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Meromorphic functions share two values with its difference operator

Yeyang Jiang · Zongxuan Chen

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Abstract In this paper, we investigate the uniqueness problem related to a meromorphic function $f(z)$ and its difference operator $\Delta f(z) = f(z+\eta) - f(z)$, and we prove that $\Delta f(z)$ and $f(z)$ share a, b CM, where $f(z)$ is a meromorphic function with $N(r, f) = S(r, f)$, then $f(z+\eta) = 2f(z)$.

Keywords sharing value · difference operator · meromorphic function

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1 Introduction and results

In this paper, we assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [6, 14]). For a meromorphic function $f(z)$, $S(r, f)$ shall always denote a quantity $o(T(r, f))$, $r \to \infty$, outside a set of finite linear measure of $r \in (0, \infty)$. A meromorphic function α is said to be a small function of f provided that $T(r, \alpha) = S(r, f)$. $S(f)$ denote the family of all meromorphic small function to $f(z)$. As for the standard notion in the uniqueness theory, two meromorphic functions f and g share a CM(IM) in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ means that $f - a$ and $g - a$ have the same zeros counting multiplicities (ignoring multiplicities).

The following two classical results, due to Nevanlinna [11], have prompted research activity on shared value problems up until today.

Theorem A *If two meromorphic functions f and g share five distinct values IM, then $f \equiv g$.*

Theorem B *If two meromorphic functions f and g share four distinct values CM, then $f \equiv g$ or $f \equiv T \circ g$, where T is a Möbious transformation.*

Yeyang Jiang, Zongxuan Chen
School of Mathematics,
South China Normal University,
P.R. China
E-mail: jyyang1018@126.com
E-mail: chzx@vip.sina.com

Uniqueness of the entire function f sharing values with its derivative f' was firstly investigated by Rubel and Yang [13], and Mues and Steinmetz [9,10] and Gundersen [4] improved their results.

Recently, many authors [1,2,7,8,12] started to consider sharing values of meromorphic functions with their shifts or difference operators.

Heittokangas et al. [7,8] proved the following theorems.

Theorem C *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions a_1, a_2, $a_3 \in \hat{S}(f) = S(f) \cup \{\infty\}$ with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem D *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$, and let a_1, a_2, $a_3 \in \hat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

It is well known that $\Delta f(z) = f(z+\eta) - f(z)$ (where $\eta \in \mathbb{C}$ is a constant satisfying $f(z+\eta) - f(z) \not\equiv 0$) is regarded as the difference counterpart of f'. So, we consider the problem that $\Delta f(z)$ and $f(z)$ share a, b CM, where $f(z)$ is a meromorphic function with $N(r, f) = S(r, f)$, and prove the following theorem.

Theorem 1.1 *Let $f(z)$ be a nonconstant meromorphic function of finite order such that $N(r, f) = S(r, f)$, let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) - f(z) \not\equiv 0$, and let a, b be two nonzero distinct finite complex constants. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share a, b CM, then $f(z+\eta) = 2f(z)$.*

Example 1.1 Let $Q(z)$ be a periodic entire function with period 1 such that $\sigma(Q) = 2$, and $f(z) = \frac{Q(z)exp(z\log 2)}{\sin 2\pi z}$. Then $\Delta f(z) = f(z+1) - f(z)$ and $f(z)$ share $1, 2$ CM, and $N(r, f) = S(r, f)$, hence $f(z+\eta) = 2f(z)$.

Corollary 1.2 *Let $f(z)$ be a nonconstant entire function of finite order, let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) - f(z) \not\equiv 0$, and let a, b be two nonzero distinct finite complex constants. If $\Delta f(z) = f(z+\eta) - f(z)$ and $f(z)$ share a, b CM, then $f(z+\eta) = 2f(z)$.*

2 Lemmas for proof of theorem 1.1

Firstly we need the following lemmas for the proof of Theorem 1.1.

