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Approximation properties of bivariate generalization of Meyer-König and Zeller type operators

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Abstract In this paper, a bivariate generalization of a general sequence of Meyer-König and Zeller (MKZ) operators based on q-integers is constructed. Approximation properties of these operators are obtained by using either Korovkin-type statistical approximation theorem or Heping-type convergence theorem for bivariate functions. Rates of statistical convergence by means of modulus of continuity and the elements of Lipschitz class functionals are also established.

Keywords linear positive operators · Heping-type convergence · statistical convergence · q-Meyer-König Zeller operators · rate of statistical convergence · modulus of continuity · Lipschitz class

Mathematics Subject Classification (2010) 41A36 · 41A25

1 Introduction

The classical Meyer-König and Zeller (MKZ) operators are defined by

$$M_n(f,x)=\begin{cases}(1-x)^{n+1}\sum\limits_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k, & \text{if } x\in[0,1)\\ f(1), & \text{if } x=1,\end{cases} \tag{1.1}$$

for $f\in C[0,1]$, $n\in\mathbb{N}$ (see [10]).

These operators were modified by Cheney and Sharma in [3] as follows:

$$H_n(f,x)=\begin{cases}(1-x)^{n+1}\sum\limits_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k, & \text{if } x\in[0,1)\\ f(1), & \text{if } x=1.\end{cases} \tag{1.2}$$

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In [3], Cheney and Sharma obtained monotonicity of the sequence of the operators defined by (1.2), for n, when the function f is convex.

The q-type generalization of positive linear operators was originated by Phillips [16]. He introduced the q-type generalization of the classical Bernstein operators and obtained the rate of convergence and Voronovskaja-type asymtotic formula for these operators. Phillips, Goodman and Oruç ([10], [14]) studied similar problems in detail. Using similar idea, Trif [18] defined the MKZ operators based on the q-integers as follows:

$$F_n(f,q,x)=\begin{cases}\prod_{s=0}^{n}(1-q^s x)\sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q x^k, & \text{if } x\in[0,1)\\ f(1), & \text{if } x=1.\end{cases} \tag{1.3}$$

Then, Trif [18] studied the approximation properties and obtained the rate of convergence by using the modulus of continuity and monotonocity properties of the operators $F_n(f,q,x)$, for $q\in(0,1]$ and $x\in[0,1)$.

Here, we recall some definitions about q-integers. For each non-negative integer k and $q\in(0,1]$, the q-integers, $[k]_q$, and the q-factorial, $[k]_q!$, are defined by [1]:

$$[k]_q:=\begin{cases}\frac{1-q^k}{1-q}, & \text{if } q\neq 1\\ k\ , & \text{if } q=1,\end{cases}$$

$$[k]_q!:=\begin{cases}[k]_q[k-1]_q\ldots[1]_q\,, & \text{if} k\geq 1\\ 1, & \text{if } k=0,\end{cases}$$

respectively.

For the integers n, k, $n\geq k\geq 0$, the q-binomial or the Gaussian coefficient is defined by [1]:

$$\begin{bmatrix}n\\k\end{bmatrix}_q:=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}\ .$$

In [5], Doğru and Duman introduced the following different kind of q-MKZ operators and studied statistical approximation properties of such operators:

$$R_n(f,q,x)=\begin{cases}\prod_{s=0}^{n}(1-q^s x)\sum_{k=0}^{\infty} f\left(\frac{q^n[k]_q}{[n+k]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q x^k, & \text{if } x\in[0,1)\\ f(1), & \text{if } x=1.\end{cases} \tag{1.4}$$

Statistical convergence was first introduced by Fast [8] nearly fifty years ago and it has become an area of active research.

Now, we give concepts of statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K):=\lim_{n\to\infty}\frac{1}{n}\left|\{k\in K:k\leq n\}\right|$ provided the limit exists [13]. So the sequence $x=(x_k)$ is said to be statistically convergent to a number L means that if for every $\varepsilon>0$, $\delta\{k:|x_k-L|\geq\varepsilon\}=0$ and it is denoted by $st-\lim_{k\to\infty}x_k=L$. It is easy to see that every convergent sequence is statistically convergent but converse is not true.

