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Existence and multiplicity results for a fourth-order boundary value problem

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Abstract In this paper we consider a class of a fourth-order boundary value problem. Using a variational method based on nonsmooth critical point theory, we prove the existence and multiplicity of solutions.

Keywords fourth-order equations · nonsmooth critical point · variational methods · locally Lipschitz
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1 Introduction

The aim of this paper is to establish the existence and multiplicity of solutions for the following fourth-order boundary value problem

$$\begin{cases} u^{iv} + \alpha u'' + \beta(x)u \in \partial F(x, u(x)), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases} \tag{1.1}$$

where α is a real constant, $\beta(x)$ is a continuous function on $[0,1]$, $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for all $t \in [0,1]$, $F(t,\cdot)$ is locally Lipschitz and $\partial F(t,\cdot)$ denotes the generalized subdifferential in the sense of Clarke [8].

The fourth-order boundary value problem have attracted much attention owing to its interest to physics, see [1–3, 11, 13, 16] and the references therein. For example in paper [4], the authors studied the following problem

$$\begin{cases} u^{iv} + \alpha u'' + \beta u = \lambda f(x, u), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases} \tag{1.2}$$

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where α, β are real constants and λ is a positive parameter. There is a wide literature that deals with multiplicity results for problem (1.2) (see, [4–6, 10, 14] and the references therein). In paper [12], the authors also considered the problem (1.2) with replaced β, $\lambda f(x,u)$ by $\beta(x)$ is a continuous function on $[0,1]$ and $\lambda f(x,u) + h(x)$, respectively.

Motivated by the above work, in this paper, we would like to investigate the existence multiplicity of results concerning (1.1). The technical tool in critical point theory for non-differentiable functionals.

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our results.

2 Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results.

Here and in the sequel, let us denote

$$\beta_1 := \min_{[0,1]} \beta(x), \qquad \beta_2 := \max_{[0,1]} \beta(x),$$

and put

$\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1$ when $\beta_2 \leq 0$ and $\alpha \geq 0$;
$\sigma_1 := 1 + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1 - \frac{\alpha}{\pi^2}$ when $\beta_2 \leq 0$ and $\alpha < 0$;
$\sigma_1 := 1 - \frac{\alpha}{\pi^2}$, $\sigma_2 := 1 + \frac{\beta_1}{\pi^4}$ when $\beta_1 \geq 0$ and $\alpha \geq 0$;
$\sigma_1 := 1$, $\sigma_2 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$ when $\beta_1 \geq 0$ and $\alpha < 0$;
$\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1\frac{\beta_2}{\pi^4}$ when $\beta_1 < 0 < \beta_2$ and $\alpha \geq 0$;
$\sigma_1 := 1 + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_2}{\pi^4}$ when $\beta_1 < 0 < \beta_2$ and $\alpha < 0$.

Moreover, in each of these cases, if $\sigma_1 > 0$, put $\theta_i := \sqrt{\sigma_i}(i = 1, 2)$

Let $X = W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ the Sobolev space endowed with the norm

$$\|u\| = \left(\int_0^1 (|u''(x)|^2 - \alpha|u'(x)|^2 + \beta(x)|u(x)|^2) dx \right)^{\frac{1}{2}}$$

that is equivalent to the usual one (see [12]).

The following Lemma is useful for proving our main result.

Lemma 2.1 (Proposition 2.2 of [12]) *Let $u \in X$. Then $\|u\|_\infty \leq \frac{1}{2\pi\theta_1}\|u\|$.*

The functional $J : X \to \mathbb{R}$ corresponding to problem (1.1) is defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F(x, u(x)) dx. \tag{2.1}$$

Now, we will establish the variational principle for problem (1.1). For this purpose our hypotheses on the nonsmooth potential $F(x,u)$ are the following:

(H1) For all $u \in \mathbb{R}$, the function $x \to F(x,u)$ is measurable;
(H2) For all $x \in [0,1]$, the function $u \to F(x,u)$ is locally Lipschitz and $F(x,0) = 0$;
(H3) There exist $a, b \in L^1([0,1]; \mathbb{R})$ and $1 \leq q < +\infty$ such that $|u^*| \leq a(x) + b(x)|u|^{q-1}$ for all $x \in [0,1]$, $x \in \mathbb{R}$ and $u^* \in \partial F(x,u)$.

