An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 1

Strongly J-clean skew triangular matrix rings*

Yosum Kurtulmaz

Received: 5.VI.2013 / Revised: 18.VI.2013 / Accepted: 21.VI.2013

Abstract Let R be an arbitrary ring with identity. An element $a \in R$ is *strongly J-clean* if there exist an idempotent $e \in R$ and element $w \in J(R)$ such that $a = e + w$ and $ew = ew$. A ring R is *strongly J-clean* in case every element in R is strongly J-clean. In this note, we investigate the strong J-cleanness of the skew triangular matrix ring $T_n(R, \sigma)$ over a local ring R, where σ is an endomorphism of R and $n = 2, 3, 4$.

Keywords strongly J-clean ring · skew triangular matrix ring · local ring

Mathematics Subject Classification (2010) 15B33 · 15B99 · 16L99

1 Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $J(R)$ and $U(R)$ will denote, respectively, the Jacobson radical and the group of units in R. An element $a \in R$ is strongly clean if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$. A ring R is strongly clean if every element in R is strongly clean. Many authors have studied such rings from very different points of view (cf. [1-9]). An element $a \in R$ is strongly J-clean provided that there exist an idempotent $e \in R$ and element $w \in J(R)$ such that $a = e + w$ and $ew = ew$. A ring R is strongly J-clean in case every element in R is strongly J-clean. Strong J-cleanness over commutative rings is studied in [1] and deduced the strong J-cleanness of $T_n(R)$ for a large class of local rings R, where $T_n(R)$ denotes the ring of all upper triangular matrices over R.

Let σ be an endomorphism of R preserving 1 and $T_n(R, \sigma)$ be the set of all upper triangular matrices over the rings R. For any $(a_{ij}), (b_{ij}) \in T_n(R, \sigma)$, we define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$, and $(a_{ij})(b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_{k=i}^{n} a_{ik}\sigma^{k-i}(b_{kj})$. Then $T_n(R, \sigma)$

* This paper is dedicated to my mother Gönül Ünalan.

Yosum Kurtulmaz
Bilkent University,
Department of Mathematics,
Bilkent, Ankara, Turkey
E-mail: yosum@fen.bilkent.edu.tr

is a ring under the preceding addition and multiplication. It is clear that $T_n(R,\sigma)$ will be $T_n(R)$ only when σ is the identity morphism. Let $a \in R$ and the maps $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$.

Strong cleanness of $T_n(R,\sigma)$ for several n was studied in [3]. In this article, we investigate the strong J-cleanness of $T_n(R,\sigma)$ over a local ring R for $n = 2, 3, 4$ and then extend strong cleanness to such properties. In this direction we show that $T_2(R,\sigma)$ is strongly J-clean if and only if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_{\sigma(b)} : R \to R$ is surjective and $R/J(R) \cong \mathbb{Z}_2$. Further if $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R), b \in J(R)$, then $T_3(R,\sigma)$ is strongly J-clean if and only if $R/J(R) \cong \mathbb{Z}_2$. The necessary condition for $T_3(R,\sigma)$ to be strongly J-clean is also discussed. In addition to these, if $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R)$, $b \in J(R)$, then $T_4(R,\sigma)$ is strongly J-clean if and only if $R/J(R) \cong \mathbb{Z}_2$.

2 The case $n = 2$

By [Theorem 4.4, 2], the triangular matrix ring $T_2(R)$ over a local ring R is strongly J-clean if and only if R is bleached and $R/J(R) \cong \mathbb{Z}_2$. We extend this result to the skew triangular matrix ring $T_2(R,\sigma)$ over a local ring R.

Remark 2.1 will be used in the sequel without reference to.

Remark 2.1 Note that if for any ring R, $R/J(R) \cong \mathbb{Z}_2$, then $2 \in J(R)$, $1 + J(R) = U(R)$ and $1 + U(R) = J(R)$. For if, f is the isomorphism $R/J(R) \cong \mathbb{Z}_2$ then $f(1 + J(R)) = 1 + 2\mathbb{Z}$. Hence $f(2 + J(R)) = 2 + 2\mathbb{Z} = 0 + 2\mathbb{Z}$. So $2 + J(R) = 0 + J(R)$, that is $2 \in J(R)$. $1 + J(R) \subseteq U(R)$. Let $u \in U(R)$. Then $f(u + J(R)) = 1 + 2\mathbb{Z} = f(1 + J(R))$. Hence $u - 1 \in J(R)$ and so $u \in 1 + J(R)$. Thus, $U(R) \subseteq 1 + J(R)$ and $U(R) = 1 + J(R)$.

