

## A survey on differential geometry of Riemannian maps between Riemannian manifolds

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**Abstract** The main aim of this paper is to state recent results in Riemannian geometry obtained by the existence of a Riemannian map between Riemannian manifolds and to introduce certain geometric objects along such maps which allow one to use the techniques of submanifolds or Riemannian submersions for Riemannian maps. The paper also contains several open problems related to the research area.

**Keywords** isometric immersion · Riemannian submersion · Riemannian map · Harmonic map · biharmonic map

**Mathematics Subject Classification (2010)** 53C40 · 53B20 · 53C43

### 1 Introduction

Smooth maps between Riemannian manifolds are useful for comparing geometric structures between two manifolds. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds and they are characterized by their Riemannian metrics and Jacobian matrices. More precisely, a smooth map  $F : (M, g_M) \rightarrow (N, g_N)$  between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$  is called an isometric immersion (submanifold) if  $F_*$  is injective and

$$g_N(F_*X, F_*Y) = g_M(X, Y), \quad (1.1)$$

for vector fields  $X, Y$  tangent to  $M$ , here  $F_*$  denotes the derivative map. A smooth map  $F : (M, g_M) \rightarrow (N, g_N)$  is called a Riemannian submersion if  $F_*$  is onto and it satisfies the equation (1.1) for vector fields tangent to the horizontal space  $(\ker F_*)^\perp$ . Riemannian submersions between Riemannian manifolds were studied by O'NEILL [27] and GRAY [18], see also [14]. We note that Riemannian submersions have their applications in spacetime of unified theory. In the theory of Klauza-Klein type, a general solution of the non-linear sigma model is given by Riemannian submersions

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from the extra dimensional space to the space in which the scalar fields of the nonlinear sigma model take values (for details, see [14]).

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [15] as a generalization of the notions of isometric immersions and Riemannian submersions. Let  $F : (M, g_M) \rightarrow (N, g_N)$  be a smooth map between Riemannian manifolds such that  $0 < \text{rank } F < \min\{m, n\}$ , where  $\dim M = m$  and  $\dim N = n$ . Then we denote the kernel space of  $F_*$  by  $\ker F_*$  and consider the orthogonal complementary space  $\mathcal{H} = (\ker F_*)^\perp$  to  $\ker F_*$ . Then the tangent bundle of  $M$  has the following decomposition

$$TM = \ker F_* \oplus \mathcal{H}.$$

We denote the range of  $F_*$  by  $\text{range } F_*$  and consider the orthogonal complementary space  $(\text{range } F_*)^\perp$  to  $\text{range } F_*$  in the tangent bundle  $TN$  of  $N$ . Since  $\text{rank } F < \min\{m, n\}$ , we always have  $(\text{range } F_*)^\perp \neq \{0\}$ . Thus the tangent bundle  $TN$  of  $N$  has the following decomposition

$$TN = (\text{range } F_*) \oplus (\text{range } F_*)^\perp.$$

Now, a smooth map  $F: (M^m, g_M) \rightarrow (N^n, g_N)$  is called Riemannian map at  $p_1 \in M$  if the horizontal restriction  $F_{*p_1}^h: (\ker F_{*p_1})^\perp \rightarrow (\text{range } F_{*p_1})$  is a linear isometry between the inner product spaces  $((\ker F_{*p_1})^\perp, g_M(p_1)|_{(\ker F_{*p_1})^\perp})$  and  $(\text{range } F_{*p_1}, g_N(p_2)|_{(\text{range } F_{*p_1})})$ ,  $p_2 = F(p_1)$ . Therefore Fischer stated in [15] that a Riemannian map is a map which is as isometric as it can be. In another words,  $F_*$  satisfies the equation (1.1) for  $X, Y$  vector fields tangent to  $\mathcal{H}$ . It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with  $\ker F_* = \{0\}$  and  $(\text{range } F_*)^\perp = \{0\}$ . It is known that a Riemannian map is a subimmersion which implies that the rank of the linear map  $F_{*p} : T_p M \rightarrow T_{F(p)} N$  is constant for  $p$  in each connected component of  $M$  (see [1] and [15]). A remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation  $\| F_* \|^2 = \text{rank } F$  which is a bridge between geometric optics and physical optics. Since the left hand side of this equation is continuous on the Riemannian manifold  $M$  and since  $\text{rank } F$  is an integer valued function, this equality implies that  $\text{rank } F$  is locally constant and globally constant on connected components. Thus if  $M$  is connected, the energy density  $e(F) = \frac{1}{2} \| F_* \|^2$  is quantized to integer and half-integer values. The eikonal equation of geometrical optics solved by using Cauchy's method of characteristics, whereby, for real valued functions  $F$ , solutions to the partial differential equation  $\| dF \|^2 = 1$  are obtained by solving the system of ordinary differential equations  $x' = \text{grad} f(x)$ . Since harmonic maps generalize geodesics, harmonic maps could be used to solve the generalized eikonal equation (see [15]).

In [15], FISCHER also proposed an approach to build a quantum model and he pointed out the succes of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps and Langrangian field theory on the mathematical side, and Maxwell's equation, Shrödinger's equation and their proposed generalization on the physical side.

Riemannian maps between semi-Riemannian manifolds have been defined in [21] by putting some regularity conditions. On the other hand, affine Riemannian maps have been also investigated and decomposition theorems related to Riemannian maps

and curvatures are obtained in [16] (For Riemannian maps and their applications in spacetime geometry, see [17].)

Recently, the present author studied Riemannian maps between almost Hermitian manifolds and Riemannian manifolds and defined invariant, anti-invariant and semi-invariant Riemannian maps. To obtain new results for such Riemannian maps, we construct Gauss-Weingarten formulas for Riemannian maps and obtain various properties by using these new formulas. Therefore it is needed to collect these results and give the techniques used in those papers. It seems that Riemannian maps deserve further investigation and they may become a research area like isometric immersions or Riemannian submersions.

In this paper, we develop certain geometric structures along a Riemannian map to investigate the geometry of such maps. More precisely we defined Gauss and Weingarten formulas for Riemannian maps by using pullback connection and the connection defined in [26]. Then we use these formulas to define totally umbilical Riemannian maps and pseudo umbilical Riemannian maps, and give characterizations. We also obtain Gauss, Codazzi and Ricci equations for Riemannian maps and obtain necessary and sufficient conditions for Riemannian maps to be totally geodesic, harmonic and biharmonic.

