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Weighted reverse order law for the weighted Moore-Penrose inverse in rings with involution

Dijana Mosić · Dragan S. Djordjević

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Abstract We investigate some necessary and sufficient conditions for the reverse order law for the weighted Moore–Penrose inverse in rings with involution.

Keywords weighted Moore-Penrose inverse · reverse order law · ring with involution

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1 Introduction

The reverse order laws is one of the most important properties of Moore–Penrose inverse. Necessary and sufficient conditions for the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ to hold were studied by many authors (see [1,2,4]). In this paper we study equivalent conditions for several generalizations of the reverse order law to hold for the weighted Moore–Penrose inverse of elements in rings with involution. As a corollary, the usual reverse order law for the weighted Moore-Penrose inverse follows.

Let $\mathcal{R}$ be an associative ring, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a;$$

$a^\#$ is a group inverse of a and it is unique, if it exists [4]. The group inverse $a^\#$ double commutes with a, that is, $ax = xa$ implies $a^\# x = xa^\#$ (see [3,4]). The set of all group invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\#$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^* = a$.

Dijana Mosić, Dragan S. Djordjević
Faculty of Sciences and Mathematics,
University of Niš,
P.O. Box 224, 18000 Niš, Serbia
E-mail: dijana@pmf.ni.ac.rs
E-mail: dragan@pmf.ni.ac.rs

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $a^\dagger \in \mathcal{R}$, if the following equations hold (see [5]):

$$(1)\ aa^\dagger a = a, \quad (2)\ a^\dagger a a^\dagger = a^\dagger, \quad (3)\ (aa^\dagger)^* = aa^\dagger, \quad (4)\ (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

Definition 1.1 *Let $\mathcal{R}$ be a ring with involution and let e, f be invertible Hermitian elements in $\mathcal{R}$. The element $a \in \mathcal{R}$ has the weighted MP-inverse with weights e, f if there exists $b \in \mathcal{R}$ such that*

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3')\ (eba)^* = eba, \quad (4')\ (fab)^* = fab.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [4]. The set of all weighted MP-invertible elements of $\mathcal{R}$ with weights e, f, will be denoted by $\mathcal{R}^\dagger_{e,f}$.

Define the mapping $x \mapsto x^{*e,f} = e^{-1}x^*f$, for all $x \in \mathcal{R}$. Notice that $(*, e, f) : \mathcal{R} \to \mathcal{R}$ is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$. Now, we formulate the following result which can be proved directly by the definition of the weighted MP-inverse.

Theorem 1.2 *Let $\mathcal{R}$ be a ring with involution and e, f be hermitian invertible elements in $\mathcal{R}$. For any $a \in \mathcal{R}^\dagger_{e,f}$, the following is satisfied:*

(a) $(a^\dagger_{e,f})^\dagger_{f,e} = a$;
(b) $(a^{*e,f})^\dagger_{f,e} = (a^\dagger_{e,f})^{*f,e}$;
(c) $(a^{*e,f}a)^\dagger_{e,e} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*f,e}$;
(d) $(aa^{*e,f})^\dagger_{f,f} = (a^\dagger_{e,f})^{*f,e}a^\dagger_{e,f}$;
(e) $a^{*e,f}(a^\dagger_{e,f})^{*f,e} = a^\dagger_{e,f}a$;
(f) $(a^\dagger_{e,f})^{*f,e}a^{*e,f} = aa^\dagger_{e,f}$;
(g) $a^{*e,f} = a^\dagger_{e,f}aa^{*e,f} = a^{*e,f}aa^\dagger_{e,f} = a^{*e,f}(a^\dagger_{e,f})^{*f,e}a^{*e,f}$;
(h) $a^\dagger_{e,f} = (a^{*e,f}a)^\dagger_{e,e}a^{*e,f} = a^{*e,f}(aa^{*e,f})^\dagger_{f,f}$;
(i) $(a^{*e,f})^\dagger_{f,e} = a(a^{*e,f}a)^\dagger_{e,e} = (aa^{*e,f})^\dagger_{f,f}a$.

