu} \in]0, \mu_{p,\overline{\lambda}}[$ and put $\nu_1 := \lambda_1$ and $\nu_2 := \frac{\lambda_2}{1 + \frac{M(b-a)}{2}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 p_\infty}$. If $p_\infty = 0$, clearly, $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\lambda \in]\nu_1, \nu_2[$. If $p_\infty \neq 0$, since $\overline{\mu} < \mu_{p,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2} + \frac{M(b-a)}{2}\overline{\mu}p_\infty < 1,$$

and so

$$\frac{\lambda_2}{1 + \frac{M(b-a)}{2}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 p_\infty} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_2$. Hence, bearing in mind that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in]\nu_1, \nu_2[$. We now introduce the functionals $\Phi, \Psi : X \to R$ given by

$$\Phi(u) = \int_a^b [G(x,u(x)) + H(x,u'(x))]dx \tag{2.3}$$

and

$$\Psi(u) = \int_a^b [F(x,u(x)) + \frac{\overline{\mu}}{\overline{\lambda}}P(x,u(x))]dx \tag{2.4}$$

for each $u \in X$. It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly lower semicontinuous, whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_a^b [f(x,u(x))v(x) + \frac{\overline{\mu}}{\overline{\lambda}}p(x,u(x))v(x)]dx,$$

for every $v \in X$. We claim that $\Psi' : X \to X^*$ is a compact operator. Indeed, for fixed $u \in X$, assume $u_n \to u$ weakly in X as $n \to +\infty$. Then $u_n \to u$ strongly in $C([a,b])$. Since $f(x,.)$ is continuous in $\mathbb{R}$ for every $x \in [a,b]$, we get that $f(x,u_n) + \frac{\overline{\mu}}{\overline{\lambda}}p(x,u_n) \to f(x,u) + \frac{\overline{\mu}}{\overline{\lambda}}p(x,u)$ strongly as $n \to +\infty$. By the Lebesgue control convergence theorem,

$\Psi'(u_n) \to \Psi'(u)$ strongly, which means that Ψ' is strongly continuous, so that it is then a compact operator. Hence the claim holds true.

Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\begin{aligned}\Phi'(u)(v) &= \int_a^b [H'(x, u'(x))v'(x) - g(x, u(x))v(x)]dx \\ &= \int_a^b \left[(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau) v'(x) - g(x, u(x))v(x) \right] dx,\end{aligned}$$

for every $v \in X$. Furthermore, Φ' is a Lipschitzian operator. Indeed, for any $u, v \in X$, taking (2.1) into account since

$$\max_{x\in[a,b]} |u(x)| \le \frac{(b-a)^{1/2}}{2} \|u\|, \tag{2.5}$$

for each $u \in X$, it holds that

$$\begin{aligned}\|\Phi'(u) - \Phi'(v)\|_{X^*} &= \sup_{\|w\|\le 1} | < \Phi'(u) - \Phi'(v), w > | \\ &\le \sup_{\|w\|\le 1} \int_a^b | \int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)} d\tau \| w'(x)| dx \\ &+ \sup_{\|w\|\le 1} \int_a^b |g(x, u(x)) - g(x, v(x))\| w(x)| dx \\ &\le (\frac{1}{m} + \frac{L}{4}(b-a)^2)\|u - v\|.\end{aligned}$$

In particular, Φ is continuously Gâteaux differentiable. Bearing (2.1) in mind, and using (2.5), we obtain

$$\begin{aligned}< \Phi'(u) - \Phi'(v), u - v > &= \int_a^b (\int_{v'(x)}^{u'(x)} \frac{1}{h(x,\tau)} d\tau)(u'(x) - v'(x))dx \\ &- \int_a^b (g(x, u(x)) - g(x, v(x)))(u(x) - v(x))dx \\ &\ge (\frac{1}{M} - \frac{L}{4}(b-a)^2)\|u - v\|^2,\end{aligned}$$

for $u,\ v \in X$. Due to the assumption $LM(b-a)^2 < 4$, it follows that Φ' is a strongly monotone operator, so Φ' is uniformly monotone, then we get that Φ is convex and continuous, and so is sequentially weakly lower semicontinuous.