The paper is organized as follows. In section 2, we recall basic facts for Riemannian maps, give examples and obtain characterizations of Riemannian maps. In section 3, we construct Gauss-Weingarten formulas for Riemannian maps and obtain the equations of Gauss, Codazzi and Ricci. In section 4, we obtain necessary and sufficient conditions for Riemannian maps to be totally geodesic. In section 5, we define totally umbilical Riemannian maps and pseudo-umbilical Riemannian maps, obtain characterizations and give methods how to obtain such maps. In section 6, we investigate the harmonicity and biharmonicity of Riemannian maps. In the last section, we propose ten open problems by giving certain background information for each problem.

## 2 Riemannian maps

In this section we give formal definition of Riemannian maps, present examples and obtain a characterization of such maps. Since every Riemannian map is a subimmersion, we show the relations among Riemannian maps, isometric immersions and Riemannian submersions, and we also show that Riemannian maps satisfy the eikonal equation.

**Definition 2.1** ([15]) Let  $(M^m, g_M)$  and  $(N^n, g_N)$  be Riemannian manifolds and  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  a smooth map between them. Then we say that  $F$  is a Riemannian map at  $p_1 \in M$  if  $0 < \text{rank } F_{*p_1} \leq \min\{m, n\}$  and  $F_{*p_1}$  maps the horizontal space  $\mathcal{H}(p_1) = (\ker(F_{*p_1}))^\perp$  isometrically onto  $\text{range}(F_{*p_1})$ , i.e.,

$$g_N(F_{*p_1}X, F_{*p_1}Y) = g_M(X, Y), \tag{2.1}$$

for  $X, Y \in \mathcal{H}(p_1)$ . Also  $F$  is called Riemannian if  $F$  is Riemannian at each  $p_1 \in M$ , (see, Figure 1).

We give some examples of Riemannian maps:

*Example 1.* Let  $I : (M^m, g_M) \rightarrow (N^n, g_N)$  be an isometric immersion between Riemannian manifolds. Then  $I$  is a Riemannian map with  $\ker F_* = \{0\}$ .

*Example 2.* Let  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  be a Riemannian submersion between Riemannian manifolds. Then  $F$  is a Riemannian map with  $(\text{range } F_*)^\perp = \{0\}$ .

*Example 3.* Consider the following map defined by

$$F : \begin{matrix} R^5 \\ (x_1, x_2, x_3, x_4, x_5) \end{matrix} \longrightarrow \begin{matrix} R^4 \\ (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_3+x_4}{\sqrt{2}}, x_5, 0) \end{matrix}.$$

Then we have

$$\ker F_* = \text{span}\{Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, Z_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\}$$

and

$$(\ker F_*)^\perp = \text{span}\{Z_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Z_4 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, Z_5 = \frac{\partial}{\partial x_5}\}.$$

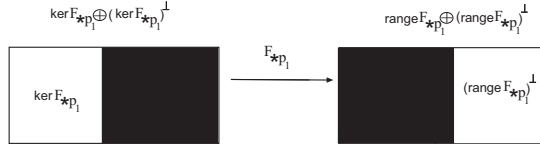
Hence it is easy to see that

$$g_{R^4}(F_*(Z_i), F_*(Z_i)) = g_{R^5}(Z_i, Z_i) = 2, g_{R^4}(F_*(Z_5), F_*(Z_5)) = g_{R^5}(Z_5, Z_5) = 1$$

and

$$g_{R^4}(F_*(Z_i), F_*(Z_j)) = g_{R^5}(Z_i, Z_j) = 0,$$

$i \neq j$ , for  $i, j = 3, 4, 5$ . Thus  $F$  is a Riemannian map.



**Fig. 1** Riemannian map, the grayscale regions are mapped isometrically to each other by  $F$  and they have the same area. The white areas are independent of each other and may have any areas (see [15]).

We recall the adjoint map of a map. Let  $F : (M, g_M) \rightarrow (N, g_N)$  be a map between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$ . Then the adjoint map  $*F_*$  of  $F_*$  is characterized by  $g_M(x, *F_{*p_1}y) = g_N(F_{*p_1}x, y)$  for  $x \in T_{p_1}M$ ,  $y \in T_{F(p_1)}N$  and  $p_1 \in M$ . Let  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  be a smooth map between Riemannian manifolds. Define linear transformations

$$\begin{aligned} \mathcal{P}_{p_1} : T_{p_1}M &\longrightarrow T_{p_1}M, \mathcal{P}_{p_1} = *F_{*p_1} \circ F_{*p_1} \\ \mathcal{Q}_{p_1} : T_{p_2}N &\longrightarrow T_{p_2}N, \mathcal{Q}_{p_1} = F_{*p_1} \circ *F_{*p_1}, \end{aligned}$$

where  $p_2 = F(p_1)$ . Using these linear transformations, we obtain the following characterizations of Riemannian maps.

**Theorem 2.2** ([15]) *Let  $F : (M, g_M) \rightarrow (N, g_N)$  be a map between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$ . Then the following are equivalent:*

- (1)  $F$  is Riemannian at  $p_1 \in M$ .
- (2)  $\mathcal{P}_{p_1}$  is projection, i.e.,  $\mathcal{P}_{p_1} \circ \mathcal{P}_{p_1} = \mathcal{P}_{p_1}$ .
- (3)  $\mathcal{Q}_{p_1}$  is projection, i.e.,  $\mathcal{Q}_{p_1} \circ \mathcal{Q}_{p_1} = \mathcal{Q}_{p_1}$ .

We now recall that a map  $F : (M^m, g_M) \rightarrow N^n$  is called subimmersion at  $p_1 \in M$  if there is a neighborhood  $U$  of  $p_1$ , a manifold  $P$ , a submersion  $S : U \rightarrow P$  and an immersion  $I : P \rightarrow N$  such that  $F|_U = F_U = I \circ S$ . A map  $F : (M^m, g_M) \rightarrow N^n$  is called subimmersion if it is subimmersion at each  $p_1 \in M$ . It is well known that  $F : (M^m, g_M) \rightarrow N^n$  is a subimmersion if and only if the rank of the linear map  $F_{*p_1} : T_{p_1}M \rightarrow T_{F(p_1)}N$  is constant for  $p_1$  in each connected component of  $M$  (see [1]), where  $M$  and  $N$  are finite dimensional manifolds. Thus by the definition, a Riemannian map is a subimmersion. Moreover we have the following:

**Theorem 2.3** ([15], [17]) *Let  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  be a Riemannian map between Riemannian manifolds  $(M^m, g_M)$  and  $(N^n, g_N)$ . Let  $U, P, S$  and  $I$  be as in the definition of subimmersion so that  $F|_U = F_U = I \circ S$ . Let  $g_{M_U}$  denotes the restriction of  $g_M$  to  $U$  and let  $g_P = I^*g_N$ . Then  $(U, g_{M_U})$  and  $(P, g_P)$  are Riemannian manifolds, the submersion  $S : (U, g_{M_U}) \rightarrow (P, g_P)$  is a Riemannian submersion and the immersion  $I : (P, g_P) \rightarrow (N^n, g_N)$  is an isometric immersion.*

Next result shows that Riemannian maps satisfy a generalized eikonal equation which is a bridge between physical optics and geometric optics.

