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$WPZI$ rings and strong regularity

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Abstract In this paper, we study the strong regularity of left SF rings and obtain the following results: Let R be a left SF ring. If R satisfies one of the following conditions, then R is a strongly regular ring: 1) R is a left $WPZI$ ring; 2) R is a right $WPZI$ ring; 3) R is a right weakly semicommutative ring; 4) R is a semicommutative ring; 5) R is a reversible ring.

Keywords $WPZI$ rings · SF rings · strongly regular rings · semicommutative rings · reduced rings · weakly semicommutative rings

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Introduction

All rings considered in this paper are associative rings with identity, and all modules are unital. The symbols $N(R)$, $Z_l(R)$ and $Z_r(R)$ will stand respectively for the set of all nilpotent elements, left and right singular ideal of R. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X, respectively. In particular, if $X = \{a\}$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

A ring R is called (von Neumann) regular (cf. Goodearl [2]) if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring R is strongly regular (cf. Rege [6]) if for every $a \in R$ there exists $b \in R$ such that $a = a^2b$. A ring R is called reduced (cf. Ramamurthi [5]) if R has no nonzero nilpotent elements. It is well known that R is a strongly regular ring if and only if R is a reduced regular ring. A ring R is called $MELT$ (resp., $MERT$) if every maximal essential left (resp., right) ideal of R is an ideal. According to Ramamurthi [5], a ring R is called left (resp., right) SF if each simple left (resp., right) $R-$module is flat. It is known that regular rings are left and right SF rings. Ramamurthi [5] initiated the study of SF rings and the question whether an SF ring is necessarily regular. For several years, SF rings have

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been studied by many authors and the regularity of SF rings which satisfy certain additional conditions is showed (cf. RAMAMURTHI [5], REGE [6], YUE CHI MING [9–11], ZHANG and DU [14,15], ZHANG [12,13], ZHOU and WANG [17,18], ZHOU [19]). But the question remains open. YUE CHI MING [11] proved the strong regularity of right SF rings whose complement left ideals are ideals, and he proposed the following question: Is R strongly regular if R is a left SF rings whose complement left ideals are ideals? ZHANG and DU [14] affirmatively answered the question. ZHOU and WANG [17] proved that if R is a right SF rings whose all maximal essential right ideals are $GW-$ideals, then R is a regular ring. ZHANG [13] proved that if R is an $MELT$ and right SF rings, then R is a regular ring. ZHOU [19] proved that if R is a left SF rings whose complement left (right) ideals are $W-$ideals, then R is a strongly regular ring.

Following ZHOU and WANG [17], a left ideal L of a ring R is called $GW-$ideal, if for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. Clearly, every ideal is $GW-$ideal, but the converse is not true, in general, by ZHOU and WANG ([17], Example 1.2).

According to ZHOU [19], a left ideal L of a ring R is called a weakly ideal ($W-$ideal), if for any $0 \neq a \in L$, there exists $n \geq 1$ such that $a^n \neq 0$ and $a^nR \subseteq L$. A right ideal K of a ring R is defined similarly to be a weakly ideal. Clearly, ideals are $W-$ideals and $W-$ideals are $GW-$ideals, but the converses are not true, in general, by ZHOU [19].

According to COHN [1], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be semicommutative (ZHAO and YANG [16]) if $ab = 0$ implies $aRb = 0$.

A ring R is called left (resp., right) $WPZI$ if for any $0 \neq a \in R$, there exists $n \geq 1$ such that $a^n \neq 0$ and $l(a^n)$ (resp., $r(a^n)$) is a $W-$ideal of R.

Clearly, semicommutative rings are left and right $WPZI$ rings.

The first purpose of this paper is to study the (strong) regularity of left $SF-$rings in terms of $WPZI$ rings. We obtain the following main results:

1) Left $WPZI$ left $SF-$rings are strongly regular;

2) Right $WPZI$ left $SF-$rings are strongly regular.

So some known results appeared in REGE [6] are extended.