In the following theorems, we extend to weighted Moore-Penrose inverse well known results for the Moore-Penrose inverse of elements in ring with involution.

Theorem 1.3 *Let e, f be Hermitian invertible elements in ring with involution $\mathcal{R}$. If $a \in \mathcal{R}^\dagger_{e,f}$, then $a^{*e,f}a$ and $aa^{*e,f}$ are group invertible and*

$$a^\dagger_{e,f} = (a^{*e,f}a)^\# a^{*e,f} = a^{*e,f}(aa^{*e,f})^\#.$$

Proof. Since $a \in \mathcal{R}^{\dagger}_{e,f}$, by Theorem 1.2, we have $a^{*e,f}a \in \mathcal{R}^{\dagger}_{e,e}$ and $(a^{*e,f}a)^{\dagger}_{e,e} = a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*f,e}$. Using again Theorem 1.2, we get

$$a^{*e,f}a(a^{*e,f}a)^{\dagger}_{e,e} = a^{*e,f}aa^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*f,e} = a^{*e,f}(a^{\dagger}_{e,f})^{*f,e} = a^{\dagger}_{e,f}a$$

and

$$(a^{*e,f}a)^{\dagger}_{e,e}a^{*e,f}a = a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*f,e}a^{*e,f}a = a^{\dagger}_{e,f}aa^{\dagger}_{e,f}a = a^{\dagger}_{e,f}a$$

implying $a^{*e,f}a(a^{*e,f}a)^{\dagger}_{e,e} = (a^{*e,f}a)^{\dagger}_{e,e}a^{*e,f}a$. Thus, $a^{*e,f}a$ is group invertible and $(a^{*e,f}a)^{\#} = (a^{*e,f}a)^{\dagger}_{e,e}$. The equality $a^{\dagger}_{e,f} = (a^{*e,f}a)^{\dagger}_{e,e}a^{*e,f}$ gives $a^{\dagger}_{e,f} = (a^{*e,f}a)^{\#}a^{*e,f}$. The second equality follows in the same way. □

Theorem 1.4 *Let e, f be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^{\dagger}_{e,f}$ and $b \in \mathcal{R}$. Then $ab = ba$ and $a^{*e,f}b = ba^{*e,f}$ if and only if $a^{\dagger}_{e,f}b = ba^{\dagger}_{e,f}$ and $(a^{\dagger}_{e,f})^{*f,e}b = b(a^{\dagger}_{e,f})^{*f,e}$.*

In addition, if $ab = ba$ and $a^{*e,f}b = ba^{*e,f}$, then

$$(a^{\dagger}_{e,f})^{*f,e}b^{*e,e} = b^{*f,f}(a^{\dagger}_{e,f})^{*f,e}$$

and $a^{*e,f}b^{*f,f} = b^{*e,e}a^{*e,f}$.

Proof. If b commutes with a and $a^{*e,f}$, then b commutes with $aa^{*e,f}$. Since the group inverse $(aa^{*e,f})^{\#}$ double commutes with $aa^{*e,f}$, we deduce that b commutes with $(aa^{*e,f})^{\#}$. From the assumption $a \in \mathcal{R}^{\dagger}_{e,f}$ and Theorem 1.3, we have $a^{\dagger}_{e,f} = a^{*e,f}(aa^{*e,f})^{\#}$ which gives that b commutes with $a^{\dagger}_{e,f}$.

Applying Theorem 1.2 and Theorem 1.3, we obtain

$$(a^{\dagger}_{e,f})^{*f,e} = (a^{*e,f})^{\dagger}_{f,e} = a(a^{*e,f}a)^{\dagger}_{e,e} = a(a^{*e,f}a)^{\#}.$$

So, b commutes with $(a^{\dagger}_{e,f})^{*f,e}$ too.

Conversely, let b commutes with $a^{\dagger}_{e,f}$ and $(a^{\dagger}_{e,f})^{*f,e}$. Now, by the first part of this proof, b commutes with $(a^{\dagger}_{e,f})^{\dagger}_{f,e} = a$ and $[(a^{\dagger}_{e,f})^{\dagger}_{f,e}]^{*e,f} = a^{*e,f}$.

