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Property (Bgw) and perturbations

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Abstract A Banach space operator T satisfies property (Bgw) if the complement in the approximate point spectrum $\sigma_a(T)$ of the semi-B-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$ coincides with the set of isolated eigenvalues of T of finite multiplicity $E^0(T)$. We find conditions for Banach Space operator to satisfy the property (Bgw). We also study the stability of property (Bgw) under perturbations by nilpotent operators, by finite rank operators, by quasi-nilpotent operators and by Riesz operators commuting with T.

Keywords Prešić type mapping · common fixed point · difference equation · equilibrium point

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1 Introduction and preliminaries

Let $B(X)$ denote the algebra of all bounded linear operator T acting on a Banach space X. For $T \in B(X)$, let T^*, $\ker(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the *adjoint*, the *null space*, the *range*, the *spectrum*, the *point spectrum* and the *approximate point spectrum* of T. Let $\mathbb{C}$ denote the set of *complex numbers*. Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in B(X)$ is said to be *upper semi-Fredholm*, $T \in SF_+(X)$, if the range of $T \in B(X)$ is closed and $\alpha(T) < \infty$, while $T \in B(X)$ is said to be *lower semi-Fredholm*, $T \in SF_-(X)$, if $\beta(T) < \infty$. An operator $T \in B(X)$ is said to be *semi-Fredholm* if $T \in SF_+(X) \cup SF_-(X)$ and Fredholm if $T \in SF_+(X) \cap SF_-(X)$. If T is semi-Fredholm then the *index* of T is defined by $\mathrm{ind}\,(\mathrm{T}) = \alpha(\mathrm{T}) - \beta(\mathrm{T})$.

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Recall that the *ascent*, $a(T)$, of an operator $T \in B(X)$ is the smallest non negative integer p such that $\ker(T^p) = \ker(T^{p+1})$ and if such integer does not exist we put $a(T) = \infty$. Analogously the *descent*, $d(T)$, of an operator $T \in B(X)$ is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$ and if such integer does not exist we put $d(T) = \infty$.

A bounded linear operator T acting on a Banach space X is *Weyl* if it is Fredholm of index zero and *Browder* if T is Fredholm of finite ascent and descent. The *Weyl spectrum* $\sigma_W(T)$ and *Browder spectrum* $\sigma_b(T)$ of T are defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$ and $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$, respectively. According to Coburn [14], *Weyl's theorem* holds for T if $\triangle(T) = \sigma(T) \setminus \sigma_W(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $\operatorname{iso}(K)$ is the set of isolated points of K.

Let $SF_+^-(X) = \{T \in SF_+ : ind(T) \leq 0\}$. The *Weyl essential approximate spectrum* is given by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(X)\}$. According to Rakočević [23], an operator $T \in B(X)$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{T}) : 0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty\}$. It is known [23] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in B(X)$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $R(T^n)$ viewed as a map from $R(T^n)$ to $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. lower) semi-Fredholm operator, then T is called *upper* (resp. *lower*) *semi-B-Fredholm operator*. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm operator*. An operator T is said to be *B-Weyl operator* if it is a B-Fredholm operator of index zero.

According to Berkani [10], an operator $T \in B(X)$ is said to be *Drazin invertible* if it has finite ascent and descent. The *Drazin spectrum* of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$. Let $\pi(T)$ be the set of poles of T, we observe that $\pi(T) = \sigma(T) \setminus \sigma_D(T)$. Define the set $LD(X) = \{T \in B(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. Following [11], an operator $T \in B(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$ [11, Definition 2.6], we observe that $\sigma_a(T) \setminus \sigma_{LD}(T) = \pi_a(T)$. Let $\pi_a(T)$ denotes the set of all left poles of T and let $\pi_a^0(T)$ denotes the set of all left poles of finite rank. It follows from [11, Theorem 2.8] that if $T \in B(X)$ is left Drazin invertible, then T is upper semi-B-Fredholm of index less than or equal to 0. The class of all *upper semi-Browder* operators is defined by $B_+(X) := \{T \in SF_+(X) : a(T) < \infty\}$, the *upper Browder spectrum* is defined by $\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_+(X)\}$. We say that *Browder's theorem* holds for $T \in B(X)$ if $\Delta(T) = \pi^0(T)$, where $\pi^0(T)$ is the set of all poles of T of finite rank and that *a-Browder's theorem* holds for T if $\Delta_a(T) = \pi_a^0(T)$, or equivalently $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. Let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Following [10], we say that *generalized Weyl's theorem* holds for $T \in B(X)$ if $\Delta^g(T) = E(T)$, $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$, and that *generalized Browder's theorem* holds for T if $\Delta^g(T) = \pi(T)$. It is proved in [9, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