Put $I_\lambda := \Phi - \lambda\Psi$. Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_\lambda(u) = 0$. Now, we want to verify that $\gamma < +\infty$. Let $\{\xi_n\}$

be a sequence of positive numbers such that $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]}[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi_n^2}$$
$$=\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi^2}.$$

Put $r_n = \frac{2\xi_n^2}{M(b-a)}$, for all $n \in \mathbb{N}$. Moreover, for each $u \in X$, since h is bounded away from zero and g is a continuous satisfies (2.1) with $g(.,0) = 0$, and bearing (2.5) in mind, from (2.3), we obtain

$$\Phi(u) \geq \frac{1}{2}(\frac{1}{M}-\frac{L}{4}(b-a)^2)\|u\|^2, \quad \text{for all } u \in X, \tag{2.6}$$

which yields

$$\begin{aligned}
&\Phi^{-1}(]-\infty,r_n[) = \{u\in X; \Phi(u)\leq r_n\}\\
&=\left\{u\in X; \frac{1}{2}(\frac{1}{M}-\frac{L}{4}(b-a)^2)\|u\|^2<r_n\right\}\\
&\subseteq \left\{u\in X; |u(x)|\leq\sqrt{\frac{2M(b-a)r_n}{4-LM(b-a)^2}},\ \text{for all } x\in[a,b]\right\},
\end{aligned}$$

Hence, taking into account that $\Phi(0) = \Psi(0) = 0$, for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v))-\Psi(u)}{r_n-\Phi(u)}\\
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)}{r_n}\\
&\leq \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]}\left[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)\right]dx}{\frac{2\xi_n^2}{M(b-a)}}\\
&\leq \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]}F(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}}\\
&+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}}.
\end{aligned}$$

Moreover, from Assumption (A2) and the condition (2.2) one has

$$\lim_{n\to\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} F(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}} + \lim_{n\to\infty} \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} P(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}} < +\infty,$$

from which follows

$$\lim_{n\to\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} [F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi_n^2} < +\infty.$$

Therefore,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \left(\frac{M(b-a)}{2}\right) \cdot \lim_{n\to\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]} [F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi_n^2} < +\infty. \tag{2.7}$$

Since

$$\frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} [F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\frac{2\xi_n^2}{M(b-a)}} \le \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} F(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}\ ]} P(x,t)dx}{\frac{2\xi_n^2}{M(b-a)}},$$

taking (2.2) into account, one has

$$\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\ ]} [F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi^2} \le \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\ ]} F(x,t)dx}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}} p_\infty. \tag{2.8}$$

Moreover, since p is nonnegative, from Assumption (A1) we obtain

$$\limsup_{\xi\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}[F(x,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,\xi)]dx}{\xi^2}\geq\limsup_{\xi\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,\xi)dx}{\xi^2}. \tag{2.9}$$

Therefore, from (2.8) and (2.9), we observe

$$\overline{\lambda}\in]\nu_1,\nu_2[\subseteq\Bigg]\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}[F(x,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,\xi)]dx}{K_1(\xi)}},$$

$$\frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi^2}}\Bigg[.$$

Assumption (A2) in conjunction with (2.7), implies

$$\Bigg]\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}[F(x,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,\xi)]dx}{K_1(\xi)}},$$

$$\frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,t)]dx}{\xi^2}}\Bigg[\subseteq\Bigg]0,\frac{1}{\gamma}\Bigg[.$$

For the fixed $\overline{\lambda}$, the inequality (2.7) implies that the condition (b) of Theorem 1.2 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the system (1.1) such that $\lim_{k\to\infty}\|u_n\|=+\infty$.

The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{K_1(\xi)}\leq\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}[F(x,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,\xi)]dx}{K_1(\xi)},$$

we can consider a real sequence $\{d_n\}$ and a positive constant τ such that $d_n\to+\infty$ as $n\to\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}[F(x,d_n)+\frac{\overline{\mu}}{\overline{\lambda}}P(x,d_n)]dx}{K_1(d_n)} \tag{2.10}$$

for each $k\in\mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by

$$w_n(x)=\begin{cases}\frac{d_n}{\mu}(x-a), & \text{if } a\leq x<a+\mu,\\ d_n, & \text{if } a+\mu\leq x\leq b-\nu,\\ \frac{d_n}{\nu}(b-x), & \text{if } b-\nu<x\leq b\end{cases} \tag{2.11}$$