1 Some properties of $WPZI$ rings

According to HWANG, JEON and PARK [3], a ring R is called NCI if $N(R) = 0$ or there exists a nonzero ideal of R contained in $N(R)$. Clearly, NI rings (that is, $N(R)$ forms an ideal of R) are NCI, but the converse is not true, in general, by HWANG, JEON and PARK [3].

Following WEI and CHEN [8], left $R-$module M is called $nil-$injective if for any $a \in N(R)$, every left $R-$homomorphism Ra to M extends to R. Evidently, $YJ-$injective modules are $nil-$injective, but the converse is not true, in general, by WEI and CHEN [8].

Proposition 1.1 (1) *Left or right $WPZI$ rings are Abelian.*

(2) *Left or right $WPZI$ rings are NCI.*

(3) *Let R be a left or right $WPZI$ ring. If every singular simple left $R-$module is $nil-$injective, then R is a reduced ring.*

(4) *If R is a left or right $WPZI$ ring, then $N_2(R) = \{a \in R | a^2 = 0\} \subseteq P(R)$.*

Proof. (1) Let R be a left $WPZI$ ring and $e \in E(R)$. Then there exists $n \geq 1$ such that $l(e) = l(e^n)$ is a $W-$ideal of R. Since $1 - e \in l(e)$, there exists $m \geq 1$ such that $(1-e)^m \neq 0$ and $(1-e)^m R \subseteq l(e)$. Therefore we obtain $(1-e)Re = 0$ for each $e \in E(R)$, so R is an Abelian ring.

Similarly, we can show that right $WPZI$ rings are Abelian.

(2) If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$. Let $n \geq 1$ be such that $a^n = 0$ and $a^{n-1} \neq 0$. Since R is a left $WPZI$ ring, there exists $m \geq 1$ such that $a^m \neq 0$ and $l(a^m)$ is an $W-$ideal. Clearly, $n > m$ and $0 \neq a^{n-m} \in l(a^m)$. Since $l(a^m)$ is a $W-$ideal, there exists $l \geq 1$ such that $(a^{n-m})^l \neq 0$ and $(a^{n-m})^l R \subseteq l(a^m)$. If $(n-m)l \geq m$, then $Ra^{(n-m)l}R$ is a nonzero nilpotent ideal of R. If $(n-m)l < m$, then $Ra^m R$ is a nonzero nilpotent ideal of R. Hence R is a NCI ring.

Similarly, we can show that right $WPZI$ rings are NCI.

(3) Let $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $l(a) \subseteq M$. If M is not essential in ${}_RR$, then $M = l(e)$ for some $e \in E(R)$. Thus $ae = 0$ because $a \in l(a) \subseteq M$. By (1), R is an Abelian ring, so $ea = 0$. This gives $e \in l(a) \subseteq l(e)$, a contradiction. Hence M is an essential left ideal of R, so R/M is a singular simple left $R-$module. By hypothesis, R/M is a $nil-$injective left $R-$module. Let $f : Ra \longrightarrow R/M$ defined by $f(ra) = r + M$. Then f is a well defined left $R-$homomorphism, so there exists a left $R-$homomorphism $g : R \longrightarrow R/M$ such that $g(a) = f(a)$. Hence there exists $c \in R$ such that $1 + M = f(a) = g(a) = ag(1) = ac + M$. Since R is a left or right $WPZI$ ring, $aRa = 0$. Thus $ac \in l(a) \subseteq M$. This leads to $1 \in M$, which is a contradiction. Hence $a = 0$.

(4) It follows from the proof of (3). □

A ring R is called directly finite if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is well known that Abelian rings are directly finite. Hence left or right $WPZI$ rings are directly finite by Proposition 1.1. According to Hwang, Jeon and Park [3], NCI rings need not be directly finite. Hence NCI rings need neither be left nor right $WPZI$.

A ring R is called left NV if every singular simple left $R-$module is $nil-$injective. Clearly, left $V-$rings and reduced rings are left NV. Since reduced rings are reversible and reversible rings are semicommutative, by Proposition 1.1, we have the following corollary.