Suppose that b commutes with a and $a^{*e,f}$. Then b commutes with $a^{\dagger}_{e,f}$ and

$$(a^{\dagger}_{e,f})^{*f,e}b^{*e,e} = f^{-1}(ba^{\dagger}_{e,f})^{*}e = f^{-1}(a^{\dagger}_{e,f}b)^{*}e = b^{*f,f}(a^{\dagger}_{e,f})^{*f,e}.$$

The equality $a^{*e,f}b^{*f,f} = b^{*e,e}a^{*e,f}$ follows in the same way. □

Now, we state a characterization of the weighted Moore–Penrose inverse in ring with involution which be used later.

Theorem 1.5 *Let e, f be Hermitian invertible elements in ring with involution $\mathcal{R}$ and let $a, b \in \mathcal{R}$. Then the following statements are equivalent:*

(a) *b is the weighted MP-inverse with weights e, f;*

(b) $a = ae^{-1}a^*b^*e$ *and* $b^* = fabf^{-1}b^*$ *(or equivalently* $a = aa^{*e,f}b^{*f,e}$ *and* $b^{*f,e} = abb^{*f,e}$*);*

(c) $a^* = ebae^{-1}a^*$ *and* $b = bf^{-1}b^*a^*f$ *(or equivalently* $a^{*e,f} = baa^{*e,f}$ *and* $b = bb^{*f,e}a^{*e,f}$*);*

(d) $a = f^{-1}b^*a^*fa$ *and* $b^* = b^*ebae^{-1}$ *(*$a = b^{*f,e}a^{*e,f}a$ *and* $b^{*f,e} = b^{*f,e}ba$*);*

(e) $a^* = a^*fabf^{-1}$ *and* $b = e^{-1}a^*b^*eb$ *(*$a^{*e,f} = a^{*e,f}ab$ *and* $b = a^{*e,f}b^{*f,e}b$*).*

Proof. (a) ⇒ (b): Since $b = a^{\dagger}_{e,f}$, then

$$a = aba = ae^{-1}eba = ae^{-1}(eba)^* = ae^{-1}a^*b^*e$$

and

$$b^* = (bab)^* = (bf^{-1}fab)^* = fabf^{-1}b^*.$$

(b) ⇒ (a): From the hypothesis $a = ae^{-1}a^*b^*e$ and $b^* = fabf^{-1}b^*$, we get

$$eba = ebae^{-1}a^*b^*e = ebae^{-1}(eba)^*$$

and

$$(fab)^* = b^*a^*f = fabf^{-1}b^*a^*f = fabf^{-1}(fab)^*$$

which yields that eba and fab are self-adjoint. Now, the equalities

$$a = ae^{-1}a^*b^*e = ae^{-1}(eba)^* = ae^{-1}eba = aba$$

and

$$b = (b^*)^* = (fabf^{-1}b^*)^* = bf^{-1}fab = bab$$

imply $b = a^{\dagger}_{e,f}$.

The equivalence (a) ⇔ (d) can be proved in the similar way as (a) ⇔ (b). Applying involution, we verify that (b) ⇔ (c), (d) ⇔ (e). □

Necessary and sufficient conditions for the reverse order rule for the weighted MP-inverse for matrices were recently given by Sun and Wei in [6] in terms of the inclusion of matrix ranges (column spaces). However, to the best of our knowledge, there are so far no articles in the literature discussing the reverse order laws for the weighted Moore-Penrose inverse in ring with involution.

Boasso [1] was proved the equivalent conditions for the reverse order law for the Moore–Penrose inverse in C^*–algebra to hold. In [2] several generalizations of the reverse order law for the Moore–Penrose inverse in rings with involution are investigated, recovering the results of [1]. We extend these results to the reverse order law for the weighted Moore–Penrose inverse in rings with involution. Notice that no additional assumption such as the $*$-cancellation property for elements in ring is needed.