**Theorem 2.4** ([15], [17]) *Let  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  be a Riemannian map between Riemannian manifolds  $(M^m, g_M)$  and  $(N^n, g_N)$ . Then  $\|F_*\|^2 = \text{rank } F$ .*

### 3 Geometric structures along Riemannian maps

In differential geometry of submanifolds, there is a set of equations that describe relationships between invariant quantities on the submanifold and ambient manifold when the Riemannian connection is used. These relationships are expressed by the Gauss formula, Weingarten formula, and the equations of Gauss, Codazzi and Ricci. These equations have such an important role for the Riemannian connection that they are called the fundamental equations for submanifolds. In this section, we extend these equations to Riemannian maps between Riemannian manifolds. To do this, we recall the pullback connection along a map and show that the second fundamental form of a Riemannian map has a special form. Then we define Gauss formula for Riemannian maps by using the second fundamental form of a Riemannian map. We also obtain Weingarten formula for Riemannian maps by using the linear connection  $\nabla^{F^\perp}$  on  $(F_*(TM))^\perp$  defined in [26]. From Gauss-Weingarten formulas, we obtain Gauss, Ricci and Codazzi equations for Riemannian maps.

#### 3.1 Gauss-Weingarten formulas for Riemannian maps

Let  $(M, g_M)$  and  $(N, g_N)$  be Riemannian manifolds and suppose that  $F : M \rightarrow N$  is a smooth map between them. Let  $p_2 = F(p_1)$  for each  $p_1 \in M$ . Suppose that  $\nabla^N$  is

the Levi-Civita connection on  $(N, g_N)$ . For  $X \in \Gamma(TM)$  and  $V \in \Gamma(TN)$ , we have

$$\nabla^F_X (V \circ F) = \nabla^N_{F_*X} V,$$

where  $\nabla^F$  is the pullback connection of  $\nabla^N$ . The differential  $F_*$  of  $F$  can be viewed as section of the bundle  $Hom(TM, F^{-1}TN) \rightarrow M$ , where  $F^{-1}TN$  is the pullback bundle which has fibres  $(F^{-1}TN)_p = T_{F(p)}N, p \in M$ .  $Hom(TM, F^{-1}TN)$  has a connection  $\nabla$  induced from the Levi-Civita connection  $\nabla^M$  and the pullback connection. Then the second fundamental form of  $F$  is given by

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*(Y) - F_*(\nabla^M_X Y), \quad (3.1)$$

for  $X, Y \in \Gamma(TM)$ . It is known that the second fundamental form is symmetric. First note that in [31] we showed that the second fundamental form  $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((\ker F_*)^\perp)$ , of a Riemannian map has no components in  $\text{range } F_*$ . More precisely we have the following.

**Lemma 3.1** *Let  $F$  be a Riemannian map from a Riemannian manifold  $(M, g_M)$  to a Riemannian manifold  $(N, g_N)$ . Then*

$$g_N((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((\ker F_*)^\perp).$$

As a result of 3.1, we have

$$(\nabla F_*)(X, Y) \in \Gamma((\text{range } F_*)^\perp), \forall X, Y \in \Gamma((\ker F_*)^\perp). \quad (3.2)$$

Thus at  $p \in M$ , we write

$$\nabla^F_X F_*(Y)(p) = F_*(\nabla^M_X Y)(p) + (\nabla F_*)(X, Y)(p), \quad (3.3)$$

for  $X, Y \in \Gamma((\ker F_*)^\perp)$ , where  $\nabla^F_X F_*(Y) \in T_F(p)N$ ,  $F_*(\nabla^M_X Y)(p) \in F_{*p}(T_pM)$  and  $(\nabla F_*)(X, Y)(p) \in (F_{*p}(T_pM))^\perp$ .

Let  $F$  be a Riemannian map from a Riemannian manifold  $(M, g_M)$  to a Riemannian manifold  $(N, g_N)$ . Then we define  $\mathcal{T}$  and  $\mathcal{A}$  as

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E} \mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E} \mathcal{H}F \quad (3.4)$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E} \mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E} \mathcal{H}F, \quad (3.5)$$

for vector fields  $E, F$  on  $M$ , where  $\nabla^M$  is the Levi-Civita connection of  $g_M$ . In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any  $E \in \Gamma(TM)$ ,  $\mathcal{T}_E$  and  $\mathcal{A}_E$  are skew-symmetric operators on  $(\Gamma(TM), g)$  reversing the horizontal and the vertical distributions. It is also easy to see that  $\mathcal{T}$  is vertical,  $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$  and  $\mathcal{A}$  is horizontal,  $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$ . We note that the tensor field  $\mathcal{T}$  satisfies

$$\mathcal{T}_U W = \mathcal{T}_W U, \forall U, W \in \Gamma(\ker F_*). \quad (3.6)$$

On the other hand, from (3.4) and (3.5) we have

$$\nabla_V^M W = \mathcal{T}_V W + \hat{\nabla}_V W, \quad (3.7)$$

$$\nabla_V^M X = \mathcal{H}\nabla_V^M X + \mathcal{T}_V X, \quad (3.8)$$

$$\nabla_X^M V = \mathcal{A}_X V + \mathcal{V}\nabla_X^M V, \quad (3.9)$$

$$\nabla_X^M Y = \mathcal{H}\nabla_X^M Y + \mathcal{A}_X Y, \quad (3.10)$$

for  $X, Y \in \Gamma((\ker F_*)^\perp)$  and  $V, W \in \Gamma(\ker F_*)$ , where  $\hat{\nabla}_V W = \mathcal{V}\nabla_V^M W$ .