Recently, combining (1.3) and (1.4) ÖZARSLAN and DUMAN [15] also introduced the following modification of q-MKZ operators:

$$T_n\left(f,q,x\right)=\begin{cases}\prod\limits_{s=0}^{n}\left(1-q^sx\right)\sum\limits_{k=0}^{\infty}f\left(\frac{a_n(q)[k]_q}{[n+k]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q x^k, & \\ & \text{if } x\in[0,1) \\ f\left(1\right), & \text{if } x=1,\end{cases} \tag{1.5}$$

where $(a_n(t))$ is a function sequence defined on the interval $[0,1]$ such that $0<a_n(t)\leq 1$ for all $n\in\mathbb{N}$. They obtained a Korovkin-type approximation theorem and computed the rates of convergence of these operators by means of modulus of continuity and the elements of Lipschitz class functionals using the following results:

$$\begin{aligned} T_n\left(e_0;q,x\right)&=1,\\ T_n\left(e_1;q,x\right)&=a_n\left(q\right)x,\\ a_n^2\left(q\right)x^2\leq T_n\left(e_2;q,x\right)&\leq qa_n^2\left(q\right)x^2+\frac{a_n^2(q)x}{[n]_q},\end{aligned} \tag{1.6}$$

where $n\in\mathbb{N}$, $x\in[0,1]$ and $q\in(0,1]$.

Replacing q in (1.6) by a sequence $(q_n)\in(0,1]$ for all $n\in\mathbb{N}$ so that

$$\lim_{n\to\infty}a_n\left(q_n\right)=\lim_{n\to\infty}q_n=1 \text{ and } \lim_{n\to\infty}[n]_{q_n}=\infty \tag{1.7}$$

and using (1.7), we realize that the Korovkin-type approximation properties are obtained for the operators (1.5). ÖZARSLAN and DUMAN gave such a sequence in [15].

Now, we recall some definitions. As usual, $C[a,b]$ denotes the space of all real valued continous functions defined on $[a,b]$. Then the space $C[a,b]$ is a Banach space with the usual norm $\|.\|$ given by $\|f\|=\sup_{x\in[a,b]}|f\left(x\right)|$, $f\in C[a,b]$. As usual, throughout the paper we use the test functions $e_i\left(x\right)=x^i$ for $i=0,1,2$.

In this paper, firstly we obtain statistically approximation properties of the (1.5) operators for $f\in C[0,b]$, $0<b<1$, with the help of Korovkin-type theorem proved by GADJIEV and ORHAN [9] and estimate the rate of statistically convergence of the sequence of the operators to the function f.

Secondly, we consider the following questions related with convergence of the (1.5) operators for $x\in[0,b]$, $0<b<1$ and positive answer to this question is given with the help of a Heping-type theorem proved by DOĞRU and GUPTA [6].

If we choose a sequence $(a_n(q_n))$ instead of a fixed $q\in(0,1]$ such that $0<q_n\leq 1$ and the following conditions:

$$\lim_{n\to\infty}\left(a_n\left(q_n\right)\right)=c,\ \ (c\in\mathbb{R}) \text{ and } \lim_{n\to\infty}q_n=1, \tag{1.8}$$

can the approximation properties of the (1.5) operators still be obtained for $x\in[0,b]$, $0<b<1$.

For instance, if we choose $(a_n(q_n))=q_n^n$, $(q_n)=\left(1-\frac{1}{n}\right)$, then the conditions (1.8) are satisfied $c=e^{-1}$. On the other hand, the conditions in (1.8) guarantee that $\lim_{n\to\infty}[n]_q=\infty$.

Finally, the aim of this paper is to construct a Stancu-type bivariate extension of the (1.5) operators, to give either the statistical or Heping-type convergence of these operators to the function f and also to compute the rate of statistical convergence of these operators.

2 Heping-type convergence

In [6], Doğru and Gupta proved the following Heping-type theorem given by Heping in [11] for any sequence of positive linear operators.

Theorem A ([6]). *Let the sequence (L_n) of positive linear operators on $C[0,b]$ satisfy:*

(i) The sequence $(L_n(e_i))$ converges to a function $L_\infty(e_i)$ for $i=1,2$ in $C[0,b]$.
(ii) $(L_n(f;x))$ is non-increasing for any convex and increasing function f and for any $x\in[0,b]$.
Then there is an operator L_∞ on $C[0,b]$ such that $\|L_n(f)-L_\infty(f)\|_{C[0,b]}\to 0$ as $n\to\infty$, for every $f\in C[0,b]$.