Lemma 2.1 ([14]) *Let $f(z)$ be a nonconstant meromorphic function, a_j ($j = 1, ..., q$) be q distinct complex numbers. Then we have*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O(1).$$

Chiang and Feng [3], Halburd and Korhonen [5] investigated the value distribution theory of difference expressions. A key result, which is a difference analogue of the logarithmic derivative lemma, reads as follows.

Lemma 2.2 *Let f be a meromorphic function of finite order and let η be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = S(r, f).$$

Lemma 2.3 *Let $f(z)$ be not constant meromorphic function in $\mathbb{C}$. Let $a_1, a_2, \cdots, a_n$ be $n \geq 1$ distinct complex numbers. Then, we have*

$$\sum_{j=1}^{n} m(r, \frac{1}{f-a_j}) \leq m(r, \frac{1}{f'}) + S(r, f), \tag{2.1}$$

$$\sum_{j=1}^{n} m(r, \frac{1}{f-a_j}) \leq m(r, \frac{1}{\Delta f(z)}) + S(r, f). \tag{2.2}$$

Proof. By Lemma 2.1 and the Lemma on the Logarithmic Derivative, we easily obtain that (2.1) holds.

By Lemma 2.1 and Lemma 2.2, we have

$$\sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-a_j}\right) = m\left(r, \sum_{j=1}^{n} \frac{1}{f(z)-a_j}\right) + O(1)$$

$$= m\left(r, \frac{1}{\Delta f} \cdot \sum_{j=1}^{n} \frac{\Delta f}{f(z)-a_j}\right) + O(1) \leq m\left(r, \frac{1}{\Delta f}\right)$$

$$+ m\left(r, \sum_{j=1}^{n} \frac{\Delta f}{f(z)-a_j}\right) + O(1)$$

$$= m\left(r, \frac{1}{\Delta f}\right) + \sum_{j=1}^{n} m\left(r, \frac{\Delta f}{f(z)-a_j}\right) + O(1) \leq m\left(r, \frac{1}{\Delta f}\right) + S(r, f).$$

□

Lemma 2.4 ([3]) *Let $f(z)$ be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and let η be a fixed non-zero complex number,then for each $\varepsilon > 0$,we have $T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f)$.*

3 Proof of theorem 1.1

By Nevanlinna's second fundamental theorem and the assumption that $\Delta f(z)$ and $f(z)$ share a, b CM, we have

$$T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f)$$

$$= N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f) \tag{3.1}$$

(3.2)

$$
\begin{aligned}
&= N\left(r, \frac{1}{\Delta f - a}\right) + N\left(r, \frac{1}{\Delta f - b}\right) + S(r,f)\\
&\leq N\left(r, \frac{1}{\Delta f - f}\right) + S(r,f).
\end{aligned}
$$

By the assumption that $N(r,f) = S(r,f)$ and the first fundamental theorem, we see that

$$
\begin{aligned}
&N\left(r, \frac{1}{\Delta f - f}\right) + S(r,f) = N\left(r, \frac{f}{\Delta f - f}\right) + S(r,f)\\
&= N\left(r, \frac{1}{\frac{\Delta f}{f} - 1}\right) + S(r,f) \leq T\left(r, \frac{1}{\frac{\Delta f}{f} - 1}\right) + S(r,f)\\
&= T\left(r, \frac{\Delta f}{f}\right) + S(r,f) = m\left(r, \frac{\Delta f}{f}\right) + N\left(r, \frac{\Delta f}{f}\right) + S(r,f) \qquad (3.3)\\
&= N\left(r, \frac{\Delta f}{f}\right) + S(r,f) \leq N\left(r, \frac{1}{f}\right) + N(r, \Delta f) + S(r,f)\\
&\leq N\left(r, \frac{1}{f}\right) + 2N(r,f) + S(r,f) = N\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leq T(r,f) + S(r,f).
\end{aligned}
$$