From now on, for simplicity, we denote by  $\nabla^N$  both the Levi-Civita connection of  $(N, g_N)$  and its pullback along  $F$ . Then according to [26], for any vector field  $X$  on  $M$  and any section  $V$  of  $(\text{range } F_*)^\perp$ , where  $(\text{range } F_*)^\perp$  is the subbundle of  $F^{-1}(TN)$  with fiber  $(F_*(T_{p_1}M))^\perp$ -orthogonal complement of  $F_*(T_{p_1}M)$  for  $g_N$  over  $p_1$ , we have  $\nabla_X^{F^\perp} V$  which is the orthogonal projection of  $\nabla_X^N V$  on  $(F_*(TM))^\perp$ . In [26], the author also showed that  $\nabla^{F^\perp}$  is a linear connection on  $(F_*(TM))^\perp$  such that  $\nabla^{F^\perp} g_N = 0$ . We now suppose that  $F$  is a Riemannian map and define  $\mathcal{S}_V$  as

$$\nabla_{F_*X}^N V = -\mathcal{S}_V F_*X + \nabla_X^{F^\perp} V, \quad (3.11)$$

where  $\mathcal{S}_V F_*X$  is the tangential component (a vector field along  $F$ ) of  $\nabla_{F_*X}^N V$ . Observe that  $\nabla_{F_*X}^N V$  is obtained from the pullback connection of  $\nabla^N$ . Thus, at  $p_1 \in M$ , we have  $\nabla_{F_*X}^N V(p_1) \in T_{F(p_1)}N$ ,  $\mathcal{S}_V F_*X(p_1) \in F_{*p_1}(T_{p_1}M)$  and  $\nabla_X^{F^\perp} V(p_1) \in (F_{*p_1}(T_{p_1}M))^\perp$ . It is easy to see that  $\mathcal{S}_V F_*X$  is bilinear in  $V$  and  $F_*X$  and  $\mathcal{S}_V F_*X$  at  $p_1$  depends only on  $V_{p_1}$  and  $F_{*p_1}X_{p_1}$ . By direct computations, we obtain

$$g_N(\mathcal{S}_V F_*X, F_*Y) = g_N(V, (\nabla F_*)(X, Y)), \quad (3.12)$$

for  $X, Y \in \Gamma((\ker F_*)^\perp)$  and  $V \in \Gamma((\text{range } F_*)^\perp)$ . Since  $(\nabla F_*)$  is symmetric, it follows that  $\mathcal{S}_V$  is a symmetric linear transformation of  $\text{range } F_*$ .

As in case of submanifolds, we call (3.3), (3.7)-(3.10) the Gauss formulae and (3.11) the Weingarten formula for the Riemannian map  $F$ .

### 3.2 The equations of Ricci, Gauss and Codazzi

In this subsection, we are going to obtain the equations of Gauss, Codazzi and Ricci for Riemannian maps. We first note that if  $F$  is a Riemannian map from a Riemannian manifold  $(M, g_M)$  to a Riemannian manifold  $(N, g_N)$ , then considering  $F_*^h$  at each  $p_1 \in M$  as a linear transformation

$$F_{*p_1}^h : ((\ker F_*)^\perp(p_1), g_{M_{p_1}}((\ker F_*)^\perp(p_1))) \longrightarrow (\text{range } F_*(p_2), g_{N_{p_2}}(\text{range } F_*(p_2))),$$

we will denote the adjoint of  $F_*^h$  by  $*F_{*p_1}^h$ . Let  $*F_{*p_1}$  be the adjoint of  $F_{*p_1} : (T_{p_1}M, g_{M_{p_1}}) \longrightarrow (T_{p_2}N, g_{N_{p_2}})$ . Then the linear transformation

$$(*F_{*p_1})^h : \text{range } F_*(p_2) \longrightarrow (\ker F_*)^\perp(p_1)$$

defined by  $({}^*F_{*p_1})^h y = {}^*F_{*p_1} y$ , where  $y \in \Gamma(\text{range } F_{*p_1})$ ,  $p_2 = F(p_1)$ , is an isomorphism and  $(F_{*p_1}^h)^{-1} = ({}^*F_{*p_1})^h = ({}^*F_{*p_1}^h)$ .

By using (3.3) and (3.11) we have

$$\begin{aligned} R^N(F_*X, F_*Y)F_*Z &= -\mathcal{S}_{(\nabla F_*)(Y,Z)}F_*X + \mathcal{S}_{(\nabla F_*)(X,Z)}F_*Y \\ &\quad + F_*(R^M(X, Y)Z) + (\nabla_X(\nabla F_*))(Y, Z) \\ &\quad - (\nabla_Y(\nabla F_*))(X, Z), \end{aligned} \quad (3.13)$$

for  $X, Y, Z \in \Gamma((\ker F_*)^\perp)$ , where  $R^M$  and  $R^N$  denote curvature tensors of  $\nabla^M$  and  $\nabla^N$  which are metric connections on  $M$  and  $N$ , respectively. Moreover,  $(\nabla_X(\nabla F_*))(Y, Z)$  is defined by

$$(\nabla_X(\nabla F_*))(Y, Z) = \nabla_X^{F^\perp}(\nabla F_*)(Y, Z) - (\nabla F_*)(\nabla_X^M Y, Z) - (\nabla F_*)(Y, \nabla_X^M Z).$$

From (3.13), for any vector field  $T \in \Gamma((\ker F_*)^\perp)$ , we have

$$\begin{aligned} g_N(R^N(F_*X, F_*Y)F_*Z, F_*T) &= g_M(R^M(X, Y)Z, T) \\ &\quad + g_N((\nabla F_*)(X, Z), (\nabla F_*)(Y, T)) \\ &\quad - g_N((\nabla F_*)(Y, Z), (\nabla F_*)(X, T)). \end{aligned} \quad (3.14)$$

Taking the  $\Gamma((\text{range } F_*)^\perp)$ - part of (3.13), we have

$$(R^N(F_*X, F_*Y)F_*Z)^\perp = (\nabla_X(\nabla F_*))(Y, Z) - (\nabla_Y(\nabla F_*))(X, Z). \quad (3.15)$$

We call (3.14) and (3.15) the Gauss equation and the Codazzi equation, respectively, for the Riemannian map  $F$ .