2 Weighted reverse order laws

In the this section the weighted reverse order laws for the weighted Moore-Penrose inverse in rings with involution will be studied. If $c = 1$ in the following theorems, then the equivalent conditions for the usual reverse order law for the weighted Moore-Penrose inverse are obtained.

Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^{\dagger}_{f,h}$ and $b \in \mathcal{R}^{\dagger}_{e,f}$. Define $s = a^{\dagger}_{f,h}a$, $p = bb^{\dagger}_{e,f}$, $q = a^{\dagger}_{f,h}(a^{\dagger}_{f,h})^{*h,f}$ and $r = bb^{*e,f}$.

Theorem 2.1 *Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^{\dagger}_{f,h}$, $b \in \mathcal{R}^{\dagger}_{e,f}$ and $ab \in \mathcal{R}^{\dagger}_{e,h}$. If $c \in \mathcal{R}$ such that c commutes with b and $b^{*e,f}$, then the following statements are equivalent:*

(1) $(ab)^{\dagger}_{e,h} = cb^{\dagger}_{e,f}a^{\dagger}_{f,h}$,
(2) $a(rsc^{*f,f} - sr)(b^{\dagger}_{e,f})^{*f,e} = 0$ *and* $a(cpq - qp)(b^{\dagger}_{e,f})^{*f,e}c^{*e,e} = 0$,
(3) $srspc^{*f,f} = sr$ *and* $scpqpc^{*f,f} = qpc^{*f,f}$.

Proof. (1) $\Rightarrow$ (2): Assume that $(ab)^{\dagger}_{e,h} = cb^{\dagger}_{e,f}a^{\dagger}_{f,h}$. By Theorem 1.2, we have $ab = ab(ab)^{*e,h}[(ab)^{\dagger}_{e,h}]^{*h,e}$ and $[(ab)^{\dagger}_{e,h}]^{*h,e} = ab(ab)^{\dagger}_{e,h}[(ab)^{\dagger}_{e,h}]^{*h,e}$. Since c commutes with b and $b^{*e,f}$, by Theorem 1.4, then

$$\begin{aligned} asr(b^{\dagger}_{e,f})^{*f,e} &= ab = abe^{-1}b^*a^*(a^{\dagger}_{f,h})^*(b^{\dagger}_{e,f})^*c^*e \\ &= abb^{*e,f}a^{*f,h}(a^{\dagger}_{f,h})^{*h,f}[(b^{\dagger}_{e,f})^{*f,e}c^{*e,e}] \\ &= abb^{*e,f}a^{\dagger}_{f,h}ac^{*f,f}(b^{\dagger}_{e,f})^{*f,e} = arsc^{*f,f}(b^{\dagger}_{e,f})^{*f,e} \end{aligned}$$

and

$$\begin{aligned} aqp(b^{\dagger}_{e,f})^{*f,e}c^{*e,e} &= (a^{\dagger}_{f,h})^{*h,f}(b^{\dagger}_{e,f})^{*f,e}c^{*e,e} = (cb^{\dagger}_{e,f}a^{\dagger}_{f,h})^{*h,e} = [(ab)^{\dagger}_{e,h}]^{*h,e} \\ &= a(bc)b^{\dagger}_{e,f}a^{\dagger}_{f,h}(a^{\dagger}_{f,h})^{*h,f}(b^{\dagger}_{e,f})^{*f,e}c^{*e,e} \\ &= acpq(b^{\dagger}_{e,f})^{*f,e}c^{*e,e}. \end{aligned}$$

Thus, the condition (2) is satisfied.

(2) $\Rightarrow$ (3): The hypothesis (2) and Theorem 1.4 imply

$$\begin{aligned} srspc^{*f,f} &= a^{\dagger}_{f,h}arsbb^{\dagger}_{e,f}c^{*f,f} = a^{\dagger}_{f,h}ars(b^{\dagger}_{e,f})^{*f,e}(b^{*e,f}c^{*f,f}) \\ &= a^{\dagger}_{f,h}ars[(b^{\dagger}_{e,f})^{*f,e}c^{*e,e}]b^{*e,f} = a^{\dagger}_{f,h}[arsc^{*f,f}(b^{\dagger}_{e,f})^{*f,e}]b^{*e,f} \\ &= a^{\dagger}_{f,h}asr(b^{\dagger}_{e,f})^{*f,e}b^{*e,f} = sr \end{aligned}$$