Proposition 2.2 *Assume that $F(x, u)$ satisfies the hypotheses $(H1)$-$(H3)$, the functional $J : X \to \mathbb{R}$ is well defined and locally Lipschitz on X. Moreover, every critical point $u \in X$ of J is a solution of problem* (1.1).

Proof. By standard arguments [8], and the hypotheses (H1)-(H3), we can prove that the functional J is locally Lipschitz on X. We assume that $u \in X$ is a critical point of J, which $0 \in \partial J(u)$. Since $\partial J(u) \subset \partial\varphi(u) + \partial\psi(u)$ and $\partial\psi(u) \subset -\int_0^1 F(x, u(x))dx$, $\partial\varphi(u) = \{u^{iv} + \alpha u'' + \beta(x)u\}$, so, by standard argument (see [9]), one can get $u^{iv} + \alpha u'' + \beta(x)u \in \partial F(x, u(x))$, a.e. $x \in [0, 1]$. Moreover, $u \in X$ implies that $u(0) = u(1) = u''(0) = u''(1) = 0$ and therefore the proof is completed. □

According to Proposition 2.2, we know that in order to find solutions of problem (1.1), it suffices to obtain the critical points of the functional J.

3 Main result

In this section we present our main results. We collect some basic notions and results of nonsmooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [8] and the monograph of Motreanu and Panagiotopoulos [15].

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \to \mathbb{R}$ be a functional. We recall that φ is locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L_U > 0$ such that $|\varphi(x) - \varphi(y)| \leq L_U\|x - y\|_X$, $\forall\, x, y \in U$. If f is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is $\varphi^\circ(u; h) = \limsup_{w \to u, t \downarrow 0^+} \frac{\varphi(w+th)-\varphi(w)}{t}$. The generalized gradient of φ at u is the set $\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v),$ for all $v \in X\}$. So $\varphi : X \to 2^{X^*}$ is a multifunction. The function $(u, v) \mapsto \varphi^\circ(u; v)$ is upper semicontinuous and $\varphi^\circ(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial\varphi(u)\}$, for all $v \in X$. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

We say that $u \in X$ is a critical point of locally Lipschitz functional φ if $0 \in \partial\varphi(u)$.

In the proof of our main results, we shall use nonsmooth critical point theory. For this, we first present an important definition.

Definition 3.1 An operator $A : X \to X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in X, $u_n \rightharpoonup u$ and $\limsup_{n \to +\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$.

Definition 3.2 A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the nonsmooth Palais-Smale condition (nonsmooth PS-condition for short) if any sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{J(u_n)\}_{n\geq 1}$ is bounded and

$$\rho(u_n) := \min\{\|u^*\|_{X^*} : u^* \in \partial\varphi(u_n)\} \to 0 \quad \text{as} \quad n \to +\infty,$$

has a strongly convergent subsequence.

Definition 3.3 A locally Lipschitz function $J : X \to \mathbb{R}$ satisfies the nonsmooth Cerami condition (nonsmooth (C) condition for short) if any sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{J(u_n)\}_{n\geq 1}$ is bounded and

$$(1 + \|u_n\|_X)\rho(u_n) \to 0 \quad \text{as} \quad n \to +\infty,$$

has a strongly convergent subsequence.

Lemma 3.4 ([15], Proposition 1.1) *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and $\varphi^\circ(u;v) = \langle \varphi'(u), v\rangle$, $\forall\, u, v \in X$, $\partial\varphi(u) = \{\varphi'(u)\}$, $\forall\, u \in X$.*

Now, we will apply some critical point theorem to obtain some existence and multiple results for problem (1.1). Our first result is as follows.