By (3.1) and (3.2), we obtain that

$$
\begin{aligned}
T(r,f) &= N\left(r, \frac{1}{f}\right) + S(r,f)\\
&= N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r,f). \qquad (3.4)
\end{aligned}
$$

Let $\phi = \frac{(\Delta f)'}{\Delta f - a} - \frac{f'}{f-a}$. Using the logarithmic derivative theorem, we obtain

$$
\begin{aligned}
m(r,\phi) &\leq m\left(r, \frac{(\Delta f)'}{\Delta f - a}\right) + m\left(r, \frac{f'}{f-a}\right) + O(1)\\
&= S(r, \Delta f) + S(r,f). \qquad (3.5)
\end{aligned}
$$

Since

$$
\begin{aligned}
T(r,\Delta f) &= m(r,\Delta f) + N(r,\Delta f) + S(r,f)\\
&\leq m(r, f \cdot \frac{\Delta f}{f}) + 2N(r,f) + S(r,f) \qquad (3.6)\\
&\leq m(r,f) + m\left(r, \frac{\Delta f}{f}\right) + 2N(r,f) + S(r,f)\\
&= m(r,f) + S(r,f) = T(r,f) + S(r,f),
\end{aligned}
$$

we have $S(r,\Delta f) = S(r,f)$. Thus, by this and (3.4), we have $m(r,\phi) = S(r,f)$. Since ϕ is the logarithmic derivative of $\frac{\Delta f - a}{f-a}$, the poles of ϕ derive from the zeros and poles of $\frac{\Delta f-a}{f-a}$. And since f, Δf share the value a CM, then $\frac{\Delta f-a}{f-a}$ has no zeros, and has at most $N(r,f)$ poles. Thus $N(r,\phi)=N(r,f)=S(r,f)$. Combining this and (3.4), we have $T(r,\phi)=S(r,f)$.

Suppose that $\phi\not\equiv 0$. Then from $\frac{\phi}{f-b}=\frac{(\Delta f)'}{(\Delta f)(\Delta f-a)}\cdot\frac{\Delta f}{f-b}-\frac{f'}{(f-a)(f-b)}$, we obtain

$$\begin{aligned}
m\left(r,\frac{1}{f-b}\right)&=m\left(r,\frac{1}{\phi}\right)+m\left(r,\frac{\phi}{f-b}\right)\\
&\le m\left(r,\frac{1}{\phi}\right)+m\left(r,\frac{(\Delta f)'}{(\Delta f)(\Delta f-a)}\right)+m\left(r,\frac{\Delta f}{f-b}\right)\\
&+m\left(r,\frac{f'}{(f-a)(f-b)}\right)+O(1)\\
&\le m\left(r,\frac{1}{\phi}\right)+m\left(r,\frac{(\Delta f)'}{a}\left(\frac{1}{\Delta f-a}-\frac{1}{\Delta f}\right)\right)+m\left(r,\frac{\Delta f}{f-b}\right)\\
&+m\left(r,\frac{f'}{(a-b)}\left(\frac{1}{f-a}-\frac{1}{f-b}\right)\right)+O(1)\\
&\le T(r,\phi)+S(r,\Delta f)+S(r,f)=S(r,f).
\end{aligned}$$

We have that $m(r,\frac{1}{f-b})=S(r,f)$. By the first fundamental theorem we obtain that

$$T(r,f)=N\left(r,\frac{1}{f-b}\right)+S(r,f). \tag{3.7}$$

Thus, by (3.3) and (3.6), we have $N(r,\frac{1}{f-a})=S(r,f)$. By the assumption that $\Delta f(z)$ and $f(z)$ share a CM, we have that $N(r,\frac{1}{\Delta f-a})=N(r,\frac{1}{f-a})=S(r,f)$. Hence $N(r,\frac{1}{\Delta f-a})=S(r,f)$. Since $\Delta f(z)$ and $f(z)$ share b CM, by (3.5), (3.6) and the first fundamental theorem, we have $m(r,\frac{1}{\Delta f-b})+N(r,\frac{1}{\Delta f-b})=T(r,\Delta f)+S(r,f)\le T(r,f)+S(r,f)=N(r,\frac{1}{f-b})+S(r,f)=N(r,\frac{1}{\Delta f-b})+S(r,f)$. Hence $m(r,\frac{1}{\Delta f-b})=S(r,\Delta f)=S(r,f)$. By Lemma 2.3 we have that