Theorem 1. *Let $a_n(q_n)$ be a real decreasing sequence such that $0<q_n\le 1$, for all $n\in\mathbb{N}$ satisying the conditions in (1.8). Then, there is an operator T_∞ on $C[0,b]$ such that $\|T_n(f,q_n)-T_\infty(f,q_n)\|_{C[0,b]}\to 0$ as $n\to\infty$, for all $f\in C[0,b]$, $(0<b<1)$.*

To obtain the proof of Theorem 1, we need the following results.

Lemma 2. ([6]) *Let $\alpha:=\frac{[n+1]_q}{[n+k+1]_q}$ and $\beta:=\frac{q^{n+1}[k]_q}{[n+k+1]_q}$. Then $\alpha+\beta=1$ and*

$$\frac{[k]_q}{[n+k+1]_q}=\alpha\frac{[k]_q}{[n+k]_q}+\beta\frac{[k-1]_q}{[n+k]_q}. \tag{2.1}$$

Theorem 3. *If $f:[0,b]\to\mathbb{R}^+$ is a convex and increasing function, $a_n(q)$ is a real decreasing sequence, for all $q\in(0,1]$, then the sequence $(T_n(f,q,x))$ is non-increasing in n, for all $q\in(0,1]$ and $x\in[0,b]$.*

Proof.

$$\begin{aligned} &T_n(f;q,x)-T_{n+1}(f;q,x)\\ &=\prod_{s=0}^{n}(1-q^s x)\sum_{k=1}^{\infty}\left\{\frac{[n+1]_q}{[n+k+1]_q}f\left(\frac{a_n(q)[k]_q}{[n+k]_q}\right)\right.\\ &\left.-f\left(\frac{a_{n+1}(q)[k]_q}{[n+k+1]_q}\right)+\frac{q^{n+1}[k]_q}{[n+k+1]_q}f\left(\frac{a_{n+1}(q)[k-1]_q}{[n+k]_q}\right)\right\}\\ &\times\begin{bmatrix} n+k+1\\ k\end{bmatrix}_q x^k. \end{aligned} \tag{2.2}$$

By choosing α and β as in Lemma 2 and substituting (2.1) into (2.2), we arrive that

$$\begin{aligned} T_n(f;q,x)-T_{n+1}(f;q,x)\ge\prod_{s=0}^{n}(1-q^s x)\sum_{k=1}^{\infty}\{\alpha f(x_1)+\beta f(x_2)\\ -f(\alpha x_1+\beta x_2)\}\times\begin{bmatrix} n+k+1\\ k\end{bmatrix}_q x^k, \end{aligned}$$

where $x_1=\frac{a_{n+1}(q)[k]_q}{[n+k]_q}$, $x_2=\frac{a_{n+1}(q)[k-1]_q}{[n+k]_q}$.

Because of convexity of f, we say that $(T_n(f; q_n, x))$ is non-increasing in n.

Proof of Theorem 1. From (1.6), we get that $(T_n(e_2; q_n, x))$ converges to $T_\infty(e_2; q_n, x) = x^2 c^2$ in $C[0, b]$, $(0 < b < 1)$. Thus we have established the property (i) of Theorem A. In addition to, (ii) of Theorem A follows from Theorem 3, so the proof is complete.□

3 Statistical approximation properties

In this section, we obtain statistical approximation properties of the operators defined by (1.5) with the help of the following statistical Korovkin-type theorem proved by Gadjiev and Orhan [9].

Theorem B ([9])**.** *If the sequence of the positive linear operators* $A_n : C[a, b] \to C[a, b]$ *satisfies the conditions* $st - \lim_{n\to\infty} \|A_n(e_i; .) - e_i\|_{C[a,b]} = 0$ *then, for all* $f \in C[a, b]$, *we have* $st - \lim_{n\to\infty} \|A_n(f; .) - f\|_{C[a,b]} = 0$, $i = 0, 1, 2$.

Now, in the definition of the operators T_n we consider a sequence $q = (q_n)$ instead of a fixed $q \in (0, 1]$ satisfying the following expression

$$st - \lim_{n\to\infty} a_n(q_n) = st - \lim_{n\to\infty} q_n = 1 \text{ and } st - \lim_{n\to\infty} [n]_{q_n} = \infty. \tag{3.1}$$

Existence of such a sequence was shown by Doğru in [4].

Theorem 4. *Let* (T_n) *be the sequence of the operators* (1.5) *and the sequence* $q = (q_n)$ *satisfies* (3.1) *for* $0 < q_n \leq 1$, *then for all* $f \in C[0, b]$, $0 < b < 1$, $st - \lim_{n\to\infty} \|T_n(f, q) - f\|_{C[0,b]} = 0$.