Let $SBF_+^-(X)$ denote the class of all *upper semi-B-Fredholm* operators such that $\text{ind}(T) \leq 0$. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(X)\}$. Let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. We say that $T \in B(X)$ satisfies *generalized a-Weyl's theorem*, if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in B(X)$ satisfies *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$ [11, Definition 2.13]. It is proved in [9, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

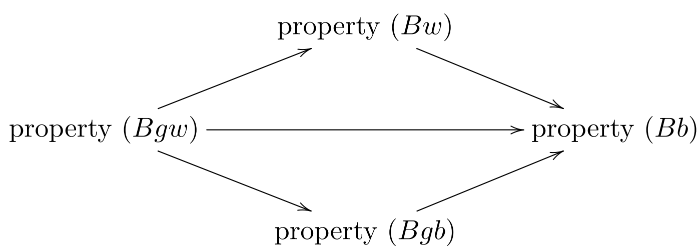
An operator $T \in B(X)$ has the *single valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T - \lambda)f(\lambda)=0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. For more information, see [1].

As a variant of generalized Weyl's theorem, property (Bw) is introduced in [17]. A bounded linear operator $T \in B(X)$ is said to satisfy *property* (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. The following variations of Weyl type theorem have been investigated in [27] by the authors.

Definition 1.1 *A bounded linear operator $T \in B(X)$ is said to satisfy*

(i) property (Bgw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$.
(ii) property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$.
(iii) property (Bgb) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$.

The following diagram resume the relationships between property (Bgw), property (Bgb), property (Bw) and property (Bb) (see [27]).



2 Property (Bgw)

Theorem 2.1 *Let $T \in B(X)$. If T satisfies property (Bgw), then the following statements are equivalent:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$;
(ii) $acc(\sigma_a(T)) \subset \sigma_{SBF_+^-}(T)$;
(iii) $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E_a(T)$;
(iv) $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E_a(T)$.

Proof. Since T satisfies property (Bgw), we conclude from [27, Theorem 2.19] that T satisfies generalized a-Browder's theorem. So, the result follows now from [26, Theorem 2.1]. □

Theorem 2.2 *Let $T \in B(X)$. If T satisfies property (Bgw), the following statements are equivalent:*

(i) *T satisfies generalized a-Weyl's theorem;*
(ii) $\sigma_{SBF_+^-}(T) \cap E_a(T) = \emptyset$;
(iii) $\pi_a(T) = E_a(T)$.

Proof. Assume that T satisfies property (Bgw), then we conclude from [27, Theorem 2.19] and [25, Theorem 2.21] that T satisfies generalized a-Browder's theorem. So the result follows now from [26, Theorem 2.2]. □

Theorem 2.3 *Let $\sigma_{SBF_+^-}(T) = \sigma_D(T)$ and let $E^0(T) = \pi(T)$ for $T \in B(X)$. Then T satisfies property (Bgw)*

Proof. By [29, Lemma 2.2], T satisfies property (gb). Since $E^0(T) = \pi(T)$, by [27, Theorem 2.7], T satisfies property (Bgw). □

The following result is immediate from [16, Corollary 3.13]

Corollary 2.4 *Let T^* has SVEP at $\lambda \notin \sigma_{SBF_+^-}(T)$ and let $E^0(T) = \pi(T)$ for $T \in B(X)$. Then T satisfies property (Bgw)*

The *quasinilpotent part* $H_0(T - \lambda I)$ and the *analytic core* $K(T - \lambda I)$ of $T - \lambda I$ are defined by

$$H_0(T - \lambda I) = \{x \in X : \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(T - \lambda I) = \{x \in X : &\text{there exists a sequence } \{x_n\} \subset X \\ &\text{and} \quad \delta > 0 \text{ for which} \\ &x = x_0, (T - \lambda I)x_{n+1} = x_n \\ &\text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{for all} \quad n = 1, 2, \cdots\}. \end{aligned}$$

We note that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are generally non-closed hyper-invariant subspaces of $T - \lambda I$ such that $\ker(T - \lambda I)^p \subseteq H_0(T - \lambda I)$ for all $p = 0, 1, \cdots$ and $(T - \lambda I)K(T - \lambda I) = K(T - \lambda I)$ (see [19]).

Let $Hol(\sigma(T))$ be the set of functions which are analytic in an open neighborhood of $\sigma(T)$.