For any fixed $n \in \mathbb{N}$, it is easy to see that $w_n \in X$ and, in particular, one has

$$\Phi(w_n) = K_1(d_n). \tag{2.12}$$

On the other hand, bearing Assumption (A1) in mind from the definition of Ψ, we infer

$$\Psi(w_n) \geq \int_{a+\mu}^{b-\nu} [F(x, d_n) + \frac{\overline{\mu}}{\overline{\lambda}} P(x, d_n)]dx. \tag{2.13}$$

So, according to (2.10), (2.12) and (2.13) (note $K_1(d_n) > 0$), we obtain

$$I_\lambda(w_n) \leq K_1(d_n) - \lambda \int_{a+\mu}^{b-\nu} [F(x, d_n) + \frac{\overline{\mu}}{\overline{\lambda}} P(x, d_n)]dx < K_1(d_n)(1 - \lambda\tau),$$

for every $n \in \mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, applying Theorem 1.2 we deduce that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{k\to\infty} \|u_n\| = +\infty$. Hence, the conclusion is achieved. □

Remark 2.1 Arguing as in [4, Remark 3.3] we notice that instead of Assumption (A2) in Theorem 2.1 we are allowed to suppose the following more general condition:

(A3) There exist two sequence $\{\alpha_n\}$ and $\{\beta_n\}$ with $K_1(\alpha_n) < \frac{2\beta_n^2}{M(b-a)}$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \beta_n = +\infty$ such that:

$$\lim_{n\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}},\, \sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}}\,]} F(x,t)dx - \int_{a+\mu}^{b-\nu} F(x,\alpha_n)dx}{\frac{2b_n^2}{M(b-a)} - K_1(\alpha_n)}$$

$$< \limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{K_1(\xi)}.$$

Obviously, from (A3) we obtain (A2), by choosing $\alpha_n = 0$, for all $n \in \mathbb{N}$. Moreover, if we assume (A3) instead of (A2) and set $r_n = \frac{2\beta_n^2}{M(b-a)}$, for all $n \in \mathbb{N}$, by the same argument as in Theorem 2.1, we obtain

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n])} \Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])} \Psi(v) - \int_\Omega F(x, w_n(x))dx}{r_n - \int_a^b [G(x,w(x)) + H(x,w'(x))]dx} \\ &\leq \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}},\, \sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}}\,]} F(x,t)dx - \int_{a+\mu}^{b-\nu} F(x,\alpha_n)dx}{\frac{2\beta_n^2}{M(b-a)} - K_1(\alpha_n)}\end{aligned}$$

where $w_n(x)$ is defined as given in (2.8), for $x \in [a,b]$, with α_n instead of d_n. We then have the same conclusion as in Theorem 2.1 with λ_2 replaced by

$$\lambda_2' := \frac{1}{\lim_{n\to+\infty} \frac{\int_a^b \sup_{t\in\left[-\sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\beta_n^2}{4-LM(b-a)^2}}\right]} F(x,t)dx - \int_{a+\mu}^{b-\nu} F(x,\alpha_n)dx}{\frac{2\beta_n^2}{M(b-a)} - K_1(\alpha_n)}}.$$

The following result is a special case of Theorem 2.1 with $\mu = 0$.

Theorem 2.2 *Assume that all assumptions of Theorem* 2.1 *hold. Then, for each*

$$\lambda \in \Lambda_1 := \left]\frac{1}{\limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{K_1(\xi)}}, \frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in\left[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\right]} F(x,t)dx}{\xi^2}}\right[$$

the problem

$$\begin{cases} -u'' = (\lambda f(x,u) + g(x,u))h(x,u'), & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{2.14}$$

has an unbounded sequence of weak solutions in $W_0^{1,2}([a,b])$.

Here we point out the following consequence of Theorem 2.2.

Corollary 2.3 *Assume there exist two positive constants μ and ν with $\mu + \nu < b - a$ such that Assumption* (*A*1) *in Theorem* 2.1 *holds. Furthermore, suppose that*

(*A*4) $\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in\left[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\right]} F(x,t)dx}{\xi^2} < \frac{2}{M(b-a)}$;

(*A*5) $\limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{K_1(\xi)} > 1$ *where $K_1(\xi)$ is given as in Assumption* (*A*2).