Lemma 2.1 *Let R be a ring and let σ be an endomorphism of R. If $T_n(R,\sigma)$ is strongly J-clean for some $n \in \mathbb{N}$, then so is R.*

Proof. Let $e = diag(1, 0, \ldots, 0) \in T_n(R,\sigma)$. Then $R \cong eT_n(R,\sigma)e$. From Corollary 3.5 in [2], R is strongly J-clean. □

Theorem 2.2 *Let R be a local ring, and let σ be an endomorphism of R. Then the following are equivalent:*

(1) $T_2(R,\sigma)$ is strongly J-clean.
(2) If $a \in 1 + J(R), b \in J(R)$, then $l_a - r_{\sigma(b)} : R \to R$ is surjective and $R/J(R) \cong \mathbb{Z}_2$

Proof. (1) ⇒ (2) From Lemma 2.2, R is strongly J-clean and by Lemma 4.2 in [2], $R/J(R) \cong \mathbb{Z}_2$. By Remark 2.1, $1 + J(R) = U(R)$. Let $a \in 1 + J(R), b \in J(R), v \in R$. Then $A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in T_2(R,\sigma)$. By hypothesis, there exists an idempotent $E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \in T_2(R,\sigma)$ such that $A - E \in J\big(T_2(R,\sigma)\big)$ and $AE = EA$. Since R is local, all idempotents in R are 0 and 1. Thus, we see that $e = 1, f = 0$; otherwise, $A -$

$E \notin J\big(T_2(R,\sigma)\big)$. So $E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. As $AE = EA$, we get $-v + x\sigma(b) = ax$. Hence, $ax - x\sigma(b) = -v$ for some $x \in R$. As a result, $l_a - r_{\sigma(b)} : R \to R$ is surjective.

(2) $\Rightarrow$ (1) Let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R,\sigma)$.

Case 1. If $a, b \in J(R)$, then $A \in J\big(T_2(R,\sigma)\big)$ is strongly J-clean.
Case 2. If $a, b \in 1 + J(R)$, then $A - I_2 \in J\big(T_2(R,\sigma)\big)$; hence, $A = I_2 + (A - I_2) \in T_2(R,\sigma)$ is strongly J-clean.
Case 3. If $a \in 1 + J(R), b \in J(R)$, by hypothesis, $l_a - r_{\sigma(b)} : R \to R$ is surjective. Thus, $ax - x\sigma(b) = v$ for some $x \in R$. Choose $E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \in T_2(R,\sigma)$. Then $E^2 = E \in T_2(R,\sigma)$, $AE = EA$ and $A - E \in J\big(T_2(R,\sigma)\big)$. That is, $A \in T_2(R,\sigma)$ is strongly J-clean.
Case 4. If $a \in J(R), b \in 1 + J(R)$, then $a + 1 \in 1 + J(R), b + 1 \in J(R)$ and by hypothesis, $l_{a+1} - r_{\sigma(b+1)} : R \to R$ is surjective. Thus $ax - x\sigma(b) = -v$ for some $x \in R$. Choose $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \in T_2(R,\sigma)$. Then $E^2 = E \in T_2(R,\sigma)$, $AE = EA$ and $A - E \in J\big(T_2(R,\sigma)\big)$. Hence, $A \in T_2(R,\sigma)$ is strongly J-clean. Therefore $A \in T_2(R,\sigma)$ is strongly J-clean. □

Corollary 2.3 *Let R be a local ring, and let σ be an endomorphism of R. Then the following are equivalent:*

