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Generalized sequence spaces in 2-normed spaces defined by ideal and a modulus function

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Abstract The main objective of this paper is to define some new kind of generalized convergent sequence spaces with respect to a modulus function, and difference operator Δ^m, $m \geq 1$ in a 2-normed space. We also examine some topological properties of the resulting sequence spaces. Finally, we have introduced a new class of generalized convergent sequences with the help of an ideal and difference sequences in the same space.

Keywords statistical convergence · $\mathcal{I}$-convergence · difference sequence · modulus function · 2-norm

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1 Introduction and background

The notion of statistical convergence has been introduced by Fast [9] in 1951 and later developed by Connor [1], Fridy [10], Maddox [21], Šalát [30] and many others. Furthermore, Kostyrko et al. [19] presented a very interesting generalization of statistical convergence called as $\mathcal{I}$-convergence. The detailed history and development in this regard can be found in [2], [3], [4] and [12].

In 1960, Gähler [11] initially introduced the concept of 2-normed space as a generalization of normed linear space. Recently, many authors have started to study summability, sequence spaces in these nonlinear spaces (see, for instance [13]). In [28], Sahiner discussed ideal summability in these spaces and defined a new type of sequence spaces.

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Kizmaz [18] introduced the notion of difference sequence spaces as follows $\Delta(X) = \{x = (x_k) : (\Delta x_k) \in X\}$, for $X = \ell_\infty, c$ and c_0, where $\Delta x = (x_k - x_{k+1})$.

Continuing on this way, the notion was further generalized by Et and Çolak [5] by introducing the sequence spaces as follows $\Delta^m(X) = \{x = (x_k) : \Delta^m x_k \in X\}$, for $X = \ell_\infty, c$ and c_0, where $m \in \mathbb{N}$, $\Delta^m x = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$, and so that $\Delta^m x_k = \sum_{v=0}^{m}(-1)^v\binom{m}{v}x_{k+v}$. More applications of the difference sequences can be seen in [6], [7], [23] and [33].

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le \sup_k p_k = H$, $C = \max(1, 2^{H-1})$. Then for a_k, $b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \le C\{|a_k|^{p_k} + |b_k|^{p_k}\}, \text{ for all } k \in \mathbb{N}. \tag{1.1}$$

We recall [25] that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$, ii) $f(x + y) \le f(x) + f(y)$ for all $x, y \ge 0$, iii) f is increasing, iv) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. A modulus function may be bounded or unbounded. Subsequently, modulus function was used to define sequence spaces by Gürdal [15], Pehlivan [26] and Savas [31].

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

In [24], Mursaleen introduced the idea of λ-statistical convergence by extending the concept of $[V, \lambda]$ summability of [20]. Further, Savas [32] unified the two approaches and gave a new concepts of $\mathcal{I}$-statistical convergence, $\mathcal{I}$-S_λ-convergence and $\mathcal{I}$-$[V, \lambda]$ convergence.

Quite recently, many authors including [8], [22], [27] and [31] have constructed some sequence spaces by using modulus function, difference sequences and investigate their properties. In the present work, we also construct some sequence spaces defined by a modulus function, generalized difference sequences with the help of an ideal in a 2-normed space.

2 Preliminaries

Throughout the paper, $\mathbb{N}$ will denote the set of all positive integers.

Let $(X, \|.\|)$ be a normed linear space. We recall that a sequence $x = (x_k) \in X$ is called statistically convergent to $L \in X$ if for each $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L\| \ge \epsilon\}$ having its natural density zero.

A family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal in $\mathbb{N}$ if and only if:
(i) $\emptyset \in \mathcal{I}$;
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$;
(iii) For $A \in \mathcal{I}$ and $B \subset A$ we have $B \in \mathcal{I}$.

A non-empty family of sets $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is called a filter on $\mathbb{N}$ if and only if:
(i) $\emptyset \notin \mathcal{F}$;
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii) For $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \ne \emptyset$ and $\mathbb{N} \notin \mathcal{I}$.

It immediately implies that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a non-trivial ideal if and only if the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{\mathbb{N} - A : A \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an admissible ideal in $\mathbb{N}$ if and only if it contains all singletons i.e., if it contains $\{\{n\} : n \in \mathbb{N}\}$. Throughout the paper, $\mathcal{I}$ is considered as a non-trivial admissible ideal.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \rightarrow \mathbb{R}$, which satisfies: (i) $\|x, y\| = 0$, if and only if x and y are linearly dependent, (ii) $\|x, y\| = \|y, x\|$, (iii) $\|\alpha.x, y\| = |\alpha|.\|x, y\|, \alpha \in \mathbb{R}$ and (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. Then the pair $(X, \|.,.\|)$ is called 2-normed space.