Theorem 3.5 *Assume that $F(x,u)$ satisfies the hypotheses $(H1)$-$(H3)$, and suppose the following conditions hold:*

$(H4)$ There exist $\mu \in (0, \frac{1}{2})$, $c_0 > 0$ and $M > 0$ such that $c_0 < F(x,u) \leq -\mu F^\circ(x,u;-u)$ for all $u \in \mathbb{R}$ with $|u| \geq M$ and $x \in [0,1]$;

$(H5)$ $\lim_{|u|\to 0} \frac{u^*}{|u|} = 0$ uniformly for all $x \in [0,1]$ and all $u^* \in \partial F(x,u)$.

Then, the problem (1.1) has at least one nonzero solution on X.

Proof. First, we claim that J satisfies the nonsmooth (PS) condition.

By (H2), (H3) and the Lebourg's mean value theorem, we have $|F(x,u)| = |F(x,u) - F(x,0)| = |(u^*, u)| \leq a(x)|u| + b(x)|u|^q \leq Ma(x) + M^q b(x)$, for all $|u| \leq M$ and $x \in [0,1]$. By the property of generalized directional derivative of locally Lipschitz function, one can get $|F^\circ(x,u;-u)| = |\max\{(u^*, -u) : u^* \in \partial F(x,u)\}| \leq a(x)|u| + b(x)|u|^q \leq Ma(x) + M^q b(x)$, for all $|u| \leq M$ and $x \in [0,1]$. Thus, we have

$$F(x,u) + \mu F^\circ(x,u;-u) \leq a_1(x),\ \forall\, u \in \mathbb{R} \text{ with } |u| \leq M \text{ and } x \in [0,1], \tag{3.1}$$

where $a_1(x) \in L^1([0,1],\mathbb{R})$.

Suppose $\{u_n\} \subset X$ satisfies

$$|J(u_n)| \leq C \quad \text{and} \quad \rho(u_n) \to 0. \tag{3.2}$$

Since $\partial J(u_n) \subset X^*$ is a weak* compact set and the norm function in a Banach space is weakly semi-continuous, by Weierstrass theorem, we can find $u_n^* \in \partial J(u_n)$ such that

$$\rho(u_n) = \|u_n^*\|_{X^*} \quad \text{and} \quad u_n^* = A(u_n) - v_n, \quad \text{for every } n \geq 1 \tag{3.3}$$

with $v_n \in L^{q'}([0,1],\mathbb{R})$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $v_n \in \partial F(x, u_n(x))$ for all $x \in [0,1]$. Here $A : X \to X^*$ is an operator defined by

$$\langle Au_n, v\rangle = \int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta(x)u(x)v(x)]dx, \quad \forall\, v \in X.$$

Thus, by (H4), (3.1) and (3.2), one can get

$$\begin{aligned} C + \mu\|u_n\| &\geq J(u_n) - \mu\langle u_n^*, u_n\rangle = \Big(\frac{1}{2} - \mu\Big)\|u\|^2 \\ &- \int_0^1 F(x,u_n(x))dx - \mu\langle v_n, -u_n\rangle \geq \Big(\frac{1}{2} - \mu\Big)\|u\|^2 \\ &- \int_{\{|u_n| \leq M\}} (F(x,u_n(x)) + \mu F^\circ(x,u_n(x);-u_n(x)))dx \\ &- \int_{\{|u_n| \geq M\}} (F(x,u_n(x)) + \mu F^\circ(x,u_n(x);-u_n(x)))dx \\ &\geq \Big(\frac{1}{2} - \mu\Big)\|u\|^2 - C_1, \end{aligned}$$

where C_1 is a constant. Therefore, the sequence $\{u_n\}$ in X is bounded and so by passing to a subsequence if necessary, by the Sobolev embedding theorem, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } W^{2,2}([0,1]), \\ u_n \to u, & \text{a.e. in } C^1([0,1]), \\ u_n \to u, & \text{a.e. in } L^2([0,1]). \end{cases} \tag{3.4}$$

For all $u, v \in X$, we have $\langle Au - Av, u - v\rangle = \int_0^1 (|u''(x) - v''(x)|^2 - \alpha|u'(x) - v'(x)|^2 + \beta(x)|u(x) - v(x)|^2)dx = \|u - v\|^2 \geq c\|u - v\|^2$ for constant $0 < c < 1$. Then, the linear operator $A : X \to X^*$ is strongly monotone. Clearly, the strongly monotonicity property implies that A satisfies $(S)_+$.