For any vector fields  $X, Y$  tangent to  $M$  and any vector field  $V$  perpendicular to  $\Gamma(\text{range } F_*)$ , we define the curvature tensor field  $R^{F^\perp}$  of the subbundle  $(\text{range } F_*)^\perp$  by

$$R^{F^\perp}(F_*(X), F_*(Y))V = \nabla_X^{F^\perp} \nabla_Y^{F^\perp} V - \nabla_Y^{F^\perp} \nabla_X^{F^\perp} V - \nabla_{[X,Y]}^{F^\perp} V. \quad (3.16)$$

Then using (3.3), (3.11) and (3.12) we obtain

$$\begin{aligned} R^N(F_*(X), F_*(Y))V &= R^{F^\perp}(F_*(X), F_*(Y))V \\ &\quad - F_*(\nabla_X^M {}^*F_*(\mathcal{S}_V F_*(Y))) + \mathcal{S}_{\nabla_X^{F^\perp} V} F_*(Y) \\ &\quad + F_*(\nabla_Y^M {}^*F_*(\mathcal{S}_V F_*(X))) - \mathcal{S}_{\nabla_Y^{F^\perp} V} F_*(X) \\ &\quad - (\nabla F_*)(X, {}^*F_*(\mathcal{S}_V F_*(Y))) + (\nabla F_*)(Y, {}^*F_*(\mathcal{S}_V F_*(X))) \\ &\quad - \mathcal{S}_V F_*([X, Y]), \end{aligned} \quad (3.17)$$

where

$$F_*([X, Y]) = \nabla_X^N F_*(Y) - \nabla_Y^N F_*(X)$$

and  ${}^*F_*$  is the adjoint map of  $F_*$ . Then for  $F_*(Z) \in \Gamma(\text{range } F_*)$ , we have

$$\begin{aligned} g_N(R^N(F_*(X), F_*(Y))V, F_*(Z)) &= g_N((\tilde{\nabla}_Y \mathcal{S})_V F_*(X), F_*(Z)) \\ &\quad - g_N((\tilde{\nabla}_X \mathcal{S})_V F_*(Y), F_*(Z)), \end{aligned} \quad (3.18)$$



where  $(\tilde{\nabla}_X \mathcal{S})_V F_*(Y)$  is defined by

$$(\tilde{\nabla}_X \mathcal{S})_V F_*(Y) = F_*(\nabla_X^M {}^*F_*(\mathcal{S}_V F_*(Y))) - \mathcal{S}_{\nabla_X^{F^\perp} V} F_*(Y) - \mathcal{S}_V P \overset{N}{\nabla^F}_X F_*(Y),$$

where  $P$  denotes the projection morphism on  $\text{range } F_*$ . On the other hand, for  $W \in \Gamma((\text{range } F_*)^\perp)$  we get

$$\begin{aligned} g_N(R^N(F_*(X), F_*(Y))V, W) &= g_N(R^{F^\perp}(F_*(X), F_*(Y))V, W) \\ &\quad - g_N((\nabla F_*)(X, {}^*F_*(\mathcal{S}_V F_*(Y))), W) \\ &\quad + g_N((\nabla F_*)(Y, {}^*F_*(\mathcal{S}_V F_*(X))), W). \end{aligned} \quad (3.19)$$

Using (3.12) we obtain

$$g_N((\nabla F_*)(X, {}^*F_*(\mathcal{S}_V F_*(Y))), W) = g_N(\mathcal{S}_W F_*(X), \mathcal{S}_V F_*(Y)).$$

Since  $\mathcal{S}_V$  is self-adjoint, we get

$$g_N((\nabla F_*)(X, {}^*F_*(\mathcal{S}_V F_*(Y))), W) = g_N(\mathcal{S}_V \mathcal{S}_W F_*(X), F_*(Y)). \quad (3.20)$$

Then using (3.20) in (3.19), we arrive at

$$\begin{aligned} g_N(R^N(F_*(X), F_*(Y))V, W) &= g_N(R^{F^\perp}(F_*(X), F_*(Y))V, W) \\ &\quad + g_N([\mathcal{S}_W, \mathcal{S}_V]F_*(X), F_*(Y)), \end{aligned} \quad (3.21)$$

where

$$[\mathcal{S}_W, \mathcal{S}_V] = \mathcal{S}_W \mathcal{S}_V - \mathcal{S}_V \mathcal{S}_W.$$

We call (3.21) the Ricci equation for the Riemannian map  $F$ .

#### 4 Totally geodesic Riemannian maps

In this section, we first give necessary and sufficient conditions for a Riemannian map to be totally geodesic in terms of (3.4) and (3.11), and then we state the results given in [16]. We recall that a differentiable map  $F$  between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$  is called a totally geodesic map if  $(\nabla F_*)(X, Y) = 0$  for all  $X, Y \in \Gamma(TM)$ .

**Theorem 4.1** ([34]) *Let  $F$  be a Riemannian map from a Riemannian manifold  $(M, g_M)$  to a Riemannian manifold  $(N, g_N)$ . Then  $F$  is totally geodesic if and only if:*

- (a)  $\mathcal{A}_X Y = 0$ ,
- (b)  $\mathcal{S}_V F_*(X) = 0$ ,
- (c) the fibers are totally geodesic,

for  $X, Y \in \Gamma((\ker f_*)^\perp)$  and  $V \in \Gamma((\text{range } F_*)^\perp)$ .

We also have the following characterizations for totally geodesic (or affine) Riemannian maps.

**Theorem 4.2** ([16]) *Let  $(M, g_M)$  be a connected Riemannian manifold with Ricci curvature  $Ric_M \geq A$  and let  $(N, g_N)$  be a Riemannian manifold with sectional curvature  $K_N \leq B$ . If  $F : M \rightarrow N$  is a Riemannian map with  $\text{rank } F \geq 2$ ,  $A \geq (\text{rank } f - 1)B$  and  $\text{div}\tau(F) \geq 0$ , then  $F$  is a totally geodesic map.*

In particular if  $\text{rank } F = 1$ , then we have:

**Theorem 4.3** ([16]) *Let  $(M, g_M)$  be a connected Riemannian manifold with Ricci curvature  $Ric_M \geq 0$  and let  $(N, g_N)$  be a Riemannian manifold. If  $F : M \rightarrow N$  is a Riemannian map with  $\text{rank } F = 1$  and  $\text{div}\tau(F) \geq 0$ , then  $F$  is a totally geodesic map, and hence  $F$  maps  $M$  into a geodesic of  $N$ .*

Since isometric immersions and Riemannian submersions are also Riemannian maps. We have the following special cases.

**Corollary 4.4** ([16]) *Let  $(M, g_M)$  be a connected Riemannian manifold with  $\dim(M) = m \geq 2$  and scalar curvature  $S_M \geq A$  and let  $(N, g_N)$  be a Riemannian manifold with sectional curvature  $K_N \leq B$  such that  $A \geq (m - 1)B$ . If  $F : M \rightarrow N$  is an isometric immersion with  $\text{div}\tau(F) \geq 0$ , then  $F$  is a totally geodesic map.*

For Riemannian submersions, above theorem gives the following result.