Proof. By (1.6) for $i = 0$, $T_n(e_0; q_n, x) = 1$. Then we have

$$st - \lim_{n\to\infty} \|T_n(e_0; q_n, .) - e_0\|_{C[0,b]} = 0. \tag{3.2}$$

By (1.6) for $i = 1$, we can write

$$\|T_n(e_1; q_n, .) - e_1\|_{C[0,b]} \leq 1 - a_n(q_n). \tag{3.3}$$

For a given $\varepsilon > 0$, define the following sets: $U := \{n : \|T_n(e_1; q_n, .) - e_1\|_{C[0,b]} \geq \varepsilon\}$ and $U' := \{n : 1 - a_n(q_n) \geq \varepsilon\}$.

It is clear that $U \subseteq U'$. Then $\delta\{k \leq n : \|T_n(e_1; q_n, .) - e_1\|_{C[0,b]} \geq \varepsilon\} \leq \delta\{k \leq n : 1 - a_n(q_n) \geq \varepsilon\}$. (3.1) yields that $st - \lim_{n\to\infty} (1 - a_n(q_n)) = 0$.

So

$$st - \lim_{n\to\infty} \|T_n(e_1; q_n, .) - e_1\|_{C[0,b]} = 0. \tag{3.4}$$

Finally, for $i = 2$ by (1.6) we get

$$\|T_n(e_2; q_n, .) - e_2\|_{C[0,b]} \leq 1 - q_n a_n^2(q_n) + \frac{a_n^2(q_n)}{[n]_{q_n}}. \tag{3.5}$$

From (3.1), we have

$$st-\lim_{n\to\infty}\left(1-q_na_n^2\left(q_n\right)\right)=st-\lim_{n\to\infty}\left(\frac{a_n^2\left(q_n\right)}{\left[n\right]_{q_n}}\right)=0. \tag{3.6}$$

Now, given $\varepsilon>0$ define the following sets: $U:=\{n:\|T_n(e_2;q_n,.)-e_2\|_{C[0,b]}\geq\varepsilon\}$, $U_1:=\{n:1-q_na_n^2(q_n)\geq\frac{\varepsilon}{2}\}$ and $U_2:=\{n:\frac{a_n^2(q_n)}{[n]_{q_n}}\geq\frac{\varepsilon}{2}\}$.

It is obvious that from (3.5) $U\subseteq U_1\cup U_2$. Then we obtain that $\delta\{n:\|T_n(e_2;q_n,.)-e_2\|_{C[0,b]}\geq\varepsilon\}\leq\delta\{n:1-q_na_n^2(q_n)\geq\frac{\varepsilon}{2}\}+\delta\{n:\frac{a_n^2(q_n)}{[n]_{q_n}}\geq\frac{\varepsilon}{2}\}$. So the right hand side of the last inequality is zero by (3.6), then

$$st-\lim_{n\to\infty}\left\|T_n\left(e_2;q_n,.\right)-e_2\right\|_{C[0,b]}=0. \tag{3.7}$$

Now using (3.2), (3.4) and (3.7), the proof follows from Theorem B. □

4 Construction of the bivariate operators

In this section, we aim to construct a bivariate extension of the operators defined in (1.5) by following technique of Barbosu in [2].

Let $I^2=[0,b]\times[0,b]$, $0<b<1$ and $a_n\left(t\right)$ be a function satisfying $0<a_n\left(t\right)\leq1$, for all $n\in\mathbb{N}$ and $t\in[0,1]$. We consider a bivariate extension of the operators (1.5) for $f\in C\left(I^2\right)$ and $0<q_1,q_2\leq1$ as follows:

$$\begin{aligned}T_{n_1,n_2}\left(f;q_1,q_2,x,y\right)=&\prod_{s_1=0}^{n_1}\left(1-q_1^{s_1}x\right)\prod_{s_2=0}^{n_2}\left(1-q_2^{s_2}y\right)\\&\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{a_{n_1}\left(q_1\right)\left[k_1\right]_{q_1}}{\left[n_1+k_1\right]_{q_1}},\frac{a_{n_2}\left(q_2\right)\left[k_2\right]_{q_2}}{\left[n_2+k_2\right]_{q_2}}\right)\\&\times\begin{bmatrix}n_1+k_1\\k_1\end{bmatrix}_{q_1}\begin{bmatrix}n_2+k_2\\k_2\end{bmatrix}_{q_2}x^{k_1}y^{k_2}.\end{aligned} \tag{4.1}$$