(1) $T_2(R,\sigma)$ is strongly J-clean.
(2) $R/J(R) \cong \mathbb{Z}_2$ and $T_2(R,\sigma)$ is strongly clean.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Let $a \in 1 + J(R), b \in J(R), v \in R$. Then $A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in T_2(R,\sigma)$. By hypothesis, there exists an idempotent $E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \in T_2(R,\sigma)$ such that $A - E \in J\big(T_2(R,\sigma)\big)$ and $AE = EA$. Since R is local, we see that $e = 0, f = 1$; otherwise, $A - E \notin J\big(T_2(R,\sigma)\big)$. So $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$. It follows from $AE = EA$ that $v + x\sigma(b) = ax$, and so $ax - v = x\sigma(b)$. Therefore $l_a - r_{\sigma(b)} : R \to R$ is surjective. By Theorem 2.3, $T_2(R,\sigma)$ is strongly J-clean as $R/J(R) \cong \mathbb{Z}_2$. □

Corollary 2.4 *Let R be a ring, and $R/J(R) \cong \mathbb{Z}_2$. If $J(R)$ is nil, then $T_2(R,\sigma)$ is strongly J-clean.*

Proof. Clearly R is local. Let $a \in 1 + J(R), b \in J(R)$. Then we can find some $n \in \mathbb{N}$ such that $b^n = 0$. For any $v \in R$, we choose $x = \big(l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}}\big)(v)$. It can be easily checked that $(l_a - r_b)(x) = (l_a - r_b)\big(l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}}\big)(v) = \big(v + a^{-1}vb + \cdots + a^{-n+1}vb^{n-1}\big) - \big(a^{-1}vb + \cdots + a^{-n}vb^n\big) = v$. Hence, $l_a - r_b : R \to R$ is surjective. Similarly, $l_a - r_{\sigma(b)}$ is surjective since $\sigma(b) \in J(R)$. This completes the proof by Theorem 2.3. □

Example 2.1 Let $\mathbb{Z}_{2^n} = \mathbb{Z}/2^n\mathbb{Z}, n \in \mathbb{N}$, and let σ be an endomorphism of $\mathbb{Z}_{2^n}$. Then, $T_2(\mathbb{Z}_{2^n}, \sigma)$ is strongly J-clean. As $\mathbb{Z}_{2^n}$ is a local ring with the Jacobson radical $2\mathbb{Z}_{2^n}$. Obviously, $J\big(\mathbb{Z}_{2^n}\big)$ is nil, and we are through by Corollary 2.4.

Example 2.2 Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, let

$$R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4\},$$

and let $\sigma : R \to R$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. Then $T_2(R, \sigma)$ is strongly J-clean. Obviously, σ is an endomorphism of R. It is easy to check that $J(R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2, b \in \mathbb{Z}_4\}$, and then $R/J(R) \cong \mathbb{Z}_2$ is a field. Thus, R is a local ring. In addition, $\big(J(R)\big)^4 = 0$, thus $J(R)$ is nil. Therefore we obtain the result by Corollary 2.4.

3 The case $n = 3$

We now extend Theorem 2.2 to the case of 3×3 skew triangular matrix rings over a local ring.

Theorem 3.1 *Let R be a local ring. If $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R), b \in J(R)$, then $T_3(R, \sigma)$ is strongly J-clean if and only if $R/J(R) \cong \mathbb{Z}_2$.*

Proof. ($\Leftarrow$) We noted in Remark 2.1, in this case we have $\sigma(J(R)) \subseteq J(R)$, $\sigma(U(R)) \subseteq U(R)$, $1 + J(R) = U(R)$ and $1 + U(R) = J(R)$ and we use them in the sequel intrinsically. Let $A = (a_{ij}) \in T_3(R, \sigma)$. We divide the proof into six cases.
Case 1. If $a_{11}, a_{22}, a_{33} \in 1 + J(R)$, then $A = I_3 + (A - I_3)$, and so $A - I_3 \in J(T_3(R, \sigma))$. Then $A \in T_3(R, \sigma)$ is strongly J-clean.
Case 2. If $a_{11} \in J(R), a_{22}, a_{33} \in 1 + J(R)$, then we have an $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$. Further, we have some $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = e_{12}\sigma(a_{23}) - a_{13}$. Choose

$$E = \begin{pmatrix} 0 & e_{12} & e_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. Furthermore,

$$EA = \begin{pmatrix} 0 & e_{12}\sigma(a_{22}) & e_{12}\sigma(a_{23}) + e_{13}\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & a_{11}e_{12} + a_{12} & a_{11}e_{13} + a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. That is, $A \in T_3(R, \sigma)$ is strongly J-clean.
Case 3. If $a_{11} \in 1 + J(R), a_{22} \in J(R), a_{33} \in 1 + J(R)$, then we have an $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$. Further, we have some $e_{23} \in R$ such that