**Corollary 4.5** ([16]) *Let  $(M, g_M)$  be a connected Riemannian manifold with Ricci curvature  $Ric_M \geq A$  and let  $(N, g_N)$  be a Riemannian manifold with  $\dim N = n$  and scalar curvature  $r_N \leq B$  such that  $nA \geq B$ . If  $F : M \rightarrow N$  is a Riemannian submersion with  $\text{div}\tau(F) \geq 0$ , then  $F$  is a totally geodesic map.*

Also by combining Corollary 4.5 with a corollary of VILMS [38, Corollary 3.7], we obtain the following splitting theorem for Riemannian manifolds:

**Theorem 4.6** ([16]) *Let  $(M^m, g_M)$  be a simply connected, connected, complete Riemannian manifold with Ricci curvature  $Ric_M \geq A$  and let  $(N^n, g_N)$  be a Riemannian manifold with scalar curvature  $r_N \leq B$  such that  $m > n$  and  $nA \geq B$ . If there is a surjective Riemannian map  $F : M \rightarrow N$  with  $\text{div}\tau(F) \geq 0$ , then  $M$  is a Riemannian product and  $F$  is a projection.*

## 5 Umbilical Riemannian maps

In this section we study umbilical Riemannian maps. Umbilical maps have been defined in [26] and [36]. In fact, in [26] the author gave many definitions of umbilical maps with respect to the Riemannian metrics of the source manifolds and target manifolds. But we note that the definition given in [36] is same with the definition of  $g$ -umbilicity map given in [26]. First note that for the tension field of a Riemannian map between Riemannian manifolds, we have the following.

**Lemma 5.1** ([30]) *Let  $F : (M, g_M) \rightarrow (N, g_N)$  be a Riemannian map between Riemannian manifolds. Then the tension field  $\tau$  of  $F$  is*

$$\tau = -m_1 F_*(\mu^{\ker F_*}) + m_2 H_2, \quad (5.1)$$

where  $m_1 = \dim((\ker F_*))$ ,  $m_2 = \text{rank } F$ ,  $\mu^{\ker F_*}$  and  $H_2$  are the mean curvature vector fields of the distribution  $\ker F_*$  and  $\text{range } F_*$ , respectively.

We now recall that a map  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  between Riemannian manifolds is called umbilical (see [36]) if

$$\nabla F_* = \frac{1}{m} g_M \otimes \tau. \quad (5.2)$$

We first show that this definition of umbilical maps has some restrictions for Riemannian maps.

**Theorem 5.2 ([32])** *Every umbilical (in the sense of [36]) Riemannian map between Riemannian manifolds is harmonic. As a result of this, it is totally geodesic.*

In fact, the above theorem tells that the definition of umbilical map does not work for Riemannian maps. Therefore we present the following definition.

**Definition 5.3 ([32])** Let  $F$  be a Riemannian map between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$ . Then we say that  $F$  is an umbilical Riemannian map at  $p_1 \in M$  if

$$\mathcal{S}_V F_* p_1(X_{p_1}) = \lambda F_* p_1(X_{p_1}), \quad (5.3)$$

for  $X \in \Gamma(\text{range } F_*)$  and  $V \in \Gamma((\text{range } F_*)^\perp)$ , where  $\lambda$  is a differentiable function on  $M$ . If  $F$  is umbilical for every  $p_1 \in M$  then we say that  $F$  is an umbilical Riemannian map.

The above definition is same with that given for isometric immersions. By using (3.12) and (5.3) we have the following lemma.

**Lemma 5.4 ([32])** *Let  $F$  be a Riemannian map between Riemannian manifolds  $(M, g_M)$  and  $(N, g_N)$ . Then  $F$  is an umbilical Riemannian map if and only if*

$$(\nabla F_*)(X, Y) = g_M(X, Y)H_2, \quad (5.4)$$

for  $X, Y \in \Gamma((\ker F_*)^\perp)$ .

The following proposition implies that it is easy to find examples of umbilical Riemannian maps if one has examples of totally umbilical isometric immersions and Riemannian submersions.

**Proposition 5.5 ([32])** *Let  $F_1$  be a Riemannian submersion from a Riemannian manifold  $(M, g_M)$  onto a Riemannian manifold  $(N, g_N)$  and  $F_2$  a totally umbilical isometric immersion from the Riemannian manifold  $(N, g_N)$  to a Riemannian manifold  $(\bar{N}, g_{\bar{N}})$ . Then the Riemannian map  $F_2 \circ F_1$  is an umbilical Riemannian map from  $(M, g_M)$  to  $(\bar{N}, g_{\bar{N}})$ .*

We now define pseudo-umbilical Riemannian maps as a generalization of pseudo-umbilical isometric immersions. Pseudo-umbilical Riemannian maps will be useful when we deal with the biharmonicity of Riemannian maps.

**Definition 5.6 ([33])** Let  $F : (M, g_M) \rightarrow (N, g_N)$  be a Riemannian map between Riemannian manifolds  $M$  and  $N$ . Then we say that  $F$  is a *pseudo-umbilical Riemannian map* if

$$\mathcal{S}_{H_2} F_*(X) = \lambda F_*(X), \quad (5.5)$$

for  $\lambda \in C^\infty(M)$  and  $X \in \Gamma((\ker F_*)^\perp)$ .

Here we present an useful formula for pseudo umbilical Riemannian maps by using (3.12) and (5.5).

**Proposition 5.7 ([33])** *Let  $F : (M, g_M) \longrightarrow (N, g_N)$  be a Riemannian map between Riemannian manifolds  $M$  and  $N$ . Then  $F$  is pseudo-umbilical if and only if*

$$g_N((\nabla F_*)(X, Y), H_2) = g_M(X, Y)g_N(H_2, H_2), \quad (5.6)$$

for  $X, Y \in \Gamma((\ker F_*)^\perp)$ .

It is known that the composition of a Riemannian submersion and an isometric immersion is a Riemannian map (see [15]). Using this we have the following.

**Theorem 5.8 ([33])** *Let  $F_1 : (M, g_M) \longrightarrow (N, g_N)$  be a Riemannian submersion and  $F_2 : (N, g_N) \longrightarrow (\bar{N}, g_{\bar{N}})$  a pseudo-umbilical isometric immersion. Then the map  $F_2 \circ F_1$  is a pseudo umbilical Riemannian map.*

**Remark 5.1** We note that above theorem gives a method to find examples of pseudo umbilical Riemannian maps. It also tells that if one has an example of pseudo-umbilical submanifolds, it is possible to find an example of pseudo umbilical Riemannian maps. For examples of pseudo umbilical submanifolds (see [9]).