It is clear that the operators (4.1) are linear and positive. By choosing $a_{n_1}\left(q_1\right)=a_{n_2}\left(q_2\right)=q_1=q_2=1$ in (4.1), we have a Stancu-type generalization of classical MKZ operators [12] and by choosing $a_{n_1}\left(q_1\right)=q_1^{n_1}$, $a_{n_2}\left(q_2\right)=q_2^{n_2}$ in (4.1), we have also the bivariate extension of q-MKZ operators from Doğru and Duman in [5].

Lemma 6. *We have*

(i) $T_{n_1,n_2}\left(f;q_1,q_2,x,y\right)=T_{n_1}^x\left(T_{n_2}^y\left(f;q_2,x,y\right)\right)$,

(ii) $T_{n_1,n_2}\left(f;q_1,q_2,x,y\right)=T_{n_2}^y\left(T_{n_1}^x\left(f;q_1,x,y\right)\right)$, *where*

$$\begin{aligned}T_{n_1}^x\left(f;q_1,x,y\right)=&\prod_{s_1=0}^{n_1}\left(1-q_1^{s_1}x\right)\sum_{k_1=0}^{\infty}f\left(\frac{a_{n_1}\left(q_1\right)\left[k_1\right]_{q_1}}{\left[n_1+k_1\right]_{q_1}},y\right)\\&\times\begin{bmatrix}n_1+k_1\\k_1\end{bmatrix}_{q_1}x^{k_1}\end{aligned} \tag{4.2}$$

and

$$T_{n_2}^{y}\left(f;q_2,x,y\right)=\prod_{s_2=0}^{n_2}\left(1-q_2^{s_2}y\right)\sum_{k_2=0}^{\infty}f\left(x,\frac{a_{n_2}\left(q_2\right)\left[k_2\right]_{q_2}}{\left[n_2+k_2\right]_{q_2}}\right)$$
$$\times\begin{bmatrix}n_2+k_2\\k_2\end{bmatrix}_{q_2}y^{k_2}. \tag{4.3}$$

Proof. We obtain (i) as follows:

$$\begin{aligned}T_{n_1}^{x}(T_{n_2}^{y}(f;q_2,x,y))&=T_{n_1}^{x}\left(\prod_{s_2=0}^{n_2}\left(1-q_2^{s_2}y\right)\sum_{k_2=0}^{\infty}f\left(x,\frac{a_{n_2}\left(q_2\right)\left[k_2\right]_{q_2}}{\left[n_2+k_2\right]_{q2}}\right)\right.\\&\left.\times\begin{bmatrix}n_2+k_2\\k_2\end{bmatrix}_{q_2}y^{k_2}\right)\\&=\prod_{s_2=0}^{n_2}\left(1-q_2^{s_2}y\right)\sum_{k_2=0}^{\infty}T_{n_1}^{x}\left(f\left(x,\frac{a_{n_2}\left(q_2\right)\left[k_2\right]_{q_2}}{\left[n_2+k_2\right]_{q2}}\right);q_1,x,y\right)\\&\times\begin{bmatrix}n_2+k_2\\k_2\end{bmatrix}_{q_2}y^{k_2}\\&=\prod_{s_2=0}^{n_2}\left(1-q_2^{s_2}y\right)\sum_{k_2=0}^{\infty}\prod_{s_1=0}^{n_1}\left(1-q_1^{s_1}x\right)\\&\times\sum_{k_1=0}^{\infty}f\left(\frac{a_{n_1}\left(q_1\right)\left[k_1\right]_{q_1}}{\left[n_1+k_1\right]_{q_1}},\frac{a_{n_2}\left(q_2\right)\left[k_2\right]_{q_2}}{\left[n_2+k_2\right]_{q2}}\right)\\&\times\begin{bmatrix}n_1+k_1\\k_1\end{bmatrix}_{q_1}x^{k_1}\begin{bmatrix}n_2+k_2\\k_2\end{bmatrix}_{q_2}y^{k_2}\\&=T_{n_1,n_2}\left(f;q_1,q_2,x,y\right).\end{aligned}$$

In a similar way, property (ii) can be proven. □

5 Approximation properties of the bivariate operators

In order to obtain Heping-type convergence of the operators (4.1) we need the following Lemma.