Theorem 2.5 *Let $T \in B(X)$. If there exist $\lambda \in \mathbb{C}$ such that $K(T - \lambda I) = 0$ and $\ker(T - \lambda I) = 0$, then property (Bgw) holds for $f(T)$ for all $f \in Hol(\sigma(T))$*

Proof. From [3, Lemma 2.4], $\sigma_p(T) = \emptyset$. So T has SVEP. We need to show that $\sigma_p(f(T)) = \emptyset$. Let $\beta \in \sigma_p(f(T))$ then we write

$$f(z) - \beta = \prod_{i=1}^{n} (z - \lambda_i)^{m_i} g(z),$$

where $\lambda_i \in \sigma(T)$ for $i = 1, 2, 3...n$ and $g(z)$ is complex valued analytic function on neighborhood of $\sigma(T)$. Since $g(z)$ is invertible, we have

$$\ker(f(T) - \beta I) = \ker(\prod_{i=1}^{n}(T - \lambda_i)^{m_i}) = \bigoplus_{i=1}^{n} \ker(T - \lambda_i)^{m_i}.$$

This implies that $ker(f(T) - \beta I) = 0$ for all $\beta \in \mathbb{C}$ and so $\sigma_p(f(T)) = \emptyset$. Applying [1, Theorem 2.40], $f(T)$ has SVEP and so $f(T)$ satisfies generalized a-Browder's theorem (see [18]).

To prove property (Bgw) holds for $f(T)$ for all $f \in Hol(\sigma(T))$, by [31, Theorem 2.11], it required to show that $E^0(f(T)) = \pi_a(f(T))$. Evidently, from the condition $\sigma_p(f(T)) = \emptyset$, $E^a(f(T)) = \pi_a(f(T)) = \emptyset$. Then, $E^0(f(T)) = \pi_a(f(T)) = \emptyset$. Then by [27, Theorem 2.11], it follows that property (Bgw) holds for $f(T)$ for all $f \in Hol(\sigma(T))$. □

An operator $T \in B(X)$ satisfy property $H(p)$ if $H_0(T - \lambda I) = ker(T - \lambda I)^p$, for all $\lambda \in \mathbb{C}$. The class $H(p)$ contains every scalar operators. Let $P(X)$ denote the class of all operators $T \in B(X)$ such that there exist $p := p(\lambda) \in \mathbb{N}$ such that $H_0(T - \lambda I) = ker(T - \lambda I)^p$ for all $\lambda \in \text{iso}(\sigma(T))$. The class $P(X)$ is large, it contains the class of operators that satisfy property $H(p)$ (see [21]).

Theorem 2.6 *Let $T^* \in P(X^*)$ and let $E^0(f(T)) = \pi(f(T))$ for $T \in B(X)$. Then $f(T)$ satisfies property (Bgw) for all $f \in Hol(\sigma(T))$.*

Proof. The result follows from [25, Theorem 2.17] and [27, Theorem 2.7]. □

Following [4], we say that an operator $T \in B(X)$ satisfies property (R) if the equality $\pi_a^0(T) = E^0(T)$ holds. As a consequence of [4, Theorem 2.4] and [27, Theorem 2.12], we have

Theorem 2.7 *Let $T \in B(X)$. If T satisfies property (Bgw), then T satisfies property (R).*

A bounded linear operator $T \in B(X)$ is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. $T \in B(X)$ is said to be *a-polaroid* if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. Since every pole is a left pole. Then it is easily seen that

$$\text{if } T \text{ is } a\text{-polaroid} \Longrightarrow T \text{ is polaroid,} \tag{2.1}$$

while, in general, the converse does not hold (see [7]).

A bounded linear operator $T \in B(X)$ is said to be *isoloid* (resp. *a-isoloid*) if every isolated point of $\sigma(T)$ (resp. $\sigma_a(T)$) is an eigenvalue of T. It is known that every polaroid operator is isoloid, but the converse is not true in general.

A bounded linear operator $T \in B(X)$ is said to be *finitely-isoloid* (resp. *finitely-a-isoloid*) if every isolated point of $\sigma(T)$ (resp. $\sigma_a(T)$) is an eigenvalue of finite multiplicity of T. $T \in B(X)$ is said to be *finitely-polaroid* (resp. *finitely-a-polaroid*) if $\text{iso}\,\sigma(\text{T}) \subseteq \pi^0(\text{T})$ (resp. $\text{iso}\,\sigma_{\text{a}}(\text{T}) \subseteq \pi^0(\text{T})$). Since $\pi^0(T) \subseteq E^0(T)$. Trivially,

$$\text{if } T \text{ is finitely polaroid} \Longrightarrow T \text{ is finitely isoloid,} \tag{2.2}$$

while, in general, the converse does not hold.

Theorem 2.8 *Let $T \in B(X)$. If T is finitely-polaroid and T^* has the SVEP, then T satisfies property (Bgw).*

Proof. Since T^* has the SVEP then by [1, Corollary 2.45] we have $\sigma(T) = \sigma_a(T)$. Suppose first that iso $\sigma(\mathrm{T}) = \emptyset$. Then $E^0(T) = \emptyset$. We show that also $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ is empty. By [2, Theorem 2.9] we have $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, T satisfies property (Bgw).