Corollary 1.2 *The following conditions are equivalent for a ring R:*
(1) *R is a reduced ring;*
(2) *R is a reversible left NV ring;*
(3) *R is a semicommutative left NV ring;*
(4) *R is a left WPZI left NV ring;*
(5) *R is a right WPZI left NV ring.*

Kim, Nam and Kim ([4], Theorem 4) proved that if R is a semicommutative ring whose every simple singular left module is $YJ-$injective, then R is a reduced weakly regular ring. Hence, by Corollary 1.2, we have the following corollary.

Corollary 1.3 *Let R be a left or right WPZI ring. If every singular simple left R−module is YJ−injective, then R is a reduced weakly regular ring.*

Wei ([7], Theorem 16) proved that a ring R is a strongly regular ring if and only if R is a semicommutative $MELT$ ring whose singular simple left modules are $YJ-$injective. Hence, by Corollary 1.2, we have the following corollary.

Corollary 1.4 *A ring R is a strongly regular ring if and only if R is a $MELT$ left or right $WPZI$ ring whose every singular simple left module is $YJ-$injective.*

It is well known that a ring R is a reduced ring if and only if R is a semiprime semicommutative ring. On the other hand, semiprime left (right) $WPZI$ rings are reversible (in fact, if $ab = 0$, then $(ba)^2 = 0$. If R is a left $WPZI$ ring, then $l(ba)$ is a $W-$ideal, so $baRba = 0$. Since R is a semiprime ring, $ba = 0$). So, we have the following proposition:

Proposition 1.5 *The following conditions are equivalent for a ring R:*
(1) *R is a reduced ring;*
(2) *R is a semiprime left $WPZI$ ring;*
(3) *R is a semiprime right $WPZI$ ring.*

2 Strong regularity of $SF-$rings

Rege ([6], Remark 3.13) pointed out that if R is a reduced left (right) SF ring, then R is a strongly regular ring. We can extend this result to right $WPZI$ rings.

Proposition 2.1 *Let R is a left SF ring. If R is right $WPZI$, then R is strongly regular.*

Proof. Assume that $a \in R$. If $a = 0$, we are done. If $a \neq 0$, then there exists $n \geq 1$ such that $a^n \neq 0$ and $r(a^n)$ is a $W-$ideal of R because R is a right $WPZI$ ring. If $Ra+r(a^nR) \neq R$, then there exists a maximal left ideal M of R containing $Ra+r(a^nR)$. Since R is a left SF ring, R/M is a flat left $R-$module, so there exists $b \in M$ such that $a = ab$ because $a \in M$. Hence $1 - b \in r(a^n)$. If $1 - b = 0$, then $1 = b \in M$, a contradiction. Therefore $1 - b \neq 0$. Since $r(a^n)$ is a $W-$ideal of R, there exists $m \geq 1$ such that $(1-b)^m \neq 0$ and $R(1-b)^m \subseteq r(a^n)$. Hence $(1-b)^m \in r(a^nR)$, which implies $(1-b)^m \in M$. Since $1-(1-b)^m = (1+(1-b)+(1-b)^2+\cdots+(1-b)^{m-1})b \in M$, $1 \in M$, which is a contradiction. Hence $Ra + r(a^nR) = R$, which implies $Ra + r(a^n) = R$. Let $1 = ca + x$, where $c \in R$ and $x \in r(a^n)$. So $a^n = a^nca$. Write $d = a^{n-1} - a^{n-1}ca$. Then $d^2 = 0$. If $d \neq 0$, then similar to the proof mentioned above, we have $Rd + r(d) = R$, so there exists $u \in R$ such that $d = dud$. Hence there exists $y \in R$ such that $a^{n-1} = a^{n-1}ya$. If $d = 0$, then $a^{n-1} = a^{n-1}ca$. Repeating the process above, we obtain that $a = awa$ for some $w \in R$. So R is a regular ring. By Proposition 1.1(1), R is an Abelian ring, so R is a strongly regular ring. □

A ring R is called right weakly semicommutative, if for any $a, b \in R$, $ab = 0$ implies $aRb^n = 0$ for some $n \geq 1$. Clearly, semicommutatve rings are right weakly semicommutative and right weakly semicommutative rings are Abelian.