Consequently, it suffices to prove the following fact

$$\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0. \tag{3.5}$$

Indeed, from (3.2) and (3.3), we have

$$\epsilon_n\|u_n - u\| \geq \langle u_n^*, u_n - u\rangle = \langle Au_n, u_n - u\rangle - \int_0^1 v_n(x)(u_n(x) - u(x))dx$$

with $\epsilon_n \downarrow 0$. By (3.4) and Hölder inequality, we can get $\int_0^1 v_n(x)(u_n(x) - u(x))dx \to 0$ as $n \to +\infty$. So, $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$. Thus (3.5) holds. Since A is of type $(S)_+$, therefore we obtain $u_n \to u$ in X.

From (H2), (H3) and (H5), using the Lebourg's mean value theorem, for all $x \in [0, 1]$, $u \in \mathbb{R}$, we obtain that

$$|F(x, u)| \leq \epsilon|u|^2 + a_2(x)|u|^\xi, \tag{3.6}$$

where $\xi > 2$, $\epsilon > 0$ is an arbitrary real number and $a_2 \in L^1([0, 1], \mathbb{R}^+)$.

Therefore, by Lemma 2.1, (3.6) and the Sobolev embedding theorem, one can get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(x, u(x))dx \\ &\geq \frac{1}{2}\|u\|^2 - \epsilon\int_0^1 |u(x)|^2dx + \int_0^1 a_2(x)|u(x)|^\xi dx \\ &\geq \frac{1}{2}\|u\|^2 - \epsilon\|u\|_\infty^2 + \|u\|_\infty^\xi \int_0^1 a_2(x)dx \\ &\geq \frac{1}{2}\|u\|^2 - \epsilon\frac{1}{4\pi^2\theta_1^2}\|u\|^2 + \frac{1}{2^\xi\pi^\xi\theta_1^\xi}\int_0^1 a_2(x)dx\|u\|^\xi \\ &\geq \left(\frac{1}{2} - \epsilon\frac{1}{4\pi^2\theta_1^2}\right)\|u\|^2 + \frac{1}{2^\xi\pi^\xi\theta_1^\xi}\int_0^1 a_2(x)dx\|u\|^\xi. \end{aligned}$$

Hence, we can find $R > 0$ and $\delta > 0$ such that

$$J(u) \geq \delta, \quad \text{for all } u \in X \text{ with } \|u\| = R. \tag{3.7}$$

We claim that $J(tu) \to -\infty$ as $t \to +\infty$. To this end, Let $\mathcal{N}$ be the Lebesgue-null set outside which the hypotheses (H3) and (H4) hold and let $x \in [0, 1]\backslash\mathcal{N}$,

$u \in \mathbb{R}$ with $|u| \geq M$. We set $\mathcal{J}(x,\lambda) = F(x,\lambda u)$, $\lambda \in \mathbb{R}$. Clearly, $\mathcal{J}(x,\cdot)$ is locally Lipschitz. By Rademarcher's theorem, we see that for every $x \in [0,1]$, $\lambda \to \mathcal{J}(x,\lambda)$ is differentiable a.e. on $\mathbb{R}$ and at a point of differentiability $\lambda \in \mathbb{R}$, we have $\frac{d}{d\lambda}\mathcal{J}(x,\lambda) \in \partial\mathcal{J}(x,\lambda)$. Moreover, by Chain rule, we have $\partial\mathcal{J}(x,\lambda) \subset (\partial_u F(x,\lambda u), u)_{\mathbb{R}}$, so $\partial\mathcal{J}(x,\lambda) \subset (\partial_u F(x,\lambda u), \lambda u)_{\mathbb{R}}$. From (H4), one can get

$$\lambda \frac{d}{d\lambda}\mathcal{J}(x,\lambda) \geq \frac{1}{\mu}\mathcal{J}(x,\lambda) \Longrightarrow \frac{\frac{d}{d\lambda}\mathcal{J}(x,\lambda)}{\mathcal{J}(x,\lambda)} \geq \frac{1}{\lambda\mu}.$$