## 6 Harmonicity of Riemannian maps

In this section, we give characterizations for Riemannian maps to be harmonic and biharmonic. We first recall that a map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. Standard arguments yield the associated Euler-Lagrange equation, the vanishing of the tension field:  $\tau(\varphi) = \text{trace}(\nabla\varphi_*)$ . Harmonic maps between Riemannian manifolds satisfy a system of quasi-linear partial differential equations. In order to have existence results one would solve PDE's on certain manifolds. It is known that Riemannian maps need not be harmonic, harmonic maps need not be Riemannian. On the other hand, the biharmonic maps are the critical points of the bienergy functional and, from this point of view, generalize harmonic maps. Let  $\varphi : (M, g_M) \longrightarrow (N, g_N)$  be a smooth map between Riemannian manifolds. Define its bienergy as

$$E^2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g.$$

Critical points of the functional  $E^2$  are called biharmonic maps and its associated Euler-Lagrange equation is the vanishing of the bitension field

$$\tau^2(\varphi) = -\Delta^\varphi \tau(\varphi) - \text{trace}_{g_M} R^N(d\varphi, \tau(\varphi))d\varphi,$$

where  $\Delta^\varphi \tau(\varphi) = -\text{trace}_{g_M} (\nabla^\varphi \nabla^\varphi - \nabla_{\nabla^\varphi}^\varphi)$  is the Laplacian on the sections of  $\varphi^{-1}(TN)$  and  $R^N$  is the Riemann curvature operator on  $(N, g_N)$ . A map between two Riemannian manifolds is said to be proper biharmonic if it is a non-harmonic biharmonic map. The notion of biharmonic map was suggested by EELLS and SAMPSON [13], see also [4]. The first variation formula and, thus, the Euler-Lagrange equation associated to the bienergy was obtained by JIANG in [19], [20]. But biharmonic maps have

been extensively studied in the last decade and there are two main research directions. In differential geometry, many authors have obtained classification results and constructed many examples. Biharmonic maps were obtained in [6], [10], [28] and biharmonic Riemannian submersions were studied in [28], for a survey on biharmonic maps (see [24]). From the analytic point of view, biharmonic maps are solutions of fourth order strongly elliptic semilinear partial differential equations. In this section we obtain necessary and sufficient conditions for Riemannian maps to be harmonic and biharmonic. First from Lemma 5.1, we have the following.

**Theorem 6.1** *Suppose  $F : (M^m, g_M) \rightarrow (N^n, g_N)$  is a non-constant Riemannian map between Riemannian manifolds. Then any two conditions below imply the third:*

- (i)  $F$  is harmonic,
- (ii) the distribution  $\ker F_*$  is minimal,
- (iii) the distribution  $\text{range } F_*$  is minimal.

For the biharmonicity of Riemannian maps, we have the following result.

**Theorem 6.2 ([33])** *Let  $F$  be a Riemannian map from a Riemannian manifold  $(M, g_M)$  to a space form  $(N(c), g_N)$ . Then  $F$  is biharmonic if and only if*

$$\begin{aligned} & m_1 \text{trace } \mathcal{S}_{(\nabla F_*)(\cdot, \mu^{\ker F_*})} F_*(\cdot) - m_1 \text{trace } F_*(\nabla_{(\cdot)} \nabla_{(\cdot)} \mu^{\ker F_*}) \\ & - m_2 \text{trace } F_*(\nabla_{(\cdot)}^* F_*(\mathcal{S}_{H_2} F_*(\cdot))) - m_2 \text{trace } \mathcal{S}_{\nabla_{(\cdot)}^{F^\perp} H_2} F_*(\cdot) \\ & - m_1 c(m_2 - 1) F_*(\mu^{\ker F_*}) = 0 \end{aligned}$$

and

$$\begin{aligned} & m_1 \text{trace } \nabla_{(\cdot)}^{F^\perp} (\nabla F_*)(\cdot, \mu^{\ker F_*}) + m_1 \text{trace} (\nabla F_*)(\cdot, \nabla_{(\cdot)} \mu^{\ker F_*}) \\ & + m_2 \text{trace} (\nabla F_*)(\cdot, F_*(\mathcal{S}_{H_2} F_*(\cdot))) - m_2 \Delta^{R^\perp} H_2 \\ & - m_2^2 c H_2 = 0. \end{aligned}$$

We also have the following result for pseudo-umbilical Riemannian maps.

**Theorem 6.3 ([33])** *Let  $F$  be a pseudo-umbilical biharmonic Riemannian map from a Riemannian manifold  $(M, g_M)$  to a space form  $(N(c), g_N)$  such that the distribution  $\ker F_*$  is minimal and the mean curvature vector field  $H_2$  is parallel. Then either  $F$  is harmonic or  $c = \|H_2\|^2$ .*

## 7 Open problems

In this section we are going to propose some open problems to the reader.

1. Simons classified compact minimal submanifolds of a sphere into three categories with respect to the length of the second fundamental form and get very important formula with respect to minimal submanifolds. By using Simons's formula, the pinching problem for the length of the second fundamental form of submanifolds in various manifolds (complex, contact) has been studied by many authors and certain conditions for submanifolds to be totally geodesic have been obtained. Since harmonic maps are a generalization of minimal submanifolds and Riemannian maps are a generalization

of isometric immersions, it would be interesting to calculate the length of the second fundamental form of a Riemannian map and obtain Simons's type formula for such maps. Also it would be interesting to use the obtained formula to obtain new conditions for Riemannian maps to be totally geodesic. For Simons's formula and its applications (see [35] and [42]).

2. A Riemannian submersion  $F : (M, g) \rightarrow (B, g')$  is called a Clairaut submersion if there exists a positive function  $r$  on  $M$  such that, for any geodesic  $\alpha$  on  $M$ , the function  $(r \circ \alpha) \sin w$  is constant, where, for any  $t$ ,  $w(t)$  is the angle between  $\dot{\alpha}(t)$  and the horizontal space at  $\alpha(t)$  (see [14]). BISHOP [5] proved the following result as a characterization of Clairaut submersion. Let  $F : (M, g) \rightarrow (B, g')$  be a Riemannian submersion with connected fibres. Then  $F$  is a Clairaut submersion with  $r = e^f$  if and only if each fibre is totally umbilical and has mean curvature vector field  $H = -gradf$ . Since a Riemannian map is a generalization of Riemannian submersion, it would be a new research area to introduce Clairaut Riemannian maps and obtain a characterization of such maps.