and

$$\begin{aligned} scpqpc^{*f,f} &= a^{\dagger}_{f,h}acpq(b^{\dagger}_{e,f})^{*f,e}(b^{*e,f}c^{*f,f}) = a^{\dagger}_{f,h}[acpq(b^{\dagger}_{e,f})^{*f,e}c^{*e,e}]b^{*e,f} \\ &= a^{\dagger}_{f,h}aqp(b^{\dagger}_{e,f})^{*f,e}(c^{*e,e}b^{*e,f}) = qp(b^{\dagger}_{e,f})^{*f,e}b^{*e,f}c^{*f,f} = qpc^{*f,f}. \end{aligned}$$

Hence, the statement (3) holds.

(2) ⇒ (1): Because c commutes with b, the condition (3) can be written as

$$a^\dagger_{f,h}abb^{*e,f}a^\dagger_{f,h}abb^\dagger_{e,f}c^{*f,f} = a^\dagger_{f,h}abb^{*e,f}$$

and

$$a^\dagger_{f,h}abcb^\dagger_{e,f}a^\dagger_{f,h}(a^\dagger_{f,h})^{*h,f}bb^\dagger_{e,f}c^{*f,f} = a^\dagger_{f,h}(a^\dagger_{f,h})^{*h,f}bb^\dagger_{e,f}c^{*f,f}.$$

Multiplying each of these equality from the left side by a and from the right side by $(b^\dagger_{e,f})^{*f,e}$ and using Theorem 1.4, we obtain

$$abb^{*e,f}a^{*f,h}(a^\dagger_{f,h})^{*h,f}(b^\dagger_{e,f})^{*f,e}c^{*e,e} = ab$$

and

$$abcb^\dagger_{e,f}a^\dagger_{f,h}(a^\dagger_{f,h})^{*h,f}(b^\dagger_{e,f})^{*f,e}c^{*e,e} = (a^\dagger_{f,h})^{*h,f}(b^\dagger_{e,f})^{*f,e}c^{*e,e},$$

that is,

$$ab(ab)^{*e,h}(cb^\dagger_{e,f}a^\dagger_{f,h})^{*h,e} = ab$$

and

$$ab(cb^\dagger_{e,f}a^\dagger_{f,h})(cb^\dagger_{e,f}a^\dagger_{f,h})^{*h,e} = (cb^\dagger_{e,f}a^\dagger_{f,h})^{*h,e}.$$

So, by Theorem 1.5, we deduce that $(ab)^\dagger_{e,h} = cb^\dagger_{e,f}a^\dagger_{f,h}$. □

Similarly as Theorem 2.1, we get characterizations for reverse order law $(ab)^\dagger_{e,h} = b^\dagger_{e,f}a^\dagger_{f,h}c$ in the following result.

Theorem 2.2 *Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^\dagger_{f,h}$, $b \in \mathcal{R}^\dagger_{e,f}$ and $ab \in \mathcal{R}^\dagger_{e,h}$. If $c \in \mathcal{R}$ such that c commutes with a and $a^{*f,h}$, then the following statements are equivalent:*

(1) $(ab)^\dagger_{e,h} = b^\dagger_{e,f}a^\dagger_{f,h}c$,

(2) $b^{*e,f}(q^\dagger_{f,f}pc - pq^\dagger_{f,f})a^\dagger_{f,h} = 0$ *and* $b^{*e,f}(c^{*f,f}sr^\dagger_{f,f} - r^\dagger_{f,f}s)a^\dagger_{f,h}c = 0$,

(3) $pq^\dagger_{f,f}psc = pq^\dagger_{f,f}$ *and* $pc^{*f,f}sr^\dagger_{f,f}sc = r^\dagger_{f,f}sc$.