Consider the other case, iso $\sigma(\mathrm{T}) \neq \emptyset$. Suppose that $\lambda \in E^0(T)$. Then λ is an isolated point of $\sigma(T)$ and hence by the finitely-polaroid condition, λ is a pole of the resolvent of T of finite rank, i.e., $a(T-\lambda I) = d(T-\lambda) < \infty$, $\alpha(T-\lambda I) < \infty$ and $\beta(T-\lambda I) < \infty$. Hence $T - \lambda I$ is Browder operator. Therefore, $\lambda \in \sigma(T) \setminus \sigma_b(T) \subseteq \Delta^g(T)$. Conversely, if $\lambda \in \Delta_a^g(T) = \Delta^g(T)$ then λ is an isolated point of $\sigma(T)$. Hence $\lambda \in \pi^0(T) \subseteq E^0(T)$ and so T satisfies property (Bgw). □

It is well-known that $f(\pi^0(T)) = \pi^0(f(T))$, for every $f \in H(\sigma(T))$. Then we have

Lemma 2.9 *Suppose that $f \in H(\sigma(T))$ is non constant on each of the components of its domain. If T is finitely-polaroid, then $f(T)$ is finitely-polaroid.*

Theorem 2.10 *Suppose that $T \in B(X)$ is finitely-polaroid and that T^* has SVEP. Then $f(T)$ satisfies property (Bgw) for all $f \in H(\sigma(T))$ such that f is not constant on each of the components of its domain.*

Proof. Suppose that T^* has the SVEP, then $f(T^*) = f(T)^*$ has the SVEP (see [1, Theorem 2.40]). Since T is finitely-polaroid then by Lemma 2.9 we have $f(T)$ is finitely-polaroid. So the result follows now by Theorem 2.8. □

3 Property (Bgw) under perturbations

In this section we are interested to study the stability of property (Bgw) under perturbations by nilpotent operators, by finite rank operators, by quasi-nilpotent operators and by Riesz operators commuting with T. We begin with this lemma:

Lemma 3.1 [28] *Let $T \in B(X)$ and let $N \in B(X)$ be a nilpotent operator commuting with T. Then $E^0(T+N) = E^0(T)$.*

Theorem 3.2 *If $T \in B(X)$ satisfies property (Bgw) and N is a nilpotent operator that commutes with T, then $T+N$ satisfies property (Bgw).*

Proof. Suppose $T \in B(X)$ satisfies property (Bgw), that is, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$. By [28, Corollary 3.1], we have $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$. Since $\sigma_a(T+N) = \sigma_a(T)$ and since $E_0(T+N) = E_0(T)$, $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E^0(T+N)$. Finally, $T+N$ satisfies property (Bgw). □

Theorem 3.3 *Suppose that $T \in B(X)$ and $N \in B(X)$ is nilpotent such that $TN = NT$. Then T is finitely-polaroid if and only if $T+N$ is finitely-polaroid.*

Proof. We conclude from proof of [8, Theorem 2.11] that $E(T) = E(T + N)$. Since T is finitely-polaroid then $E(T) \subseteq \pi^0(T)$. But we know that $\pi^0(T) = \pi^0(T + N)$. Hence $E(T + N) \subset \pi^0(T + N)$. That is, $T + N$ is finitely-polaroid. $\square$

Theorem 3.4 *Suppose that $T \in B(X)$ and $N \in B(X)$ is nilpotent such that $TN = NT$. If T is finitely-polaroid and T^* has SVEP, then $T + N$ satisfies property (Bgw).*

Proof. If T is polaroid then by [5, Theorem 2.5] T^* is polaroid. Clearly, N^* is nilpotent, since $(N^*)^n = (N^n)^* = 0$ for some $n \in \mathbb{N}$. Therefore $T^* + N^*$ is finitely-polaroid, by Theorem 3.3. Since $T^* + N^*$ has SVEP, by [1, Corollary 2.12], it then follows, by Theorem 2.8, that $T + N$ satisfies property (Bgw). $\square$

Theorem 3.5 *Suppose that T is finitely-polaroid and $N \in B(X)$ a nilpotent operator commuting with T. If T^* has SVEP and $f \in Hol(\sigma(T))$ such that f is not constant on each of the components of its domain, then property (Bgw) holds for $f(T) + N$.*

Proof. By Theorem 2.10, $f(T)$ satisfies property (Bgw). Since $f(T^*) = f(T)^*$ has SVEP (see [1, Theorem 2.40]), by Theorem 2.9 we have $f(T)$ is finitely-polaroid, by Theorem 3.4 it then follows that property (Bgw) holds for $f(T) + N$. $\square$

Theorem 3.6 *Let $T \in B(X)$ and $F \in B(X)$ be a finite rank operator commuting with T. If T satisfies property (Bgw), then the following assertions are equivalent:*

(i) $T + F$ *satisfies property* (Bgw);
(ii) $E^0(T + F) = \pi_a(T + F)$;
(iii) $E^0(T + F) \cap \sigma_a(T) \subset E^0(T)$.