By integrating from 1 to λ_0 from above inequality, we get $\ln \frac{\mathcal{J}(x,\lambda_0)}{\mathcal{J}(x,1)} \geq \ln \lambda_0^{\frac{1}{\mu}}$. So, we have proved that for $x \in [0,1]\backslash\mathcal{N}$, $|u| \geq M$ and $\lambda \geq 1$, we have $\lambda_0^{-\frac{1}{\mu}} F(x,\lambda u) \geq \lambda_0^{\frac{1}{\mu}} F(x,u)$.

Let $z(x) = \min\{F(x,u): \ |u| = M\}$, clearly $z \in L^2([0,1],\mathbb{R}^+)$ and $z(x) \geq c_0$ for every $x \in [0,1]$. Therefore, for every $x \in [0,1]\backslash\mathcal{N}$ and $|u| \geq M$, we have

$$F(x,u)=F(x,|u|M^{-1}Mu|u|^{-1})\geq\Big(\frac{|u|}{M}\Big)^{\frac{1}{\mu}} F\Big(x,\frac{u}{|u|}M\Big)\geq z(x)\Big(\frac{|u|}{M}\Big)^{\frac{1}{\mu}}. \tag{3.8}$$

Combing with (H3) and (3.8), it is easy to prove that for any $u \in X\backslash\{0\}$, we have $J(tu) \to -\infty$ as $t \to +\infty$. Hence the claim is true. Then, for large $t_0 > 0$, we have $J(t_0 u) < 0$ with $u \in X\backslash\{0\}$ fixed. Then, noting that $J(0) = 0$, combing with (3.7) and using the non-smooth mountain pass theorem (see [7,9]), we obtain $u \in X$, $u \neq 0$ such that $0 \in \partial J(u)$. By Proposition 2.2, we finish the proof. □

In the following result we replace condition (H4) by conditions (H6)-(H8).

Theorem 3.6 *Assume that* $\pi^4 + \beta_2 > |\alpha|\pi^2$, *there exist two positive constants* ϑ, γ *with* $\gamma > 2$ *and* $\vartheta > \gamma - 2$ *and* $F(x,u)$ *satisfies the hypotheses* $(H1)$-$(H3)$, *and suppose the following conditions hold:*

$(H6)$ $\lim_{|u|\to+\infty} \frac{F(x,u)}{|u|^2} = +\infty$ *uniformly for all* $x \in [0,1]$;

$(H7)$ $\limsup_{|u|\to+\infty} \frac{F(x,u)}{|u|^\gamma} \leq M < +\infty$ *uniformly for some* $M > 0$ *and all* $x \in [0,1]$;

$(H8)$ $\liminf_{|u|\to+\infty} \frac{2F(x,u+F^\circ(x,u;-u)}{|u|^\vartheta} > 0$ *uniformly for all* $x \in [0,1]$.

Then, the problem (1.1) *has at least one nonzero solution on* X.

Proof. First, we will prove that J satisfies the nonsmooth (C) condition (see Definition 3.3). Let $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{J(u_n)\}_{n\geq 1}$ is bounded and

$$(1 + \|u_n\|)\rho(u_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

Then, there exist $C > 0$ such that

$$|J(u_n)| \leq C \quad \text{and} \quad (1 + \|u_n\|)\rho(u_n) \leq C, \quad \text{for all} \ \ n \in \mathbb{N} \tag{3.9}$$

by (H7), there exist $\varrho_1 > 0$ and $\delta_1 > 0$ such that $F(x,u) \leq \varrho_1|u|^\gamma$ for all $|u| \geq \delta_1$ and $x \in [0,1]$. It follows from (H1), (H2) and the Lebourg's mean value theorem that $|F(x,u)| \leq \overline{a}_2(x)$ for all $|u| \leq \delta_1$ and $x \in [0,1]$, where $\overline{a}_2(x) \in L^1([0,1],\mathbb{R}^+)$. Therefore, we obtain

$$|F(x,u)| \leq \varrho_1 |u|^\gamma + \overline{a}_2(x), \qquad \text{for all } u \in \mathbb{R},\ x \in [0,1]. \tag{3.10}$$