Lemma 7. *Let $e_{ij}=I^2\to I$, $e_{ij}=x^iy^j$, be the two-dimensional test functions. Then the following results hold for the operators* (4.1):

(i) $T_{n_1,n_2}\left(e_{00};q_1,q_2,x,y\right)=1.$
(ii) $T_{n_1,n_2}\left(e_{10};q_1,q_2,x,y\right)=a_{n_1}\left(q_1\right)x.$
(iii) $T_{n_1,n_2}\left(e_{01};q_1,q_2,x,y\right)=a_{n_2}\left(q_2\right)y.$
(iv) $a_{n_1}^2\left(q_1\right)x^2\leq T_{n_1,n_2}\left(e_{20};q_1,q_2,x,y\right)\leq q_1a_{n_1}^2\left(q_1\right)x^2+\frac{a_{n_1}^2\left(q_1\right)x}{\left[n_1\right]_{q_1}}.$

(v) $a_{n_2}^2(q_2)\,y^2 \le T_{n_1,n_2}(e_{02};q_1,q_2,x,y) \le q_2 a_{n_2}^2(q_2)\,y^2 + \frac{a_{n_2}^2(q_2)y}{[n_2]_{q_2}}$.

Proof. By using Lemma 6 and (1.6), the results (i)-(v) can easily be proven as in [2].□

Theorem 8. *If the sequence* $(q_{n_1}),(q_{n_2}),(a_{n_1}(q_{n_1}))$ *and* $(a_{n_2}(q_{n_2}))$ *in the interval* $(0,1]$ *satisfy the conditions* (1.7), *then the sequence of operators* (4.1) *converge uniformly to* $f(x,y)$ *on* I^2, *for any* $f\in C(I^2)$.

Proof. Using the linearity of the operators (4.1) and combining the items (iv) and (v) of Lemma 7, we obtain

$$\begin{aligned} &a_{n_1}^2(q_{n_1})\,x^2 + a_{n_2}^2(q_{n_2})\,y^2 \\ &\le T_{n_1,n_2}(e_{20};q_{n_1},q_{n_2},x,y) + T_{n_1,n_2}(e_{02};q_{n_1},q_{n_2},x,y) \\ &\le q_{n_1}a_{n_1}^2(q_{n_1})\,x^2 + q_{n_2}a_{n_2}^2(q_{n_2})\,y^2 + \frac{a_{n_1}^2(q_{n_1})\,x}{[n_1]_{q_{n_1}}} + \frac{a_{n_2}^2(q_{n_2})\,y}{[n_2]_{q_{n_2}}}. \end{aligned} \tag{5.1}$$

Using (1.7) and (5.1), we see that

$$T_{n_1,n_2}(e_{20}+e_{02};q_{n_1},q_{n_2},x,y) \to x^2+y^2 \text{ as } n_1\to\infty \text{ and } n_2\to\infty \tag{5.2}$$

uniformly. By using (5.2) and (i)-(iii) of Lemma 7, the proof is complete from Volkov's theorem (see [19]). □

In view of Theorems 1 and 8, we can give the following result.

Theorem 9. *Let* (q_{n_1}) *and* (q_{n_2}) *be the sequences satisfying the conditions* (1.8) *and* (T_{n_1,n_2}) *be sequence of linear positive operators defined by* (4.1). *Then, for all* $f\in C(I^2)$, $\lim_{n_1,n_2\to\infty}\|T_{n_1,n_2}(f)-T_{\infty,\infty}(f)\|_{C(I^2)}=0$.

Now, we give the following theorem which we shall use for the statistical convergence of the (4.1) operators.

Theorem E ([7]). *Let* (L_n) *be a sequence of positive linear operators from* $C(K)$ *into* $C(K)$, *where* $K=[a,b]\times[c,d]$, *for* $a,b,c,d\in\mathbb{R}$. *Then, for all* $f\in C(K)$, $st-\lim_{n\to\infty}\|L_n(f,.)-f\|_{C(K)}=0$ *if and only if, the followings hold:*

$$st-\lim_{n\to\infty}\|L_n(e_i,.)-e_i\|_{C(K)}=0,\ i=0,1,2.$$

Let (q_{n_1}) and (q_{n_2}) be the sequences that converge statistically to 1 but are not convergent in ordinary sense, so it can be written as for $0<q_{n_1},q_{n_2}\le 1$,

$$\begin{aligned} st-\lim_{n_1\to\infty} a_{n_1}(q_{n_1}) &= st-\lim_{n_1\to\infty} q_{n_1} \\ &= st-\lim_{n_2\to\infty} a_{n_2}(q_{n_2}) = st-\lim_{n_2\to\infty} q_{n_2} = 1. \end{aligned} \tag{5.3}$$

Now under the condition in (5.3), let us show the statistical convergence of the bivariate operators (4.1) with the help of the proof of Theorem 4.