Proof. (i)$\Longleftrightarrow$(ii) If $T + F$ satisfies property (Bgw), then from [27, Theorem 2.10], we have $E^0(T + F) = \pi_a(T + F)$. Conversely, assume that $E^0(T + F) = \pi_a(T + F)$, since T satisfies property (Bgw), then it follows from [27, Theorem 2.10] that T satisfies generalized a-Browder's theorem and hence $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Since F is a finite rank operator, from [13, Lemma 2.3] we have $\sigma_{SBF_+^-}(T + F) = \sigma_{SBF_+^-}(T)$. As F commutes with T, from [12, Theorem 2.1] we have $\sigma_{LD}(T) = \sigma_{LD}(T + F)$. Therefore, $\sigma_{SBF_+^-}(T + F) = \sigma_{LD}(T + F)$ and so $T + F$ satisfies generalized a-Browder's theorem. As $\pi_a(T + F) = E^0(T + F)$, then from [27, Theorem 2.10], $T + F$ satisfies property (Bgw).

(iii)$\Longrightarrow$(ii) Let $\lambda \in E^0(T + F)$. Then $\lambda \in \mathrm{iso}\sigma_\mathrm{a}(\mathrm{T + F})$. If $\lambda \notin \sigma(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T + F)$. As $\lambda \in \mathrm{iso}\sigma_\mathrm{a}(\mathrm{T+F})$, it follows from [11, Theorem 2.8] that $\lambda \in \pi^a(T + F)$. If $\lambda \in \sigma_a(T)$, then $\lambda \in E^0(T + F) \cap \sigma_a(T)$ and by assumption $\lambda \in E^0(T)$. Since T satisfies property (Bgw), so by [27, Theorem 2.10] T satisfies generalized a-Browder's theorem and it follows then from [11, Theorem 4.3] that $T + F$ satisfies generalized a-Browder's theorem. Hence $\lambda \notin \sigma_{SBF_+^-}(T + F)$ and so $\lambda \in \pi_a(T + F)$. In the two cases, we have $E^0(T + F) \subset \pi_a(T + F)$. Conversely, let $\lambda \in \pi_a(T + F)$, then $\lambda \notin \sigma_{LD}(T + F)$ and so $\lambda \notin \sigma_{LD}(T)$. As T satisfies property (Bgw), then $\lambda \in \pi_a(T) = E^0(T)$ and so $\lambda \in E^0(T + F)$.

(ii)$\Longrightarrow$(iii) Assume that $E^0(T + F) = \pi_a(T + F)$ and let $\lambda \in E^0(T + F) \cap \sigma_a(T)$, then $\lambda \in \pi_a(T + F) \cap \sigma_a(T)$. So $\lambda \notin \sigma_{LD}(T + F)$. As $\sigma_{LD}(T) = \sigma_{LD}(T + F)$ and $\lambda \in \sigma_a(T)$, then $\lambda \in \pi_a(T)$. Since T satisfies property (Bgw), then $\lambda \in E^0(T)$. Therefore, $E^0(T + F) \cap \sigma_a(T) \subset E^0(T)$. $\square$

Theorem 3.7 *Let $T \in B(X)$ and $N \in B(X)$ be a nilpotent operator commuting with T. If T satisfies property (Bgw), then the following assertions are equivalent:*

(i) $T+N$ *satisfies property* (Bgw);

(ii) $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$;

(iii) $E^0(T) = \pi_a(T+N)$.

Proof. (i)$\Longleftrightarrow$ (ii) Assume that $T+N$ satisfies property (Bgw), then $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E^0(T+N)$. As $\sigma_a(T+N) = \sigma_a(T)$ and $E^0(T+N) = E^0(T)$ then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T+N) = E^0(T)$. Since T satisfies property (Bgw), then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$ and so $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$. Conversely, assume that $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$. Since T satisfies property (Bgw), then

$$E^0(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E^0(T+N).$$

So $T+N$ satisfies property (Bgw).

(i)$\Longleftrightarrow$(iii) Assume that $T+N$ satisfies property (Bgw), then from [27, Theorem 2.10], we have $E^0(T+N) = \pi_a(T+N)$. Therefore, by Lemma 3.1, we have $\pi(T+N) = E^0(T)$. Conversely, assume that $E^0(T) = \pi_a(T+N)$. Since T satisfies property (Bgw), by [27, Theorem 2.10] we then have T satisfies generalized a-Browder's theorem. As we know from [9, Theorem 2.2] that a-Browder's theorem is equivalent to generalized a-Browder's theorem. So $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. From [15, Theorem 2.13], we know that $\sigma_{SF_+^-}(T+N) = \sigma_{SF_+^-}(T)$. By [1, Theorem 3.65], we know that $\sigma_{ub}(T) = \sigma_{SF_+^-}(T) \cup \mathrm{acc}\sigma_{\mathrm{a}}(\mathrm{T})$. Hence $\sigma_{ub}(T+N) = \sigma_{ub}(T)$. Therefore $\sigma_{SF_+^-}(T+N) = \sigma_{ub}(T+N)$ and $T+N$ satisfies a-Browder's theorem. So it satisfies generalized a-Browder's theorem, that is, $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \pi_a(T+N)$. As by assumption $E^0(T) = \pi_a(T+N)$, it follows that $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E^0(T+N)$ and so $T+N$ satisfies property (Bgw). □