Then, the problem (2.14) *has an unbounded sequence of weak solutions in X.*

Remark 2.2 Theorem 1.1 in the Introduction is an immediately consequence of Corollary 2.3 by setting $[a,b] = [0,1]$, $f(x,t) = f(t)$, $g(t) = 0$ and $h(x,t) = h(t)$, for all $x \in [0,1]$ and $t \in \mathbb{R}$.

It is of interest to list some special cases of Theorem 2.1.

Let f and F be as before, and let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e. $|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$, for all $t_1, t_2 \in \mathbb{R}$, satisfying $g(0) = 0$. Let $h : \mathbb{R} \to]0,+\infty[$ be a bounded and continuous function with $m := \inf h > 0$. Denote $M := \sup h$ and suppose that the Lipschitz constant $L > 0$ satisfies $LM(b-a)^2 < 4$. We introduce the functions $H : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$H(t) = \int_0^t \int_0^\tau \frac{1}{h(\delta)} d\delta d\tau, \quad \text{for all } t \in \mathbb{R}$$

and

$$G(t) = -\int_0^t g(\xi)d\xi, \quad \text{for all } t \in \mathbb{R}.$$

Then, we have the following result:

Theorem 2.4 *Assume there exist two positive constants μ and ν with $\mu + \nu < b - a$ such that Assumption $(A1)$ in Theorem 2.1 holds. Furthermore, suppose that:*

$(A6)$ $\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} F(x,t)dx}{\xi^2}$ $< \frac{2}{M(b-a)} \limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{K_2(\xi)}$ *where* $K_2(\xi) := \frac{\mu+\nu}{\xi}\int_0^\xi G(t)dt + (b-a-\mu-\nu)G(\xi) + \mu H(\frac{\xi}{\mu}) + \nu H(-\frac{\xi}{\nu})$*. Then, for each*

$$\lambda \in \Lambda_2 := \left] \frac{1}{\limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{K_2(\xi)}}, \right.$$

$$\left. \frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} F(x,t)dx}{\xi^2}} \right[,$$

for every non-negative L^1-Carathéodory function $p : [a,b] \times \mathbb{R} \to \mathbb{R}$ such that:

$$p_\infty := lim_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} P(x,t)dx}{\xi^2} < +\infty$$

and for every $\mu \in [0, \mu_{p,\lambda}[$ where

$$\mu_{p,\lambda} := \frac{2}{M(b-a)p_\infty} \cdot$$

$$\cdot \left(1 - \lambda \frac{M(b-a)}{2} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} P(x,t)dx}{\xi^2}\right),$$

the problem

$$\begin{cases} -u'' = (\lambda f(x,u) + \mu p(x,u) + g(u))h(u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{2.15}$$

has an unbounded sequence of weak solutions in X.

Proof. Setting $g(x,t) = g(t)$ and $h(x,t) = h(t)$, for all $(x,t) \in [a,b] \times \mathbb{R}$, then from Theorem 2.1 we have the conclusion. □

Now, we consider a special situation of the result. Set $p(x,t)=0$ and $g(t)=0$, for all $x\in[a,b]$ and $t\in\mathbb{R}$. Then, we have the following consequence of Theorem 2.4.

Theorem 2.5 *Assume there exist two positive constants μ and ν with $\mu+\nu<b-a$ such that Assumption (A1) in Theorem 2.1 holds. Furthermore, suppose that:*

(*A7*) $\liminf_{\xi\to+\infty}\frac{\int_a^b \sup_{|t|\le\xi}F(x,t)dx}{\xi^2}<\frac{2}{M(b-a)}\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})}$. *Then, for each*

$$\lambda\in\Lambda_3:=\left]\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})}},\ \frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{|t|\le\xi}F(x,t)dx}{\xi^2}}\right[$$

for every non-negative L^1-Carathéodory function $p:[a,b]\times\mathbb{R}\to\mathbb{R}$ such that:

$$p_\infty:=lim_{\xi\to+\infty}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}<+\infty$$

and for every $\mu\in[0,\mu_{p,\lambda}[$ where

$$\mu_{p,\lambda}:=\frac{2}{M(b-a)p_\infty}\cdot$$
$$\cdot\left(1-\lambda\frac{M(b-a)}{2}\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}\right),$$

the problem

$$\begin{cases}-u''=(\lambda f(x,u)+\mu p(x,u))h(u') & \text{in } (a,b),\\ u(a)=u(b)=0,\end{cases}\tag{2.16}$$

has an unbounded sequence of weak solutions in X.