Theorem 10. *If the sequence* $(q_{n_1}), (q_{n_2}), (a_{n_1}(q_{n_1}))$ *and* $(a_{n_2}(q_{n_2}))$ *in the interval* $(0,1]$ *satisfy the conditions* (5.3)*, and* (T_{n_1,n_2}) *be the sequence of linear positive operators defined by* (4.1)*. Then, for all* $f \in C(I^2)$

$$st - \lim_{n_1,n_2 \to \infty} \|T_{n_1,n_2}(f; q_{n_1}, q_{n_2}, .) - f\|_{C(I^2)} = 0.$$

Proof. By using Theorem 4, Lemma 6 and Theorem E, the proof can be obtained. □

6 Rate of convergence of the bivariate operators

Recall that the modulus of continuity for the bivariate functions is defined by $\omega(f;\delta_1, \delta_2) = \sup\{|f(t,s) - f(x,y)| : (t,s),(x,y) \in I^2,\ |t-x| \le \delta_1,\ |s-y| \le \delta_2\}$ (see [2] also [17]).

It is clear that, if $f \in C\left(I^2\right)$, then $\omega(f;\delta_1,\delta_2) \to 0$ for $\delta_1 \to 0$ and $\delta_2 \to 0$.

Also by the monotonicity of $\omega\left(f;\delta_1,\delta_2\right)$, we obtain

$$\begin{aligned} |f(t,s) - f(x,y)| &\le \omega\left(f; |t-x|, |s-y|\right) \\ &\le \omega\left(f;\delta_1,\delta_2\right)\left(\frac{|t-x|}{\delta_1}+1\right)\left(\frac{|s-y|}{\delta_2}+1\right). \end{aligned} \tag{6.1}$$

Recall that Özarslan and Duman obtained the following inequality for the (1.5) operators in [15];

$$|T_n(f;q,x) - f(x)| \le 2\omega\left(f;\delta_n(x,q)\right),\ \ f \in C[0,1] \text{ and } n \in \mathbb{N}, \tag{6.2}$$

where

$$\delta_n(x,q) = \left\{\left(3 - 2a_n(q) - qa_n^2(q)\right)x^2 + \frac{a_n^2(q)\,x}{[n]_q}\right\}^{1/2}. \tag{6.3}$$

Theorem 11. *If the sequence* $(q_{n_1}), (q_{n_2}), \left(a_{n_1}\left(q_{n_1}\right)\right)$ *and* $(a_{n_2}(q_{n_2}))$ *in the interval* $(0,1]$ *satisfy the conditions* (1.7)*, then*

$$\|T_{n_1,n_2}(f;q_{n_1},q_{n_2}) - f\|_{C(I^2)} \le 4\omega\left(f;\delta_1(q_{n_1}),\delta_2(q_{n_2})\right), \tag{6.4}$$

where

$$\delta_{n_1}(q_{n_1}) = \left\{\left(3 - 2a_{n_1}(q_{n_1}) - q_{n_1}a_{n_1}^2(q_{n_1})\right)b^2 + \frac{a_{n_1}^2(q_{n_1})\,b}{[n_1]_{q_{n_1}}}\right\}^{1/2}, \tag{6.5}$$

$$\delta_{n_2}(q_{n_2}) = \left\{\left(3 - 2a_{n_2}(q_{n_2}) - q_{n_2}a_{n_2}^2(q_{n_2})\right)b^2 + \frac{a_{n_2}^2(q_{n_2})\,b}{[n_2]_{q_{n_2}}}\right\}^{1/2}. \tag{6.6}$$

Proof. Using the Cauchy-Schwarz inequality in (6.1), we see that, in view of (6.2) and (6.3), the proof immediatelly follows.

Recall that the Lipschitz class for the bivariate functions is defined as $Lip_M(f;\alpha) = \{f : |f(t,s) - f(x,y)| \le M[|t-x|^2 + |s-y|^2]^{\alpha/2}(t,s),(x,y) \in I^2\}$, where $0 < \alpha \le 1$.