Proof. By Theorem 1.2, the equality $(ab)^\dagger_{e,h} = b^\dagger_{e,f}a^\dagger_{f,h}c$ is equivalent to $[(ab)^{*e,h}]^\dagger_{h,e} = [(ab)^\dagger_{e,h}]^{*h,e} = (b^\dagger_{e,f}a^\dagger_{f,h}c)^{*h,e}$ which can be written as

$$(b^{*e,f}a^{*f,h})^\dagger_{h,e} = c^{*h,h}(a^{*f,h})^\dagger_{h,f}(b^{*e,f})^\dagger_{f,e}.$$

Now, we can show that the previous equality is equivalent to the statements (2) and (3) of this theorem in the same way as in the proof of Theorem 2.1. Notice that $q^\dagger_{f,f} = a^{*f,h}a$ and $r^\dagger_{f,f} = (b^\dagger_{e,f})^{*f,e}b^\dagger_{e,f}$. □

Using Theorem 2.1, the following generalization of the reverse order law for the weighted Moore-Penrose inverse can be characterized.

Theorem 2.3 *Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^\dagger_{f,h}$, $b \in \mathcal{R}^\dagger_{e,f}$ and $c \in \mathcal{R}$ such that $cab \in \mathcal{R}^\dagger_{e,h}$. If c commutes with a and $a^{*f,h}$, then the following statements are equivalent:*

(1) $(cab)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$,
(2) $b^{\dagger}_{e,f}(qpc^{*f,f} - pq)a^{*f,h} = 0$ *and* $b^{\dagger}_{e,f}(csr - rs)a^{*f,h}c^{*h,h} = 0$,
(3) $pqpsc^{*f,f} = pq$ *and* $pcsrsc^{*f,f} = rsc^{*f,f}$.

Proof. Observe that, $cab \in \mathcal{R}^{\dagger}_{e,h}$ and $(cab)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$ if and only if $b^{\dagger}_{e,f}a^{\dagger}_{f,h} \in \mathcal{R}^{\dagger}_{h,e}$ and $(b^{\dagger}_{e,f}a^{\dagger}_{f,h})^{\dagger}_{h,e} = cab$, by Theorem 1.2.

If we denote by p_1, q_1, r_1, s_1 the elements of $\mathcal{R}$ corresponding to p, q, r, s defined using $b^{\dagger}_{e,f} \in \mathcal{R}^{\dagger}_{f,e}$ and $a^{\dagger}_{f,h} \in \mathcal{R}^{\dagger}_{h,f}$ instead of a and b, it follows $p_1 = s$, $q_1 = r$, $r_1 = q$, $s_1 = p$. Applying Theorem 2.1 to $b^{\dagger}_{e,f}$, $a^{\dagger}_{f,h}$, $b^{\dagger}_{e,f}a^{\dagger}_{f,h}$ and c, we prove this theorem. □

In the following theorem, the equivalent conditions for the reverse order law $(abc)^{\dagger}_{e,h}= b^{\dagger}_{e,f}a^{\dagger}_{f,h}$ are given.

Theorem 2.4 *Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^{\dagger}_{f,h}$, $b \in \mathcal{R}^{\dagger}_{e,f}$ and $c \in \mathcal{R}$ such that $abc \in \mathcal{R}^{\dagger}_{e,h}$. If c commutes with b and $b^{*e,f}$, then the following statements are equivalent:*

(1) $(abc)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$,
(2) $(a^{\dagger}_{f,h})^{*h,f}(r^{\dagger}_{f,f}sc - sr^{\dagger}_{f,f})b = 0$ *and* $(a^{\dagger}_{f,h})^{*h,f}(c^{*f,f}pq^{\dagger}_{f,f} - q^{\dagger}_{f,f}p)bc = 0$,
(3) $sr^{\dagger}_{f,f}spc = sr^{\dagger}_{f,f}$ *and* $sc^{*f,f}pq^{\dagger}_{f,f}pc = q^{\dagger}_{f,f}pc$.