$a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. Thus $-a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) = -e_{12}\sigma(a_{22})\sigma(e_{23}) = e_{12}\sigma(a_{23}) - e_{12}\sigma(e_{23})\sigma^2(a_{33})$. Choose

$$E = \begin{pmatrix} 1 & e_{12} & -e_{12}\sigma(e_{23}) \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R,\sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R,\sigma))$. Furthermore,

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) & a_{13} + e_{12}\sigma(a_{23}) - e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & 0 & e_{23}\sigma(a_{33}) \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & -a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. Thus, $A \in T_3(R,\sigma)$ is strongly J-clean.

Case 4. If $a_{11}, a_{22} \in 1 + J(R), a_{33} \in J(R)$, then we find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Thus, there exists $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} - a_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R,\sigma))$. Furthermore,

$$EA = \begin{pmatrix} a_{11} & a_{12} & a_{13} + e_{13}\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{12} & a_{11}e_{13} + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $EA = AE$. Therefore $A \in T_3(R,\sigma)$ is strongly J-clean.

Case 5. If $a_{11} \in 1 + J(R), a_{22}, a_{33} \in J(R)$, then we have some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$. Further, there exists $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R,\sigma))$. Hence

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) & a_{13} + e_{12}\sigma(a_{23}) + e_{13}\sigma^2(a_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & a_{11}e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $EA = AE$. Thus $A \in T_3(R,\sigma)$ is strongly J-clean.
Case 6. If $a_{11} \in J(R), a_{22} \in 1+J(R), a_{33} \in J(R)$, then we find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Hence there is $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$. It is easy to verify that

$$e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) = e_{12}\sigma(a_{22}e_{23}) = a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}).$$

Choose

$$E = \begin{pmatrix} 0 & e_{12} & e_{12}\sigma(e_{23}) \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} 0 & e_{12}\sigma(a_{22}) & e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & a_{11}e_{12} + a_{12} & a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

and so $EA = AE$. Consequently, $A \in T_3(R,\sigma)$ is strongly J-clean.
Case 7. If $a_{11}, a_{22} \in J(R), a_{33} \in 1 + J(R)$, then we find $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. Further, we have an $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = -a_{13} - a_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R,\sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R,\sigma))$. Furthermore,

$$EA = \begin{pmatrix} 0 & 0 & e_{13}\sigma^2(a_{33}) \\ 0 & 0 & e_{23}\sigma(a_{33}) \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & 0 & a_{11}e_{13} + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. As a result, $A \in T_3(R,\sigma)$ is strongly J-clean.
Case 8. If $a_{11}, a_{22}, a_{33} \in J(R)$, then $A = 0 + A$, where $A \in J(T_3(R,\sigma))$. Hence, $A \in T_3(R,\sigma)$ is strongly J-clean.

Thus, $T_3(R,\sigma)$ is strongly J-clean.

($\Rightarrow$) Similar to Theorem 2.2, we easily complete the proof. □

Corollary 3.2 *Let R be a ring, and $R/J(R) \cong \mathbb{Z}_2$. If $J(R)$ is nil, then $T_3(R,\sigma)$ is strongly J-clean.*