Using the above terminology, GÜRDAL [14] defined $\mathcal{I}$-convergence in 2-nomed space which were further investigated in [17], [29] and [34].

Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and $(X, \|.,.\|)$ is a 2-normed space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}$-convergent to $L \in X$ if for each $\epsilon > 0$ and nonzero $z \in X$, the set $A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z\| \geq \epsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim_{k\rightarrow\infty} \|x_k, z\| = \|L, z\|$.

Definition 2.1 ([8]) *A sequence $x = (x_k)$ is said to be λ_X^m-statistically convergent to the number L if, for every $\epsilon > 0$, $\lim_{n\rightarrow\infty} \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| = 0$. In this case, we write $S_\lambda(\Delta^m, X) - \lim_{k\rightarrow\infty} x_k = L$.*

Recently, SAVAŞ ET AL. [32] combined the ideas of λ-statistical convergence and ideal convergence to introduce new concepts of $\mathcal{I} - S_\lambda$-convergence, $\mathcal{I} - [V, \lambda]$ convergence and later some pioneer works have been extended in this direction by numerous authors such as [2] and [16].

Definition 2.2 ([32]) *A sequence $x = (x_k)$ is said to be $\mathcal{I} - [V, \lambda]$ summable to L, if for any $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} \|x_k - L\| \geq \delta\} \in \mathcal{I}$, where $I_n = [n - \lambda_n + 1, n]$.*

Definition 2.3 ([32]) *A sequence $x = (x_k)$ is said to be $\mathcal{I} - \lambda$-statistically convergent or $\mathcal{I} - S_\lambda$ convergent to L, if for every $\epsilon > 0$ and $\delta > 0$, $\{r \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}$. In this case, we write $x_k \rightarrow L(\mathcal{I} - S_\lambda)$ or $\mathcal{I} - S_\lambda - \lim_{k\rightarrow\infty} x_k = L$.*

The following well-known lemma is required for establishing a very important result in our article.

Lemma 2.4 *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f(x) \leq \frac{2.f(1)x}{\delta}$.*

3 Main results

In this section, we introduce a certain type of sequence spaces using modulus function and generalized difference operator Δ^m in a 2-normed space, where S_X^2 stands for the space of all sequences defined over 2-normed space $(X, \|.,.\|)$.

Definition 3.1 *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, f be a modulus function and $p = (p_k)$ be a bounded sequence of positive (strictly) real numbers, then for each $\epsilon > 0$ and*

$z \in X$, we define the following sequence spaces:

$$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0 = \left\{ x = (x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta\} \in \mathcal{I} \right\},$$

$$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] = \left\{ x=(x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta\} \in \mathcal{I} \right\}, \text{ for } L > 0,$$

$$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_\infty = \left\{ x=(x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq K\} \in \mathcal{I} \right\}, \text{ for } K > 0.$$

We can write it as $x = (x_k) \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ or $x_k \to L(V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.

Remark 3.1 If we take $f(x) = x$ in the above definition, then we obtain $V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$, $V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ instead of $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ and $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ respectively.

Theorem 3.2 *$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$, $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ and $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_\infty$ are linear spaces over $\mathbb{C}$.*

Proof. We will prove the assertion for $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ only and the others can be proved similarly.

We assume $x = (x_k)$, $y = (y_k) \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ and $\alpha, \beta \in \mathbb{C}$. Then for any $\delta > 0$ and for each $z \in X$, the sets

$$A_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\}, \tag{3.1}$$

$$B_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\} \tag{3.2}$$

belong to $\mathcal{I}$.

Since f be a modulus function and $\|.,.\|$ is a 2-norm function, then the following inequality holds

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k, z\|]^{p_k} \\
&\leq \frac{1}{\lambda_n} \sum_{k \in I_n} [f(|\alpha|.\|\Delta^m x_k, z\|) + f(|\beta|.\|\Delta^m y_k, z\|]^{p_k} \\
&\leq C.(M_\alpha)^H.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \\
&+ C.(M_\beta)^H.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k}, \quad \text{by (1.1)}
\end{aligned}$$

where M_α, M_β are positive integers such that $|\alpha| \le M_\alpha$ and $|\beta| \le M_\beta$. For given $\delta > 0$ and for all $z \in X$, we have the following containment