Proof. Notice that, $(abc)^{\dagger}_{e,h}=b^{\dagger}_{e,f}a^{\dagger}_{f,h}$ if and only if $(b^{\dagger}_{e,f}a^{\dagger}_{f,h})^{\dagger}_{h,e}=abc$ which is equivalent to $[(b^{\dagger}_{e,f}a^{\dagger}_{f,h})^{*h,e}]^{\dagger}_{e,h} = (abc)^{*e,h}$, that is, $[(a^{\dagger}_{f,h})^{*h,f}(b^{\dagger}_{e,f})^{*f,e}]^{\dagger}_{e,h} = c^{*e,e}b^{*e,f}a^{*f,h}$. Similarly as in the proof of Theorem 2.1, we can verify that this equality is satisfied if and only if the conditions (2) and (3) of this theorem hold. □

In the previous theorems, we generalized the results [2] for the weighted reverse order law for the Moore-Penrose inverse to the weighted reverse order law for the weighted Moore-Penrose inverse.

For $c = 1$, as a corollary of the previous results, we get necessary and sufficient conditions related to the reverse order law for the weighted Moore-Penrose inverse in ring with involution. In the following result, we extend the results [1] for the reverse order law for the Moore-Penrose inverse in C^*-algebra to the reverse order law for the weighted Moore-Penrsoe inverse in ring with involution.

Theorem 2.5 *Let e, f, h be Hermitian invertible elements in ring with involution $\mathcal{R}$, $a \in \mathcal{R}^{\dagger}_{f,h}$, $b \in \mathcal{R}^{\dagger}_{e,f}$ and $ab \in \mathcal{R}^{\dagger}_{e,h}$. Then the following statements are equivalent:*

(1) $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$,
(2) $a(rs - sr)(b^{\dagger}_{e,f})^{*f,e} = 0$ *and* $a(pq - qp)(b^{\dagger}_{e,f})^{*f,e} = 0$,
(3) $srsp = sr$ *and* $spqp = qp$;
(4) $b^{*e,f}(q^{\dagger}_{f,f}p - pq^{\dagger}_{f,f})a^{\dagger}_{f,h} = 0$ *and* $b^{*e,f}(sr^{\dagger}_{f,f} - r^{\dagger}_{f,f}s)a^{\dagger}_{f,h} = 0$,
(5) $pq^{\dagger}_{f,f}ps = pq^{\dagger}_{f,f}$ *and* $psr^{\dagger}_{f,f}s = r^{\dagger}_{f,f}s$.

(6) $b^{\dagger}_{e,f}(qp - pq)a^{*f,h} = 0$ *and* $b^{\dagger}_{e,f}(sr - rs)a^{*f,h} = 0$,
(7) $pqps = pq$ *and* $psrs = rs$.
(8) $(a^{\dagger}_{f,h})^{*h,f}(r^{\dagger}_{f,f}s - sr^{\dagger}_{f,f})b = 0$ *and* $(a^{\dagger}_{f,h})^{*h,f}(pq^{\dagger}_{f,f} - q^{\dagger}_{f,f}p)b = 0$,
(9) $sr^{\dagger}_{f,f}sp = sr^{\dagger}_{f,f}$ *and* $spq^{\dagger}_{f,f}p = q^{\dagger}_{f,f}p$.

As a consequence of Theorem 2.1, we obtain the following result in a complex C^*-algebra $\mathcal{A}$. An element $a \in \mathcal{R}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. Recall that if a is regular in C^*-algebra $\mathcal{A}$ then the unique weighted Moore-Penrose inverse $a^{\dagger}_{e,f}$ exists [4].

Corollary 2.6 *Let e, f, h be positive invertible elements in a complex C^*-algebra $\mathcal{A}$, $a, b, ab \in \mathcal{A}$ are regular and $\lambda \in C$. Then the following statements are equivalent:*

(1) $(ab)^{\dagger}_{e,h} = \lambda b^{\dagger}_{e,f}a^{\dagger}_{f,h}$,
(2) $a(\overline{\lambda}rs - sr)(b^{\dagger}_{e,f})^{*f,e} = 0$ *and* $a(\lambda pq - qp)(b^{\dagger}_{e,f})^{*f,e} = 0$,
(3) $\overline{\lambda}srsp = sr$ *and* $\lambda spqp = qp$.

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