Proof. Obviously R is local. Let $a \in U(R), b \in J(R)$. Then we can find some $n \in \mathbb{N}$ such that $b^n = 0$; hence, $\big(\sigma(b)\big)^n = 0$. For any $v \in R$, we choose $x = \big(l_{a^{-1}} + l_{a^{-2}}r_{\sigma(b)} + \cdots + l_{a^{-n}}r_{\sigma(b)^{n-1}}\big)(v)$. It can be easily checked that $(l_a - r_{\sigma(b)})(x) = (l_a - r_{\sigma(b)})\big(l_{a^{-1}} + l_{a^{-2}}r_{\sigma(b)} + \cdots + l_{a^{-n}}r_{\sigma(b)^{n-1}}\big)(v) = \big(v + a^{-1}v\sigma(b) + \cdots + a^{-n+1}v\sigma(b)^{n-1}\big) - \big(a^{-1}v\sigma(b) + \cdots + a^{-n}v\sigma(b)^n\big) = v$. Thus, $l_a - r_{\sigma(b)} : R \to R$ is surjective. Likewise, $l_b - r_{\sigma(a)} : R \to R$ is surjective. Consequently, $T_3(R,\sigma)$ is strongly J-clean by Theorem 3.1. □

4 A characterization

We will consider the necessary and sufficient conditions under which the skew triangular matrix ring $T_3(R,\sigma)$ is strongly J-clean.

Lemma 4.1 *Let R be a local ring. If $T_3(R,\sigma)$ is strongly J-clean, then $l_a - r_{\sigma(b)}$, $l_a - r_{\sigma^2(b)}$, $l_b - r_{\sigma(a)}$ and $l_b - r_{\sigma^2(a)}$ are surjective for any $a \in 1+J(R), b \in J(R)$.*

Proof. Let $a \in 1+J(R), b \in J(R)$. Clearly, $T_2(R,\sigma)$ is strongly J-clean. By Theorem 2.2, $l_a - r_{\sigma(b)}$ is surjective. As $1-b \in 1+J(R)$ and $1-a \in J(R)$, we get $l_{1-b} - r_{\sigma(1-a)} : R \to R$ is surjective. For any $v \in R$, we have an $x \in R$ such that $(1-b)x - x\sigma(1-a) = -v$. Thus, $bx - x\sigma(a) = v$ and so $l_b - r_{\sigma(a)} : R \to R$ is surjective.

Let $v \in R$ and let

$$A = \begin{pmatrix} b & 0 & v \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} \in T_3(R,\sigma).$$

We have an idempotent $E = (e_{ij}) \in T_3(R,\sigma)$ such that $A - E \in J\big(T_3(R,\sigma)\big)$ and $EA = AE$. This implies that $e_{11}, e_{22}, e_{33} \in R$ are all idempotents. As $a \in 1+J(R), b \in J(R)$, we have $e_{11} = 0, e_{22} = 0$ and $e_{33} = 1$; otherwise, $A - E \notin J(T_3(R,\sigma))$. As $E^{=}E$, we have

$$E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

for some $e_{13}, e_{23} \in R$. Observing that

$$\begin{pmatrix} 0 & 0 & be_{13}+v \\ 0 & 0 & be_{23} \\ 0 & 0 & a \end{pmatrix} = AE = EA = \begin{pmatrix} 0 & 0 & e_{13}\sigma^2(a) \\ 0 & 0 & e_{23}\sigma(a) \\ 0 & 0 & a \end{pmatrix},$$

we have $be_{13} - e_{13}\sigma^2(a) = -v$. Thus, $l_b - r_{\sigma^2}(a) : R \to R$ is surjective. Since $1-a \in J(R)$ and $1-b \in 1+J(R)$, we have, $l_{1-a} - r_{\sigma^2(1-b)} : R \to R$ is surjective. Thus, we can find some $x \in R$ such that $(1-a)x - x\sigma^2(1-b) = -v$. This implies that $ax - x\sigma^2(b) = v$, hence $l_a - r_{\sigma^2(b)}$ is surjective. □

Theorem 4.2 *Let R be a local ring and let σ be an endomorphism of R. Then the following are equivalent:*

(1) $T_3(R,\sigma)$ is strongly J-clean.
(2) $R/J(R) \cong \mathbb{Z}_2$, and $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1+J(R), b \in J(R)$.

Proof. (1) ⇒ (2) is obvious from Lemma 4.1.

(2) ⇒ (1) Clear from Theorem 3.1. □

Corollary 4.3 *Let R be a local ring and let σ be an endomorphism of R.Then the following are equivalent:*

(1) $T_2(R,\sigma)$ is strongly J-clean.
(2) $T_3(R,\sigma)$ is strongly J-clean.
(3) $R/J(R) \cong \mathbb{Z}_2$ and $l_a - r_{\sigma(b)}$ is surjective for any $a \in 1+J(R), b \in J(R)$.