In [15], recall that Özarslan and Duman obtained the following inequality for the (1.5) operators

$$|T_n(f;q,x) - f(x)| \leq M\{\delta_n(x,q)\}^{\alpha}, \ f \in Lip_M(\alpha) \text{ and } n \in \mathbb{N}, \tag{6.7}$$

where $\delta_n(x,q)$ is defined as in (6.3). □

Theorem 12. *Let the sequence* $(q_{n_1}), (q_{n_2}), (a_{n_1}(q_{n_1}))$ *and* $(a_{n_2}(q_{n_2}))$ *satisfy conditions* (1.7) *in the interval* $(0,1]$. *If* $f \in Lip_M(f;\alpha)$, *then* $\|T_{n_1,n_2}(f;q_{n_1},q_{n_2},.) - f\|_{C(I^2)} \leq M[\delta_{n_1}^{\alpha}(q_{n_1}) + \delta_{n_2}^{\alpha}(q_{n_2})]$, *where* $(\delta_{n_1}(q_{n_1}))$ *and* $(\delta_{n_2}(q_{n_2}))$ *are defined as in* (6.5) and (6.6).

Proof. By using (i) of Lemma 7, we have

$$\begin{aligned}
&|T_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \\
&\leq \prod_{s_1=0}^{n_1}\left(1-q_{n_1}^{s_1}x\right)\prod_{s_2=0}^{n_2}\left(1-q_{n_2}^{s_2}y\right) \\
&\times \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left|f\left(\frac{a_{n_1}(q_{n_1})[k_1]_{q_{n_1}}}{[n_1+k_1]_{q_{n_1}}}, \frac{a_{n_2}(q_{n_2})[k_2]_{q_{n_2}}}{[n_2+k_2]_{q_{n_2}}}\right)\right. \\
&\left.-f(x,y)\right|\begin{bmatrix} n_1+k_1 \\ k_1 \end{bmatrix}_{q_{n_1}}\begin{bmatrix} n_2+k_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1}y^{k_2}.
\end{aligned} \tag{6.8}$$

Let us first add and drop the function $f(\frac{a_{n_1}(q_{n_1})[k_1]_{q_{n_1}}}{[n_1+k_1]_{q_{n_1}}}, y)$ inside the absolute value sign on the right-hand side of (6.8). Using the triangle inequality and the fact that $f \in Lip_M(f;\alpha)$, finally applying Hölder's inequality, with $p = \frac{2}{\alpha}$ and $r = \frac{2}{2-\alpha}$ such that $\frac{1}{p} + \frac{1}{r} = 1$, to the resulting term, we reach to

$$\begin{aligned}
&|T_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \\
&\leq M\left\{\left[T_{n_1}\left((t-x)^2;q_{n_1},x\right)\right]^{\alpha/2} + \left[T_{n_2}\left((s-y)^2;q_{n_2},y\right)\right]^{\alpha/2}\right\} \\
&= M\left[(\varphi_{n_1,2})^{\alpha/2} + (\varphi_{n_2,2})^{\alpha/2}\right],
\end{aligned} \tag{6.9}$$

where $\varphi_{n_1,2}$ and $\varphi_{n_2,2}$ are the second central moments of the operators (1.5). By (1.6) can be obtained as follows:

$$\varphi_{n,2}(x) = T_n\left((t-x)^2;q_n,x\right) \leq \left[3 - 2a_n(q_n) - q_n a_n^2(q_n)x^2\right] + \frac{a_n^2(q_n)x}{[n]_{q_n}}.$$

Then $\|\varphi_{n_1,2}(x)\| \leq \delta_{n_1}^2(q_{n_1})$ and $\|\varphi_{n_2,2}(x)\| \leq \delta_{n_2}^2(q_{n_2})$, where $\delta_{n_1}(q_{n_1})$ and $\delta_{n_2}(q_{n_2})$ are defined as in (6.5) and (6.6), respectively.

If we take supremum over $(x,y) \in I^2$ from the right hand side of (6.9), the proof follows. □

Remark. Using (5.3), it is easily verified that $st - \lim_{n_1 \to \infty} \delta_{n_1}(q_{n_1}) = 0$ and $st - \lim_{n_2 \to \infty} \delta_{n_2}(q_{n_2}) = 0$. Hence this result helps us to estimate rate of statistical approximation of the operators (4.1) by means of modulus of continuity and the elements of Lipschitz class.

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