We now exhibit an example in which the hypotheses of Theorem 2.5 are satisfied.

Example 2.1 Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x,t)=\begin{cases}f^*(x)te^t(2+t-\cos(\ln(|t|))-(2+t)\sin(\ln(|t|))),\\ \qquad \text{if } (x,t)\in[a,b]\times(\mathbb{R}-\{0\}),\\ 0,\quad \text{if } (x,t)\in[a,b]\times\{0\},\end{cases}$$

where $f^*:[a,b]\to\mathbb{R}$ is a non-negative continuous function, and let $h(y)=\frac{1}{2-\sin y}$ for each $y\in\mathbb{R}$. A direct calculation shows

$$F(x,t)=\begin{cases}f^*(x)t^2e^t(1-\sin(\ln(|t|))), & \text{if } (x,t)\in[a,b]\times(\mathbb{R}-\{0\}),\\ 0, & \text{if } (x,t)\in[a,b]\times\{0\},\end{cases}$$

and $H(y) = y^2 - y + \sin y$ for each $y \in \mathbb{R}$. So,

$$\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{|t|\le\xi} F(x,t)dx}{\xi^2} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{\mu H(\frac{\xi}{\mu}) + \nu H(-\frac{\xi}{\nu})} = +\infty$$

for some positive constants μ and ν with $\mu+\nu < b-a$. Hence, using Theorem 2.5, the problem (2.16), in this case, with $p(x,t) = e^{x-t^+}(t^+)^\gamma$ where $t^+ = \max\{t,0\}$ and γ is a positive real number, for all $(x,t) \in [0,1]\times\mathbb{R}$ for every $(\lambda,\mu) \in]0,+\infty[\times[0,+\infty[$ has an unbounded sequence of weak solutions in X.

Set $h(t) = 1$, for all $t \in \mathbb{R}$. Then, we have the following consequence of Theorem 2.5.

Theorem 2.6 *Assume there exist two positive constants μ and ν with $\mu+\nu < b-a$ such that Assumption (A1) in Theorem 2.1 holds. Furthermore, suppose that*

(A8) $\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{|t|\le\xi} F(x,t)dx}{\xi^2} < \frac{4\mu\nu}{(b-a)(\mu+\nu)} \limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{\xi^2}$. *Then, for each*

$$\lambda \in \Lambda_4 := \left] \frac{\frac{\mu+\nu}{2\mu\nu}}{\limsup_{\xi\to+\infty} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{\xi^2}}, \frac{\frac{2}{(b-a)}}{\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{|t|\le\xi} F(x,t)dx}{\xi^2}} \right[,$$

for every non-negative L^1-Carathéodory function $p : [a,b]\times\mathbb{R}\to\mathbb{R}$ such that:

$$p_\infty := lim_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} P(x,t)dx}{\xi^2} < +\infty$$

and for every $\mu \in [0,\mu_{p,\lambda}[$ where

$$\mu_{p,\lambda} := \frac{2}{M(b-a)p_\infty}\cdot$$
$$\cdot\left(1 - \lambda\frac{M(b-a)}{2} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]} P(x,t)dx}{\xi^2}\right),$$

the problem

$$\begin{cases} -u'' = \lambda f(x,u) + \mu p(x,u), & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{2.17}$$

has an unbounded sequence of weak solutions in X.

We end this paper by giving the following consequence:

Corollary 2.7 *Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, and denote that $G_1(t) = \int_0^t g_1(\xi)d\xi$ for all $t \in \mathbb{R}$. Assume there exist two positive constants μ and ν with $\mu + \nu < b - a$ such that:*

(A9) $\liminf_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^2} < +\infty$ *;*

(A10) $\limsup_{\xi\to+\infty} \frac{G_1(\xi)}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})} = +\infty.$