Theorem 3.8 *Let $T \in B(X)$ and let $K \in B(X)$ be a compact operator commutating with T. If T satisfies property (Bgw), then $T+K$ satisfies property (Bgw) if and only if $E^0(T+K) = \pi_a(T+K)$.*

Proof. If T satisfies property (Bgw), then from [27, Theorem 2.10], $E^0(T+k) = \pi_a(T+K)$. Conversely, suppose $E_0(T+K) = \pi_a(T+K)$. Since T satisfies property (Bgw), it follows from [27, Theorem 2.20] and [27, Theorem 2.22] that T satisfies generalized a-Browder's theorem. Thus from [9], T satisfies a-Browder's theorem. Hence it follows from [1, Corollary 3.39] and [1, Corollary 3.45] that $T+K$ satisfies a Browder's theorem. Again by [9], $T+K$ satisfies generalized a-Browder's theorem. Thus, $\sigma_a(T+K) \setminus \sigma_{SBF_+^-}(T+K) = \pi_a(T+K) = E^0(T+K)$. □

Theorem 3.9 *If $T \in B(X)$ satisfies property (Bgw) and F is finite rank operator commutates with Tsuch that $\pi_a(T+F) \subseteq \sigma_a(T)$ and $E_0(T+F) \subseteq \pi_a(T+F)$, then $T+F$ satisfies property (Bgw).*

Proof. From [27, Theorem 2.10], it suffices to prove that $\pi_a(T+F) \subseteq E^0(T+F)$. Let $\lambda \in \pi_a(T+F)$, then $\lambda \in \sigma_a(T+F)$ and $\lambda \notin \sigma_{SBF_+^-}(T+F)$. Since $\lambda \notin \sigma_{SBF_+^-}(T+F)$, $\lambda \notin$

$\sigma_{SBF_+^-}(T)$. Also the hypothesis $\pi_a(T+F) \subseteq \sigma_a(T)$ implies that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then by the property (Bgw) of T we have $\lambda \in E_0(T)$ and $T - \lambda I$ is Drazin invertible. Thus, $T + F - \lambda I$ is Drazin invertible by [10, Theorem 2.7]. Since $\lambda \in \sigma(T + F)$ and since $\alpha(T + F - \lambda I) < \infty$, $\lambda \in E^0(T + F)$. This completes the proof. □

Theorem 3.10 *Let $T \in B(X)$ such that $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$ and let F is finite rank operator commutates with T such that $E^0(T + F) \subseteq \pi(T + F)$. Then $T + F$ satisfies property (Bgw).*

Proof. Since $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$ and F is finite rank operator commutates with T, from [29, Theorem 2.11], $T + F$ satisfies property (gb). Since $E_0(T + F) \subseteq \pi(T + F)$, the required result follows from [27, Theorem 2.7]. □

Recall that $T \in B(X)$ is said to be *Riesz operator* if $T - \lambda I$ is Fredholm for all $\lambda \in C \setminus \{0\}$. Consequently compact operators and quasi-nilpotent operators are Riesz operators.

As a consequence of Theorem 2.7 and [6, Theorem 2.4], we have

Proposition 3.11 *If $T \in B(X)$ has property (Bgw) and K is a Riesz operator for which $TK = KT$ and $\sigma_a(T) = \sigma_a(T + K)$ then $E^0(T) \subseteq E^0(T + K)$.*

Theorem 3.12 *Suppose that $T \in B(X)$ is an isoloid operator for which property (Bgw) holds and let $K \in B(X)$ be a bounded operator commuting with T such that K^n is a finite rank operator for some $n \in \mathbb{N}$ and $\sigma_a(T) = \sigma_a(T + K)$. Then the following assertions hold:*

(i) $E^0(T) = E^0(T + K)$;
(ii) *$T + K$ satisfies property (Bgw).*

Proof. (i) This follows from Theorem 2.7 and [6, Theorem 2.6].