Proof. (1) ⇔ (3) is proved by Theorem 2.2.

(2) ⇔ (3) is obvious from Theorem 4.2. □

5 The case $n = 4$

We now extend the preceding discussion to the case of 4×4 skew triangular matrix rings over a local ring.

Theorem 5.1 *Let R be a local ring. If $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R)$, $b \in J(R)$, then $T_4(R, \sigma)$ is strongly J-clean if and only if $R/J(R) \cong \mathbb{Z}_2$.*

Proof. ($\Leftarrow$) As $R/J(R) \cong \mathbb{Z}_2$, $\sigma(J(R)) \subseteq J(R)$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \in T_4(R, \sigma).$$

We show the existence of

$$E = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ 0 & e_{22} & e_{23} & e_{24} \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 0 & 0 & e_{44} \end{pmatrix} \in T_4(R, \sigma),$$

such that $E^2 = E, AE = EA$ and $A - E \in J(T_4(R, \sigma))$. One can easily derive from $E^2 = E$ that

(a) $e_{12} = e_{11}e_{12} + e_{12}\sigma(e_{22})$
(b) $e_{13} = e_{11}e_{13} + e_{12}\sigma(e_{23}) + e_{13}\sigma^2(e_{33})$
(c) $e_{23} = e_{22}e_{23} + e_{23}\sigma(e_{33})$

and from $AE = EA$ that

(d) $a_{11}e_{12} - e_{12}\sigma(a_{22}) = e_{11}a_{12} - a_{12}\sigma(e_{22})$
(e) $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = e_{11}a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23}) - a_{13}\sigma^2(e_{33})$
(f) $a_{22}e_{23} - e_{23}\sigma(a_{33}) = e_{22}a_{23} - a_{23}\sigma(e_{33})$

Case 1. If $a_{22} \in J(R)$, $a_{11} \in 1 + J(R)$ then $e_{22} = 0$, $e_{11} = 1$. Hence, (d) implies that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$ and by assumption there exists $e_{12} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{22})})(e_{12}) = a_{12}$.

(**A**) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (f), $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$ and (b) implies that $e_{13} = -e_{12}\sigma(e_{23})$.

(**B**) If $a_{33} \in J(R)$, then $e_{33} = 0$. By (c), $e_{23} = 0$. From (e), we have $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$ and by assumption there exists $e_{13} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{33})})(e_{13}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$.

Case 2. If $a_{22} \in 1 + J(R)$, $a_{11} \in 1 + J(R)$, then $e_{22} = 1$, $e_{11} = 1$. By (a) implies that $e_{12} = 0$.

(**C**) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (b), we have $e_{13} = 0$ and (c) implies that $e_{23} = 0$.

(**D**) If $a_{33} \in J(R)$, then $e_{33} = 0$. By (f), we have $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$, and (e) gives rise to $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$ and by assumption there exists $e_{13} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{33})})(e_{13}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$.

Case 3. If $a_{22} \in 1 + J(R)$, $a_{11} \in J(R)$, then $e_{22} = 1$, $e_{11} = 0$. By (d), $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$ and there exists $e_{12} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{22})})(e_{12}) = -a_{12}$.

(**E**) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (c), we have $e_{23} = 0$. Then from (e), we have $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = e_{12}\sigma(a_{23}) - a_{13}$

(**F**) If $a_{33} \in J(R)$, then $e_{33} = 0$. From (f), we have $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$ and there exists $e_{23} \in R$ such that $(l_{a_{22}} - r_{\sigma(a_{33})})(e_{23}) = a_{23}$. Then (b) implies that $e_{13} = e_{12}\sigma(e_{23})$.

Case 4. If $a_{22} \in J(R)$, $a_{11} \in J(R)$, then $e_{22} = 0$, $e_{11} = 0$. Hence, (a) implies that $e_{12} = 0$.