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k, z\|)]^{p_k} \ge \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \ge \frac{\delta}{2C.(M_\alpha)^H}\right\}$$
$$\cup \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k} \ge \frac{\delta}{2C.(M_\beta)^H}\right\}.$$

By using (3.1) and (3.2), the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k,\ z\|)]^{p_k} \ge \delta\} \in \mathcal{I}$. This completes the proof. □

Theorem 3.3 *Let $p = (p_k)$ be a sequence of strictly positive real numbers, then for $m \ge 1$ the inclusion $V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_{0,\infty} \subset V_f^{\mathcal{I}}[\|.,.\|, \Delta^m,\ \lambda, p]_{0,\infty}$ is strict.*

Proof. We will prove the result for $V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$ only. The others can be proved similarly.

Suppose $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$, by definition for every $\delta > 0$ and $z \in X$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} \ge \delta\right\} \in \mathcal{I}. \tag{3.3}$$

By the property of modulus function, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \le \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|) + f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k}$$
$$\le C.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} + C.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k} \text{ by (1.1).}$$

Now for given $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \ge \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} \ge \frac{\delta}{2C}\right\}$$
$$\cup \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k} \ge \frac{\delta}{2C}\right\},$$

for each $z \in X$.

Since $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$, it follows that the sets on the right hand side in the above containment belong to $\mathcal{I}$. Hence $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$. To show that the inclusion is strict, we give the following example:

We take $f(x) = x$, $\lambda_n = n$ and consider a sequence $x = (x_k) = k^{m-1}$, then $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ but does not belong to $V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$ for $p_k = 1$, $k \in \mathbb{N}$. This shows that the inclusion is strict. □

Theorem 3.4 *Let f', f'' are modulus functions. If $\limsup_{t\to\infty} \dfrac{f'(t)}{f''(t)} = P > 0$, then $V_{f'}^{\mathcal{I}}[\|.,.\|,\ \Delta^m,\ \lambda,\ p] \subset V_{f''}^{\mathcal{I}}[\|.,.\|,\ \Delta^m,\ \lambda, p]$.*

Proof. Let $\limsup_{t\to\infty} \frac{f'(t)}{f''(t)} = P$, then there exists a constant $M > 0$ such that $f'(t) \geq M.f''(t)$, for all $t \geq 0$. Therefore for each $z \in X$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq (M)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L, Z\|)]^{p_k}.$$

Then for every $\delta > 0$ and $z \in X$, we have following relationship

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L, Z\|)]^{p_k} \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta.(K)^H\right\}.$$

Since $x \in V_{f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$, it follows that the set on left side of the above containment belong to $\mathcal{I}$. Which gives that $x \in V_{f''}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$. □

Theorem 3.5 *If f, f' and f'' are modulus functions, then*:
(*i*) $V_{f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] \subset V_{f\circ f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$,
(*ii*) $V_{f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] \cap V_{f''}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] \subset V_{f'+f''}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.

Proof. (i) Let $x = (x_k) \in V_{f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$, then for every $\epsilon > 0$ and for some $L > 0$ such that

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \epsilon\right\} \in \mathcal{I}, \tag{3.4}$$

for each $z \in X$. For given $\epsilon > 0$, we choose $\delta \in (0, 1)$ such that $f(t) < \epsilon$ for all $0 < t < \delta$. On the other hand, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}[f\circ f'(\|\Delta^m x_k - L, z\|)]^{p_k}\\
&= \frac{1}{\lambda_n}\sum_{k\in I_n \& [f'(\|\Delta^m x_k - L, z\|)]^{p_k} < \delta}[f\circ f'(\|\Delta^m x_k - L, z\|)]^{p_k}\\
&+ \frac{1}{\lambda_n}\sum_{k\in I_n \& [f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta}[f\circ f'(\|\Delta^m x_k - L\|)]^{p_k}\\
&\leq (\epsilon)^H + \max\left(1, \left(2.\frac{f(1)}{\delta}\right)^H\right).\frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L, z\|)]^{p_k} \text{ by Lemma 2.4.}
\end{aligned}$$

By using (3.4), we obtain $x \in V^{\mathcal{I}}_{f \circ f'}[\|.,.\|, \Delta^m, \lambda, p]$.