(ii) We conclude from [28, Theorem 2.8] that $\sigma_{SBF_+^-}(T + K) = \sigma_{SBF_+^-}(T)$. By assumption we have $\sigma_a(T + K) = \sigma_a(T)$, so

$$E^0(T + K) = E^0(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T + K) \setminus \sigma_{SBF_+^-}(T + K).$$

That is, $T + K$ satisfies property (Bgw). □

Theorem 3.13 *Suppose that $T \in B(X)$ is a finite-isoloid operator for which property (Bgw) holds. If K is a Riesz operator which commutes with T and such that $\sigma_a(T) = \sigma_a(T + K)$, then the following assertions hold:*

(i) $E^0(T) = E^0(T + K)$;
(ii) *$T + K$ satisfies property (Bgw) if and only if $\sigma_{SBF_+^-}(T + K) = \sigma_{SBF_+^-}(T)$.*

Proof. The proof of the first part follows from Theorem 2.7 and [6, Theorem 2.11]. The proof of second part is similar to the proof of part (ii) of Theorem 3.12. □

Theorem 3.14 *Suppose that $T \in B(X)$ is a-polaroid and finitely-isoloid. If T has the SVEP and $Q \in B(X)$ is quasinilpotent which commutes with T, then $T + Q$ satisfies property (Bgw).*

Proof. If $\lambda \in \mathrm{iso}\sigma_a(T+Q)$ then $\lambda \in \mathrm{iso}\sigma_a(T)$ and hence, since T is finitely a-polaroid, λ is a pole of the resolvent of T, in particular an isolated point of the spectrum. Therefore, $a(T-\lambda I) = d(T-\lambda I) < \infty$ and since by assumption $\alpha(T-\lambda I) < \infty$ we then have $\alpha(T-\lambda I) = \beta(T-\lambda I)$ (see [1, Theorem 3.4]), so $T-\lambda I$ is Browder. Since by [24], Browder operators are invariant under Riesz commuting perturbation, in particular under quasinilpotent commuting perturbations, hence $T+Q-\lambda I$ is Browder, and consequently, λ is a pole of the resolvent of $T+Q$ of finite rank. Therefore, $T+Q$ is finitely a-polaroid, that is, $\mathrm{iso}\sigma_a(T+Q) \subseteq \pi^0(T+Q)$. Since T has the SVEP and $QT = TQ$ for Q quasinilpotent then, by [20, Proposition 3.4.11], $T+Q$ has the SVEP. Hence $T+Q$ satisfies generalized a-Browder's theorem. Thus $\sigma_a(T+Q) \setminus \sigma_{SBF_+^-}(T+Q) = \pi_a(T+Q) \subseteq \pi^0(T+Q) \subseteq E^0(T+Q)$. So $T+Q$ satisfies generalized a-Browder's theorem and $\pi_a(T+Q) = E^0(T+Q)$. Therefore, by [27, Theorem 2.10], $T+Q$ satisfies property (Bgw). □

Theorem 3.15 *Suppose that $T \in B(X)$ is finitely a-polaroid. If T has the SVEP and $Q \in B(X)$ is quasinilpotent which commutes with T, then*

(i) $E^0(T) = E^0(T+Q)$.
(ii) $T+Q$ *satisfies property* (Bgw).

Proof. (i) Let $\lambda \in E^0(T)$, then λ is isolated in $\sigma(T)$ and hence isolated in $\sigma_a(T)$. Since T is finitely a-polaroid then $\lambda \in \pi^0(T)$. Hence $\lambda \notin \sigma_b(T)$. Thus from [24], $\lambda \notin \sigma_b(T+Q)$. This implies that $\lambda \in \pi^0(T+Q)$. The reverse inclusion follows by symmetry.

(ii) Since T has the SVEP and $QT = TQ$ for Q quasinilpotent then, by [20, Proposition 3.4.11], $T+Q$ has the SVEP. Hence $T+Q$ satisfies generalized a-Browder's theorem. Thus $\sigma_a(T+Q) \setminus \sigma_{SBF_+^-}(T+Q) = \pi_a(T+Q)$. By [27, Theorem 2.10], it suffices to show that $E^0(T+Q) = \pi_a(T+Q)$. Let $\lambda \in E^0(T+Q) = E^0(T)$. Then by the proof of the first part $\lambda \in \pi^0(T+Q) \subseteq \pi_a(T+Q)$ and so $E^0(T+Q) \subseteq \pi_a(T+Q)$. For the reverse inclusion, let $\lambda \in \pi_a(T+Q)$ then, by [8, Lemma 3.5], $\lambda \notin \sigma_{LD}(T+Q) = \sigma_{LD}(T)$. Hence $\lambda \in \pi_a(T)$ and since T is finitely a-polaroid then $\lambda \in \pi^0(T) \subseteq E^0(T) = E^0(T+Q)$. Therefore, $T+Q$ satisfies property (Bgw). □

Theorem 3.16 *Let $T \in B(X)$ and let $R \in B(H)$ be a Riesz operator commutating with T. If T satisfies property (Bgw), then $T+R$ satisfies property (Bgw) if and only if $E^0(T+R) = \pi_a(T+R)$.*

Proof. If $T+R$ satisfies property (Bgw), then from [27, Theorem 2.10], we have $E^0(T+R) = \pi_a(T+R)$. Conversely, suppose $E^0(T+R) = \pi_a(T+R)$. Since T satisfies property (Bgw), T satisfies generalized a-Browder's theorem. Then it follows from [22, Corollary 2.3] that $T+R$ satisfies generalized a-Browder's theorem. By hypothesis $E^0(T+R) = \pi_a(T+R)$, and so $T+R$ satisfies property (Bgw). □

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References

1. Aiena, P. – *Fredholm and Local Spectral Theory, With Applications to Multipliers*, Kluwer Academic Publishers, Dordrecht, 2004.