(**G**) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (f), $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$ and there exists $e_{23} \in R$ such that $(l_{a_{22}} - r_{\sigma(a_{33})})(e_{23}) = a_{23}$. So (e) gives us $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = -a_{12}\sigma(e_{23}) - a_{13}$. Hence there exists $e_{13} \in R$ such that $(l_{a_{11}} - r_{\sigma^2(a_{33})})(e_{13}) = -a_{12}\sigma(e_{23}) - a_{13}$.

(**H**) If $a_{33} \in J(R)$, then $e_{33} = 0$. From (c), we have $e_{23} = 0$ and by (b) we obtain $e_{13} = 0$.

Similar to preceding calculations from $E^2 = E$ we have

(1) $e_{14} = e_{11}e_{14} + e_{12}\sigma(e_{24}) + e_{13}\sigma^2(e_{34}) + e_{14}\sigma^3(e_{44})$
(2) $e_{24} = e_{22}e_{24} + e_{23}\sigma(e_{34}) + e_{24}\sigma^2(e_{44})$
(3) $e_{34} = e_{33}e_{34} + e_{34}\sigma(e_{44})$

and from $AE = EA$ we have

(4) $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}\sigma^3(e_{44}) + e_{11}a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$
(5) $a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = -a_{23}\sigma(e_{34}) - a_{24}\sigma^2(e_{44}) + e_{22}a_{24} + e_{23}\sigma(a_{34})$
(6) $a_{33}e_{34} - e_{34}\sigma(a_{44}) = -a_{34}\sigma(e_{44}) + e_{33}a_{34} + e_{34}\sigma(a_{44})$

To complete the proof we only need to show the existence of e_{14}, e_{24} and e_{34} in R satisfying preceding conditions (1)-(6).

Case 1. If $a_{44} \in J(R)$, $a_{33} \in 1 + J(R)$, then $e_{44} = 0$ and $e_{33} = 1$, otherwise $A - E \notin J(T_4(R, \sigma))$. By (6), $a_{33}e_{34} - e_{34}\sigma(a_{44}) = a_{34}$ and by hypothesis there exists e_{34} such that $(l_{a_{33}} - r_{\sigma(a_{44})})(e_{34}) = a_{34}$. Then by (5), $a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = -a_{23}\sigma(e_{34}) + e_{22}a_{24} + e_{23}\sigma(a_{34})$. There are two possibilities:

(**A**) If $a_{22} \in 1 + J(R)$, then $e_{22} = 1$ otherwise $A - E \notin J(T_4(R, \sigma))$. Then there exists $e_{24} \in R$ such that $(l_{a_{22}} - r_{\sigma^2(a_{44})})(e_{24}) = a_{24} - a_{23}\sigma(e_{34}) + e_{23}\sigma(a_{34})$. From (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) + e_{11}a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. If $a_{11} \in U(R)$, then $e_{11} = 1$, otherwise $A - E \notin J(T_4(R, \sigma))$. Hence, there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = -a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) + a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. If $a_{11} \in J(R)$, then $e_{11} = 0$ and by (1), $e_{14} = e_{12}\sigma(e_{24}) + e_{13}\sigma^2(e_{34})$.

(**B**) If $a_{22} \in J(R)$, then $e_{22} = 0$ otherwise $A - E \notin J(T_4(R, \sigma))$. By (2), $e_{24} = e_{23}\sigma(e_{34})$. From equation (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) + e_{11}a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. If $a_{11} \in U(R)$, then $e_{11} = 1$. By hypothesis, there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = -a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) + a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. If $a_{11} \in J(R)$, then $e_{11} = 0$ and by (1), $e_{14} = e_{12}\sigma(e_{24}) + e_{13}\sigma^2(e_{34})$.