Theorem 2.2 *If R is a right weakly semicommutative left SF ring, then R is a strongly regular ring.*

Proof. Assume that $a \in R$. If $Ra + r(aR) \neq R$, then there exists a maximal left ideal M of R containing $Ra + r(aR)$. Since R is a left SF ring and R/M is a simple left $R-$module, R/M is flat. Hence $a = ab$ for some $b \in M$. Since R is a right weakly semicommutative ring, there exists $n \geq 1$ such that $aR(1-b)^n = 0$, hence $(1-b)^n \subseteq M$.

Thus $1 \in M$, a contradiction. Therefore $Ra + r(aR) = R$, which implies $a \in aRa$. Therefore R is a regular ring. Since R is an Abelian ring, R is a strongly regular ring. □

Lemma 2.3 *If R is a left SF ring and left $WPZI$ ring, then $Z_l(R) = 0$.*

Proof. If $Z_l(R) \neq 0$, then there exists $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. If $Z_l(R) + r(aR) \neq R$, then there exists a maximal left ideal M of R containing $Z_l(R) + r(aR)$. Since R is a left SF ring, R/M is a flat left $R-$module. Since $a \in Z_l(R) \subseteq M$, $a = ab$ for some $b \in M$. So $b \neq 1$ and $a \in l(1-b)$. Since R is a left $WPZI$ ring, there exists $n \geq 1$ such that $(1-b)^n \neq 0$ and $l((1-b)^n)$ is a $W-$ideal of R. Hence $aR \subseteq l((1-b)^n)$, which shows that $(1-b)^n \subseteq r(aR) \subseteq M$, which implies $1 \in M$. This contradiction leads to $Z_l(R) + r(aR) = R$. Let $1 = x + y$ where $x \in Z_l(R)$ and $y \in r(aR)$. Thus $ay = 0$ and $a = ax$. Since $x \in Z_l(R)$, $l(1-x) = 0$, which implies $a = 0$, a contradiction. Therefore $Z_l(R) = 0$. □

Lemma 2.4 *Let R be a left SF ring. If R is a left $WPZI$ ring, then R is a right weakly semicommutative ring.*

Proof. Assume that $a, b \in R$ with $ab = 0$. If $b = 0$, then $aRb = 0$, we are done. If $b \neq 0$, then there exists $n \geq 1$ such that $b^n \neq 0$ and $l(b^n)$ is an $W-$ideal of R. If $aRb^n \neq 0$, then $acb^n \neq 0$ for some $c \in R$. By Lemma 2.3, $Z_l(R) = 0$, so there exists a nonzero left ideal L of R such that $L \cap l(acb^n) = 0$. Let $0 \neq x \in L$. Then $xab = 0$ because $ab = 0$. If $xa = 0$, then $xacb^n = 0$, which implies $x \in L \cap l(acb^n)$. Thus $x = 0$, which is a contradiction. Hence $xa \neq 0$. Since $l(b^n)$ is a $W-$ideal of R, there exists $m \geq 1$ such that $(xa)^m \neq 0$ and $(xa)^m Rb^n = 0$. Hence $(xa)^{m-1}x \in l(acb^n) \cap L$, so $(xa)^{m-1}x = 0$, which implies $(xa)^m = 0$, a contradiction. Hence $aRb^n = 0$, we are done. □

Using Theorem 2.2 and Lemma 2.4, we have the following theorem:

Theorem 2.5 *If R is a left SF ring and left $WPZI$ ring, then R is a strongly regular ring.*

By Theorem 2.5, we obtain the following two corollaries which are generalization of Rege ([6], Remark 3.13).

Corollary 2.6 *If R is a left SF ring and semicommutative ring, then R is a strongly regular ring.*

Corollary 2.7 *If R is a left SF ring and reversible ring, then R is a strongly regular ring.*

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