2. Aiena, P.; Miller, T.L. – *On generalized a-Browder's theorem*, Studia Math., 180 (2007), 285–300.
3. Aiena, P.; Bondi, M.T.; Villafane, F. – *Property* (w) *and perturbations III*, J. Math. Anal. Appl., 353 (2009), 205–214.
4. Aiena, P.; Guillén, J.R.; Peña, P. – *Property* (R) *for bounded linear operators*, Mediterr. J. Math., 8 (2011), 491–508.
5. Aiena, P.; Guillén, J.R.; Peña, P. – *Property* (w) *for perturbations of polaroid operators*, Linear Algebra Appl., 428 (2008), 1791–1802.
6. Aiena, P.; Aponte, E.; Guillén, J.R.; Peña, P. – *Property* (R) *under perturbations*, Mediterr. J. Math., 10 (2013), 367–382.
7. Aiena, P.; Aponte, E.; Balzan, E. – *Weyl type theorems for left and right polaroid operators*, Integral Equations Operator Theory, 66 (2010), 1–20.
8. Amouch, M. – *Polaroid operators with SVEP and perturbations of property* (gw), Mediterr. J. Math., 6 (2009), 461–470.
9. Amouch, M.; Zguitti, H. – *On the equivalence of Browder's and generalized Browder's theorem*, Glasg. Math. J., 48 (2006), 179–185.
10. Berkani, M. – *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl., 272 (2002), 596–603.
11. Berkani, M.; Koliha, J.J. – *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged), 69 (2003), 359–376.
12. Berkani, M.; Zariouh, H. – *Generalized a-Weyl's theorem and perturbations*, Funct. Anal. Approx. Comput., 2 (2010), 7–18.
13. Berkani, M.; Amouch, M. – *Preservation of property* (gw) *under perturbations*, Acta Sci. Math. (Szeged), 74 (2008), 769–781.
14. Cuburn, L.A. – *Weyl's theorem for non-normal operators*, Michigan. Math. J., 13 (1966), 285–288.
15. Djordjević, D.S.; Djordjević, S.V. – *On a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl., 44 (1999), 361–369 (2000).
16. Duggal, B.P. – *Polaroid operators and generalized Browder-Weyl theorems*, Math. Proc. R. Ir. Acad., 108 (2008), 149–164.
17. Gupta, A.; Kashyap, N. – *Property* (Bw) *and Weyl type theorems*, Bull. Math. Anal. Appl., 3 (2011), 1–7.
18. Heuser, H.G. – *Functional Analysis*, Translated from the German by John Horváth. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1982.
19. Lahrouz, M.; Zohry, M. – *Weyl type theorems and the approximate point spectrum*, Irish Math. Soc. Bull., 55 (2005), 41–51.
20. Laursen, K.B.; Neumann, M.M. – *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs. New Series, 20, The Clarendon Press, Oxford University Press, New York, 2000.
21. Oudghiri, M. – *Weyl's and Browder's theorems for operators satisfying the SVEP*, Studia Math., 163 (2004), 85–101.
22. Oudghiri, M. – *a-Weyl's theorem and perturbations*, Studia Math., 173 (2006), 193–201.
23. Rakočević, V. – *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl., 34 (1989), 915–919.
24. Rakočević, V. – *Semi-Browder operators and perturbations*, Studia Math., 122 (1997), 131–137.
25. Rashid, M.H.M. – *Property* (gb) *and perturbations*, J. Math. Anal. Appl., 383 (2011), 82–94.
26. Rashid, M.H.M. – *Property* (gw) *and perturbations*, Bull. Belg. Math. Soc. Simon Stevin, 18 (2011), 635–654.
27. Rashid, M.H.M.; Prasad, T. – *Variations of Weyl type theorems*, Ann. Funct. Anal., 4 (2013), 40–52.
28. Zeng, Q.; Jiang, Q.; Zhong, H. – *Spectra originated from semi-B-Fredholm theory and commuting perturbations*, arXiv: 1203.2442vl [math. FA], 12 Mar 2012.
29. Zeng, Q.; Zhong, H. – *A note on property* (gb) *and perturbations*, arXiv:1203.1134vl [math. FA], 6 Mar 2012.