Then, for every $\alpha_i \in L^1([a,b])$ for $1 \le i \le n$, with $\min_{x\in[a,b]}\{\alpha_i(x);\ 1 \le i \le n\} \ge 0$ and with $\alpha_1 \neq 0$, and for every non-negative continuous $g_i : \mathbb{R} \to \mathbb{R}$, for $2 \le i \le n$, satisfying

$$\max\left\{\sup_{\xi\in\mathbb{R}} G_i(\xi)dt;\ 2 \le i \le n\right\} \le 0$$

and

$$\min\left\{\liminf_{\xi\to+\infty} \frac{G_i(\xi)}{\xi^2};\ 2 \le i \le n\right\} > -\infty,$$

where $G_i(t) = \int_0^t g_i(\xi)d\xi$, for all $t \in \mathbb{R}$ for $2 \le i \le n$, for each

$$\lambda \in \Lambda_5 := \left]0,\ \frac{2}{(M(b-a)\int_a^b \alpha_1(x)dx)\liminf_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^2}}\right[,$$

for every non-negative L^1-Carathéodory function $p : [a,b]\times\mathbb{R} \to \mathbb{R}$ such that:

$$p_\infty := lim_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\ ]} P(x,t)dx}{\xi^2} < +\infty$$

and for every $\mu \in [0, \mu_{p,\lambda}[$ where

$$\mu_{p,\lambda} := \frac{2}{M(b-a)p_\infty}\cdot$$
$$\cdot\left(1 - \lambda\frac{M(b-a)}{2}\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\ \sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}\ ]} P(x,t)dx}{\xi^2}\right),$$

the problem

$$\begin{cases} -u'' = \lambda h(u')\sum_{i=1}^n \alpha_i(x)g_i(u) + \mu p(x,u) & \text{in } (a,b),\\ u(a) = u(b) = 0, \end{cases} \tag{2.18}$$

has an unbounded sequence of weak solutions in $W_0^{1,2}([a,b])$.

Proof. Set $f(x,t) = \sum_{i=1}^n \alpha_i(x)g_i(t)$, for all $(x,t) \in [a,b]\times\mathbb{R}$. The assumption (A10) together with the condition

$$\min\left\{\liminf_{\xi\to+\infty} \frac{G_i(\xi)}{\xi^2};\ 2 \le i \le n\right\} > -\infty$$

yields

$$\limsup_{\xi\to+\infty}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})}=\limsup_{\xi\to+\infty}\frac{\sum_{i=1}^{n}(G_i(\xi)\int_{a+\mu}^{b-\nu}\alpha_i(x)dx)}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})}=+\infty.$$

Moreover, the assumption (A9) in conjunction with the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}}G_i(\xi)dt;\ 2\le i\le n\right\}\le 0$$

implies

$$\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{|t|\le\xi}F(x,t)dx}{\xi^2}\le\left(\int_a^b\alpha_1(x)dx\right)\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2}<+\infty.$$

Hence, applying Theorem 2.5 we have the result. □

Arguing as in the proof of Theorem 2.1, but using conclusion (c) of Theorem 1.2 instead of (b), the following result holds.

Theorem 2.8 *Assume there exist two positive constants μ and ν with $\mu+\nu<b-a$ such that Assumption (A1) in Theorem 2.1 holds and*

$(B1)$ $\liminf_{\xi\to 0^+}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}F(x,t)dx}{\xi^2}$
$<\frac{2}{M(b-a)}\limsup_{\xi\to 0^+}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{K_1(\xi)}$ *where $K_1(\xi)$ is given as in Assumption (A2). Then, for each*

$$\lambda\in\Lambda_7:=\left]\frac{1}{\limsup_{\xi\to 0^+}\frac{\int_{a+\mu}^{b-\nu}F(x,\xi)dx}{K_1(\xi)}},\right.$$

$$\left.\frac{\frac{2}{M(b-a)}}{\liminf_{\xi\to 0^+}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}F(x,t)dx}{\xi^2}}\right[,$$

for every non-negative L^1-Carathéodory function $p:[a,b]\times\mathbb{R}\to\mathbb{R}$ such that:

$$p_0:=lim_{\xi\to 0^+}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}<+\infty$$

and for every $\mu\in[0,\mu_{p,\lambda}[$ where

$$\mu_{p,\lambda}:=\frac{2}{M(b-a)p_0}\cdot$$
$$\cdot\left(1-\lambda\frac{M(b-a)}{2}\liminf_{\xi\to 0^+}\frac{\int_a^b\sup_{t\in[-\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi^2}{4-LM(b-a)^2}}]}P(x,t)dx}{\xi^2}\right),$$

the problem (1.1) *has a sequence of weak solutions, which strongly converges to 0 in* X.