(ii) The result of the theorem is proved by using the following inequality

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[(f'+f'')(\|\Delta^m x_k - L, z\|)]^{p_k} \leq \frac{C}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L, z\|)]^{p_k} + \frac{C}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L, z\|)]^{p_k},$$

where $\sup_k p_k = H$ and $C = \max(1, 2^{H-1})$. □

Theorem 3.6 *Let f be a modulus function and $p = (p_k)$ be a sequence of positive real numbers, then $V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] \subseteq V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. This can be proved by using the techniques similar to those used in Theorem 3.4 (i). □

Theorem 3.7 *Let f be a modulus function. If $\limsup_{t\to\infty} \dfrac{f(t)}{t} = M > 0$, then $V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p] \subseteq V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. Suppose $x = (x_k) \in V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]$ and $\limsup_{t\to\infty} \frac{f(t)}{t} = M > 0$, then there exists a constant $K > 0$ such that $f(t) \geq K.t$, for all $t \geq 0$. Which implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq (K)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}\|\Delta^m x_k - L, z\|^{p_k},$$

for each $z \in X$. Which gives the result. □

Theorem 3.8 *If $0 < p_k \leq q_k$ and $(\frac{q_k}{p_k})$ be bounded, then $V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, q] \subset V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. The proof of this theorem is omitted. □

4 $S^{\Delta^m}_{\lambda}(\|.,.\|, \mathcal{I})$-convergence

In this section, we define a new class of generalized statistical convergent sequences with the help of an ideal and difference sequences. Furthermore, we also establish a strong connection between this convergence and the sequence space $V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]$.

Definition 4.1 *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence. A sequence $x = (x_k) \in X$ is said to be $S^{\Delta^m}_{\lambda}(\mathcal{I})$-convergent to a number L provided that for every $\epsilon > 0$, $\delta > 0$ and $z \in X$, the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}$. In this case, we write $S^{\Delta^m}_{\lambda}(\mathcal{I}) - \lim_{k\to\infty} \|x_k, z\| = \|L, z\|$. Let $S^{\Delta^m}_{\lambda}(\|.,.\|, \mathcal{I})$ denotes the set of all $S^{\Delta^m}_{\lambda}(\mathcal{I})$-convergent sequences in X.*

Theorem 4.2 *Let f be a modulus function and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p] \subset S^{\Delta^m}_{\lambda}(\|.,.\|, \mathcal{I})$.*

Proof. Suppose $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ and $\epsilon > 0$ be given. Then for each $z \in X$, we obtain

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L, z\|)]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in I_n \|\Delta^m x_k - L, z\| \geq \epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k}$$
$$+\frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L, z\| < \epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k}$$
$$\geq \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L, z\| \geq \epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\epsilon)]^{p_k}$$
$$\geq \frac{1}{\lambda_n}\sum_{k\in I_n}\min([f(\epsilon)]^h, [f(\epsilon)]^H) \geq K.\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}|,$$

where $K = \min([f(\epsilon)]^h, [f(\epsilon)]^H)$. Then for every $\delta > 0$ and $z \in X$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}| \geq \delta\right\}$$
$$\subseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq K.\delta\}.$$

Since $x_k \to L(V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p])$ so that $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty}\|x_k, z\| = \|L, z\|$. □

Theorem 4.3 *Let $p = (p_k)$ be a sequence of strictly positive real numbers and f e a bounded modulus function. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) \subset V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. Using the same technique of [8, Theorem 3.5], it is easy to prove this theorem. □

Theorem 4.4 *If f be bounded and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) = V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ if and only if f is bounded.*

Proof. This part can be obtained by combining Theorems 4.1 and 4.2. □

Conversely. Suppose f is unbounded defined by $f(k) = k$ for all $k \in \mathbb{N}$. We take a fixed set $A \in \mathcal{I}$, where $\mathcal{I}$ be an admissible ideal and define $x = (x_k)$ as follows:

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n,\ n \notin A, \\ k^{m+1}, & \text{for } n - \lambda_n + 1 \leq k \leq n,\ n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For given $\epsilon > 0$ and for each $z \in X$, we have $\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0$ for $n \notin A$. Hence for $\delta > 0$, there exists a positive integer n_0 such that $\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| < \delta$ for $n \notin A$ and $n \geq n_0$. Now, we have

$\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| \geq \delta\} \subset \{A \cup (1, 2, \cdots n_0 - 1)\}$. Since $\mathcal{I}$ be an admissible ideal. It follows that $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} \|x_k, z\| \to 0$ for each $z \in X$.

On the other hand, if we take $p_k = 1$, for all $k = 1, 2, \cdots$, then $x_k \notin V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$. This contradicts the fact $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) = V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$, so our supposition is wrong.

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