Case 2. If $a_{44} \in 1 + J(R), a_{33} \in 1 + J(R)$, then $e_{44} = e_{33} = 1$. Then by (3), $e_{34} = 0$. Again there are two possibilities:

(**C**) If $a_{22} \in U(R)$, then $e_{22} = 1$ and by (2), $e_{24} = 0$. If $a_{11} \in U(R)$, then $e_{11} = 1$ and by (1), $e_{14} = 0$. If $a_{11} \in J(R)$, then $e_{11} = 0$. Then by equation (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. Hence, there exists $e_{14} \in J(R)$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$

(**D**) If $a_{22} \in J(R)$, then $e_{22} = 0$ and by (5), $a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = -a_{24} + e_{23}\sigma(a_{34})$. So, there exists $e_{24} \in R$ such that $(l_{a_{22}} - r_{\sigma_{(a_{34})}}(e_{24}) = -a_{24} + e_{23}\sigma(a_{34})$. If $a_{11} \in J(R)$, then $e_{11} = 0$. From equation (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma(a_{34})$. By assumption, there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3_{(a_{44})}}) = -a_{12}\sigma(e_{24}) - a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma(a_{34})$. If $a_{11} \in U(R)$, then $e_{11} = 1$. By equation (1), $e_{14} = -e_{12}\sigma(e_{24})$.

Case 3. If $a_{44} \in 1 + J(R), a_{33} \in J(R)$. In this case $e_{33} = 0$ and $e_{44} = 1$. By (6), $a_{33}e_{34} - e_{34}\sigma(a_{44}) = -a_{34}$. Hence, there exists $e_{34} \in R$ such that $(l_{a_{33}} - r_{\sigma(a_{44})})(e_{34}) = -a_{34}$. Using (5), $a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = e_{22}a_{24} + e_{23}\sigma(a_{34}) - a_{23}\sigma(e_{34}) - a_{24}$. Then there are two possibilities:

(**E**) If $a_{22} \in 1 + J(R)$, then $e_{22} = 1$ and from (2), $e_{24} = -e_{23}\sigma(e_{34})$. Then by (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = e_{11}a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}$. If $a_{11} \in J(R)$, then $e_{11} = 0$. So there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}$. If $a_{11} \in U(R)$, then $e_{11} = 1$ and by (1), $e_{14} = -e_{12}\sigma(e_{24}) - e_{13}\sigma^2(e_{34})$.

(**F**) If $a_{22} \in J(R)$, then $e_{22} = 0$ and by hypothesis there exists $e_{24} \in R$ such that $(l_{a_{22}} - r_{\sigma^2(a_{44})})(e_{24}) = -a_{24} + e_{23}\sigma(a_{34}) - a_{23}\sigma(e_{34})$. From equation (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = e_{11}a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}$. If $a_{11} \in J(R)$, then $e_{11} = 0$. From (4) and by hypothesis, there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}$. If $a_{11} \in U(R)$, then $e_{11} = 1$ and by (1), $e_{14} = -e_{12}\sigma(e_{24}) - e_{13}\sigma^2(e_{34})$.

Case 4. If $a_{44} \in J(R), a_{33} \in J(R)$. In this case $e_{33} = e_{44} = 0$. By (3), $e_{34} = 0$.

(**G**) If $a_{22} \in J(R)$, then $e_{22} = 0$. By (2), $e_{24} = 0$. If $a_{11} \in J(R)$, then $e_{11} = 0$ and from (1), $e_{14} = 0$. If $a_{11} \in U(R)$, then $e_{11} = 1$. Hence, equation (4) becomes $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. By hypothesis there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$.

(**H**) If $a_{22} \in 1 + J(R)$, then $e_{22} = 1$ and from (5), $a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = a_{24} + e_{23}\sigma(a_{34})$. By assumption, there exists $e_{24} \in R$ such that $(l_{a_{22}} - r_{\sigma^2(a_{44})})(e_{24}) = a_{24} + e_{23}\sigma(a_{34})$. If $a_{11} \in U(R)$, then $e_{11} = 1$ and by (4), $a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) + a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$.

Hence, there exists $e_{14} \in R$ such that $(l_{a_{11}} - r_{\sigma^3(a_{44})})(e_{14}) = -a_{12}\sigma(e_{24}) + a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})$. If $a_{11} \in J(R)$, then $e_{11} = 0$ and from (1), $e_{14} = e_{12}\sigma(e_{24})$. Thus, we always find e_{14}, e_{24} and e_{34} in R.

($\Rightarrow$) Analogous to Theorem 2.2 we easily obtain the result. $\square$

Acknowledgements I would like to thank the referee for his/her careful reading and valuable comments.

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