Also, By (H8), there exist $\varrho_2 > 0$ and $\delta_2 > 0$ such that $2F(x,u) + F^\circ(x,u;-u) \geq \varrho_2 |u|^\vartheta$ for all $|u| \geq \delta_2$ and $x \in [0,1]$. By the similar argument as (3.1), we have $|2F(x,u) + F^\circ(x,u;-u)| \leq \overline{a}_3(x)$, for all $|u| \leq \delta_2$ and $x \in [0,1]$. Thus, one can get

$$2F(x,u)+F^\circ(x,u;-u)\geq\varrho_2|u|^\vartheta - \varrho_1\delta_2^\vartheta - \overline{a}_3(x), \text{ for all } u\in\mathbb{R},\ x\in[0,1], \tag{3.11}$$

where $\overline{a}_3(x) \in L^1([0,1],\mathbb{R}^+)$, Therefore, by (3.9) and (3.10), we get

$$C \geq |J(u_n)| \geq \frac{1}{2}\|u_n\|^2 - \varrho_1 \int_0^1 |u_n(x)|^\gamma dx - \int_0^1 \overline{a}_2(x)dx.$$

Thus, we have

$$\frac{1}{2}\|u_n\|^2 \leq \varrho_1 \int_0^1 |u_n(x)|^\gamma dx + \int_0^1 \overline{a}_2(x)dx + C. \tag{3.12}$$

From (3.9) and (3.10), one can get

$$3C \geq 2J(u_n) - \langle u_n^*, u_n \rangle \geq \varrho_2 \int_0^1 |u_n(x)|^\vartheta dx - \int_0^1 \overline{a}_3(x)dx - \varrho_1\delta_2^\vartheta,$$

where $u_n^* \in \partial J(u_n)$ and $v_n \in \partial F(x,u_n)$. Therefore, $\{u_n\}$ is bounded in $L^\vartheta([0,1],\mathbb{R})$.

Since $\gamma > 2$ and $\vartheta > \gamma - 2$, if $\gamma \leq \vartheta$, then by Hölders inequality, we have $\int_0^1 |u_n(x)|^\gamma dx \leq \left(\int_0^1 |u_n(x)|^\vartheta dx\right)^{\frac{\gamma}{\vartheta}}$, which combining with (3.12) implies that $\{u_n\}$ is bounded in X. If $\vartheta < \gamma$, by Lemma 2.1, we have

$$\int_0^1 |u_n(x)|^\gamma dx \leq \|u_n\|_\infty^{\gamma-\vartheta} \int_0^1 |u_n(x)|^\vartheta dx \leq \frac{1}{2\pi\theta_1}\|u_n\|^{\gamma-\vartheta} \int_0^1 |u_n(x)|^\vartheta dx.$$

Combining with (3.12) implies that $\{u_n\}$ is bounded in X. By the same argument of Theorem 3.5, we can obtain that $\{u_n\}$ strongly converges in X.

By (H3) and (H5), we can find $R > 0$ and $\delta > 0$ such that

$$J(u) \geq \delta, \quad \text{for all } u \in X \text{ with } \|u\| = R. \tag{3.13}$$