$$m\left(r,\frac{1}{\Delta f}\right)+m\left(r,\frac{1}{\Delta f-a}\right)+m\left(r,\frac{1}{\Delta f-b}\right)\le m\left(r,\frac{1}{(\Delta f)'}\right)+S(r,f), \tag{3.8}$$

$$m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)\le m\left(r,\frac{1}{\Delta f}\right)+S(r,f). \tag{3.9}$$

By (3.3) and the previous equalities for the counting functions, we have

$$N\left(r,\frac{1}{\Delta f-a}\right)+N\left(r,\frac{1}{\Delta f-b}\right)=N\left(r,\frac{1}{\Delta f-b}\right)+S(r,f), \tag{3.10}$$

$$N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f). \tag{3.11}$$

By (3.7)-(3.10), we obtain

$$\begin{aligned}
&T\left(r,\frac{1}{\Delta f-a}\right)+T\left(r,\frac{1}{\Delta f-b}\right)+T\left(r,\frac{1}{f-a}\right)+T\left(r,\frac{1}{f-b}\right)\\
&\le m\left(r,\frac{1}{(\Delta f)'}\right)+N\left(r,\frac{1}{\Delta f-b}\right)+T(r,f)+S(r,f)\\
&\le T\left(r,\frac{1}{(\Delta f)'}\right)+T\left(r,\frac{1}{\Delta f-b}\right)+T(r,f)+S(r,f).
\end{aligned}\tag{3.12}$$

By $N(r,f)=S(r,f)$, we obtain

$$\begin{aligned}
T\left(r,\frac{1}{(\Delta f)'}\right)&=m(r,(\Delta f)')+N(r,(\Delta f)')+O(1)\\
&=m\left(r,\Delta f\cdot\frac{(\Delta f)'}{\Delta f}\right)+N(r,\Delta f)+\overline{N}(r,\Delta f)+O(1)\\
&=m(r,\Delta f)+m\left(r,\frac{(\Delta f)'}{\Delta f}\right)+S(r,f)=T(r,\Delta f)+S(r,f).
\end{aligned}\tag{3.13}$$

By (3.5), (3.11), (3.12) and the first fundamental theorem, we have that $T(r,f)=S(r,f)$. A contradiction.

Therefore $\phi\equiv 0$, that is

$$\frac{(\Delta f)'}{\Delta f-a}=\frac{f'}{f-a}.\tag{3.14}$$

Integrating (3.13), we get

$$\frac{\Delta f-a}{f-a}\equiv C_1,\tag{3.15}$$

where C_1 is some nonzero constant.

Using the same method as above, by the assumption $\Delta f(z)$ and $f(z)$ share b CM, we obtain

$$\frac{\Delta f-b}{f-b}\equiv C_2,\tag{3.16}$$

where C_2 is some nonzero constant.

If $C_1=1$ (or $C_2=1$), then by (3.14) (or (3.15)), we obtain $f(z+c)=2f(z)$, that is the conclusion holds. If $C_1\neq 1$ and $C_2\neq 1$, then by (3.14) and (3.15), we obtain

$$(C_1-C_2)\,f(z)=-a+b+C_1\,a-C_2\,b.\tag{3.17}$$

If $C_1\neq C_2$, then f is a constant. A contradiction. Hence $C_1=C_2$. Thus, by (3.16), we have $a-b=C_1(a-b)$. Thus $C_1=C_2=1$. A contradiction.

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