Proof. We take Φ, Ψ and I_λ to be as before. By a similar way as in the proof of Theorem 2.1 we verify that $\delta < +\infty$ For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_a^b \sup_{t\in[-\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}},\sqrt{\frac{4\xi_n^2}{4-LM(b-a)^2}}]}[F(x,t)dx + \frac{\mu}{\lambda}P(x,t)]}{\xi_n^2} < +\infty.$$

Put $r_n = \frac{2\xi_n^2}{M(b-a)}$, for all $n \in \mathbb{N}$, arguing as in the proof of Theorem 2.1, it follows that $\delta < +\infty$. Fix $\lambda \in \Lambda_7$. We claim that the functional I_λ does not have a local minimum at zero. Let $\{d_n\}$ be a sequence of positive numbers and $\tau > 0$ such that $d_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\lambda} < \tau < \frac{\int_{a+\mu}^{b-\nu} F(x,d_n)dx}{K_1(d_n)} \tag{2.19}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined as given in (2.8). According to (2.12), (2.13) and (2.19), we obtain $I_\lambda(w_n) \le K_1(d_n) - \lambda \int_{a+\mu}^{b-\nu} F(x,d_n)dx < K_1(d_n)(1-\lambda\tau) < 0$, for every $n \in \mathbb{N}$ large enough. Since $I_\lambda(0) = 0$, this derives our claim. Hence, the part (c) of Theorem 1.2 ensures that there exists a sequence $\{u_n\}$ in X of critical points of I_λ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete. □

Remark 2.3 We explicitly observe that in Remark 2.1 and Corollary 2.3, by Theorem 2.8 and replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same reasoning, we have that the problem (1.1) for every $\lambda \in \Lambda_2$, in this case, and the problem (2.14), respectively, has a sequence of weak solutions, which strongly converges to 0 in X. Also, we notice that in Theorems 2.2- 2.6 by Theorem 2.8 and replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same argument, we deduce that the problem (2.14) for every $\lambda \in \Lambda_1$, in this case, the problem (2.15) for every $\lambda \in \Lambda_2$, in this case, and the problem (1.1) for every $\lambda \in \Lambda_3$, in this case, and the problem (2.16) for every $\lambda \in \Lambda_4$, in this case, and the problem (2.17) for every $\lambda \in \Lambda_5$, respectively, as well as Corollary 2.7, replacing $\xi \to +\infty$ with $\xi \to 0^+$, for every $\lambda \in \Lambda_6$, has a sequence of weak solutions, which strongly converges to 0 in X.

Finally, we give an application of Theorem 2.8.

Example 2.2 Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) = \begin{cases} f^*(x)(1-\cos(\ln(|t|)) - \sin(\ln(|t|))), & \text{if } (x,t) \in [a,b]\times(\mathbb{R}-\{0\}), \\ 0, & \text{if } (x,t) \in [a,b]\times\{0\}, \end{cases}$$

where $f^* : [a,b] \to \mathbb{R}$ is a non-negative continuous function, and let $h(y) = \frac{1}{2+\cos y}$ for each $y \in \mathbb{R}$. A direct calculation shows

$$F(x,t) = \begin{cases} f^*(x)t(1-\sin(\ln(|t|))), & \text{if } (x,t) \in [a,b]\times(\mathbb{R}-\{0\}), \\ 0, & \text{if } (x,t) \in [a,b]\times\{0\}, \end{cases}$$

and $H(y) = y^2 - \cos y + 1$ for each $y \in \mathbb{R}$. So, $\liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{|t|\le\xi} F(x,t)dx}{\xi^2} = 0$ and $\limsup_{\xi\to 0^+} \frac{\int_{a+\mu}^{b-\nu} F(x,\xi)dx}{\mu H(\frac{\xi}{\mu})+\nu H(-\frac{\xi}{\nu})} = +\infty$, for some positive constants μ and ν with $\mu+\nu < b-a$. Hence, using Theorem 2.8, and taking Remark 2.3 into account, the problem (2.14), in this case, for every $\lambda \in]0,+\infty[$, has a sequence of weak solutions, which strongly converges to 0 in X.

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