Next, we prove that there exists $u_0 \in X$ such that $J(u_0) < 0$. By (H6), for $\varrho_3 = \frac{2(\pi^4 - \alpha\pi^2 + \beta_2)}{3} > 0$, there exists $\delta_3 > 0$ such that $F(x,u) \geq \varrho_3|u|^2$, for all $|u| \geq \delta_3,\ x \in [0,1]$. It follows from (H2), (H3) and the Lebourg's mean value theorem that

$$F(x,u) \geq \varrho_3|u|^2 - \varrho_3\delta_3^2 - \overline{a}_4(x), \qquad \text{for all } u \in \mathbb{R},\ x \in [0,1], \tag{3.14}$$

where $\overline{a}_4(x) \in L^1([0,1], \mathbb{R}^+)$. Therefore, by (3.14), (H3) and Lemma 2.1, choose $u_0 = \sin(\pi x) \in X$, we have

$$\begin{aligned} J(su_0) &= \frac{s^2}{2}\int_0^1 (|u_0''(x)|^2 - \alpha|u_0'(x)|^2 + \beta(x)|u_0(x)|^2)dx - \int_0^1 F(x, su_0(x))dx \\ &\le \frac{s^2}{4}(\pi^4 - \alpha\pi^2 + \beta_2) - \frac{s^2}{2}\varrho_3 + C' \\ &= \frac{s^2}{2}\Big(\frac{\pi^4 - \alpha\pi^2 + \beta_2}{2} - \varrho_3\Big) + C'. \end{aligned}$$

Here C' is a positive constant. Since

$$\frac{\pi^4 - \alpha\pi^2 + \beta_2}{2} - \varrho_3 = -\frac{\pi^4 - \alpha\pi^2 + \beta_2}{6} < 0,$$

then there exists a sufficiently large $s_0 > 0$ such that $J(s_0u_0) < 0$.

Finally, noting that $J(0) = 0$, combing with (3.13) and using the nonsmooth mountain pass theorem under the nonsmooth (C) condition (see [9]), combing with Proposition 2.2, we complete the proof. $\square$

Theorem 3.7 *Assume that $F(x,u)$ satisfies the hypotheses $(H1)$-$(H5)$, and $(H9)$ $F(x,u) = F(x,-u)$, for all $x \in [0,1]$, $u \in \mathbb{R}$. Then, the problem (1.1) has an unbounded sequence of solutions $\{u_n\} \subset X$ such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.*

Proof. By condition (H9), we have J is even. Combing with the proof of Theorem 3.5 and using the nonsmooth symmetric mountain pass theorem [7], we obtain that J possesses an unbounded sequence $\{c_n\}$ of critical values with $J(u_n) = c_n$, where $0 \in \partial J(u_n)$ for $n = 1, 2, \ldots$.

It follows from $0 \in \partial J(u_n)$ and (3.3) that

$$\|u_n\|^2 - \int_0^1 (v_n(x), u_n(x))dx = 0, \tag{3.15}$$

where $v_n \in \partial F(x, u_n)$. Now, by (3.8), (3.15), (H5) and (H9), we can obtain

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 &= \frac{3}{2}\|u_n\|^2 - \int_0^1 (v_n(x), u_n(x))dx \\ &\ge 3c_n + \int_0^1 (3F(x, u_n(x)) - F^\circ(x, u_n(x); u_n(x)))dx \\ &\ge 3c_n + \Big(3 + \frac{1}{\mu}\Big)\int_0^1 (F(x, u_n(x)) + F(x, -u_n(x))dx \\ &\ge 3c_n + \Big(3 + \frac{1}{\mu}\Big)\int_{\{|u_n(x)| \le M\}} F(x, u_n(x)dx. \end{aligned}$$

Since $c_n \to +\infty$ as $n \to +\infty$, it follows the above inequality that $\|u_n\| \to +\infty$ as $n \to +\infty$.

From the above inequality, we have

$$3c_n \leq \frac{1}{2}\|u_n\|^2 + C'', \tag{3.16}$$

where C'' is a constant. By condition (H3) and (3.16), one can get

$$2c_n \leq \frac{1}{2}\|u_n\|^2 - c_n + C'' = \int_0^1 F(x, u_n(x)dx + C'' = \int_0^1 (u_n^*(x), u_n(x)dx + C''$$
$$\leq \|u\|_\infty \int_0^1 a(x)dx + \|u\|_\infty^q \int_0^1 b(x)dx + C'',$$

where $u_n^* \in \partial F(x, su_n)$ with $s \in (0,1)$. Therefore, we obtain $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Thus, we have the conclusion. □

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