3. A Riemannian manifold is called locally symmetric manifold if  $\nabla R = 0$ , where  $\nabla$  is the Levi-Civita connection of Riemannian manifold  $(M, g)$  and  $R$  is its curvature tensor field. Manifolds satisfying this condition are called locally symmetric. This is equivalent with the condition that every local geodesic symmetry is an isometry. It is known that such manifolds are a generalization of space forms. For a  $(0, k)$  tensor field  $T$  on manifold  $M$ ,  $(0, k + 2)$  tensor field  $R \bullet T$  is defined by

$$(R \bullet T)(X_1, \dots, X_k; X, Y) = -T(R(X, Y)X_1, \dots, X_k) \\ - \dots - T(X_1, \dots, R(X, Y)X_k),$$

for  $X, Y, X_1, \dots, X_k \in \Gamma(TM)$ . A Riemannian manifold is called a semi-symmetric manifold if  $R \bullet R = 0$ . Semi-symmetric manifolds are a generalization of locally symmetric manifolds. Similar conditions have been extended to other tensors like Ricci tensor, Weyl tensor etc. Curvature conditions involving tensors of the form  $R \bullet T$  only, are called curvature conditions of semisymmetric type; examples are  $R \bullet R = 0$ ,  $R \bullet Ric = 0$ . Such symmetry conditions were studied for immersions, specially for hypersurfaces. It will be interesting problem to investigate effect of a Riemannian map on the symmetry conditions (see [25], [37]).

4. For a symmetric  $(0, 2)$  tensor field  $G$  on a manifold  $M$ , the endomorphism  $X \wedge_G Y$  is defined by

$$(X \wedge_G Y)Z = G(Z, Y)X - G(Z, X)Y.$$

A Riemannian manifold  $M$  is said to be pseudosymmetric if at every point of  $M$  the tensor fields  $R \bullet R$  and  $Q(g, R)$  are linearly dependent, where  $Q(g, R)$  is defined as

$$Q(g, R)(X_1, \dots, X_k; X, Y) = -R((X \wedge_g Y)X_1, \dots, X_k) \\ - \dots - R(X_1, \dots, (X \wedge_g Y)X_k),$$

for  $X, Y, X_1, \dots, X_k \in \Gamma(TM)$ . The above curvature condition arose in [2] for a study on totally umbilical submanifolds and such curvature conditions have been applied to various submanifolds and different tensor fields. It will bring new ideas and new results if one studies the effect of Riemannian maps on pseudo symmetry.

5. An immersion  $f$  is said to be parallel if  $\nabla h = 0$ . An isometric immersion  $f$  from a Riemannian manifold  $M^n$  into a space form  $Q^N(c)$  of constant sectional curvature

$c$  is called pseudo-parallel if  $R^Q(X \wedge Y)h = \varphi(X \wedge Y)h$  holds for all  $X, Y \in \Gamma(TM)$ , where  $R^Q$  denotes the Riemannian curvature tensor of  $Q^N(c)$  and  $h$  the second fundamental form of  $f$  and  $\varphi$  is a smooth real-valued function  $M^n$ . If  $\varphi = 0$ , then the immersion becomes semi-parallel immersion (see [11]). Pseudo parallel immersions were defined in [3] and the authors of that paper obtained characterizations of such submanifolds. Since the second fundamental form of a map is a generalization of the second fundamental form of an immersion and a Riemannian map is a generalization of isometric immersion, one can define such conditions for the second fundamental form of a Riemannian map and obtain many new results by following [23].

6. A submanifold  $M$  of a manifold  $\bar{M}$  has parallel mean curvature vector field if  $\nabla_X^\perp H = 0$ , for  $X \in \Gamma(TM)$ , where  $\nabla^\perp$  and  $H$  are the normal connection and mean curvature vector field of  $M$ , respectively. If  $M$  is a surface, then  $M$  has parallel mean curvature vector field if and only if either  $M$  is minimal or the mean curvature is a nonzero constant and the normalized mean curvature vector is parallel (see [7]). On the other hand, RUH and VILMS [29] proved the following; A submanifold  $M$  of  $E^n$  has parallel mean curvature vector if and only if the Gauss map of  $M$  is harmonic. So it would be an important problem to characterize Riemannian maps whose tension field is parallel.

7. For a submersion  $\pi : E \rightarrow M$ , a vector field  $X$  is called a horizontal vector field if  $x \in E$ ,  $X(x) \in H(x)$ . A smooth map  $F : E_1 \rightarrow E_2$  is called horizontal map if  $F_*$  maps any horizontal vector field in  $E_1$  into a horizontal vector field in  $E_2$ . Let  $\pi_1 : E_1 \rightarrow M_1$ ,  $\pi_2 : E_2 \rightarrow M_2$  be Riemannian submersions and  $F : E_1 \rightarrow E_2$  a smooth map. If it maps each fiber submanifold  $F_1$  of  $E_1$  into a corresponding fiber submanifold  $F_2$  of  $E_2$ , then  $F$  is called equivariant map with respect to  $\pi_1$  and  $\pi_2$  (see [39]). This means that if  $x_1$  and  $x'_1$  are in a same fiber, then their images are in same fiber, i.e.,  $\pi_1(x_1) = \pi_1(x'_1)$  implies  $\pi_2(F(x_1)) = \pi_2(F(x'_1))$ . An equivariant map  $F$  induces a map  $\bar{F}$  between base manifolds. By using the notion of horizontal equivariant map, Xin obtained a necessary and sufficient condition for harmonic maps (see [41]). For Riemannian maps, one can introduce equivariant maps with respect to Riemannian maps and this notion will be a useful argument to obtain new conditions for harmonicity of Riemannian maps.

8. The theory of total mean curvature is the study of the integral of the  $n$ -th power of the mean curvature of a compact  $n$ -dimensional submanifold in a Euclidean  $m$ -space. Inspiring this concept, B. Y. Chen introduced the notion of the order of a submanifold and he used this idea to introduce and study submanifolds of finite type (see [8]). By using the tension field of a Riemannian manifold instead of the mean curvature vector field of a submanifold, one can study total mean curvature and Riemannian maps of finite type.

9. Some special vector fields play important roles in Riemannian geometry, for instance Killing vector fields. A smooth vector field  $X$  on a Riemannian manifold  $M$  is said to be Killing if its local flow consists of local isometries of the Riemannian manifold  $(M, g)$ . The presence of a nonzero Killing vector field on a compact Riemannian manifold constrains its geometry as well as topology. Recently, DESHMUKH [12] defined the Jacobi-type vector field on a Riemannian manifold  $M$  as a generalization of a Killing field: A vector field  $\xi$  is said to be a Jacobi-type vector field if the following condition is satisfied

$$\nabla_X \nabla_X \xi - \nabla_{\nabla_X X} \xi + R(\xi, X)X = 0, \forall X \in \Gamma(TM),$$

where  $R$  is the curvature tensor field on  $M$ . Such a vector field is a Jacobi field along any geodesic. One can investigate the effect of Killing vector fields and Jacobi-type vector fields (especially for the harmonicity or biharmonicity of such maps) on the geometry of Riemannian maps.

10. The index of a harmonic map  $F : (M, g) \rightarrow (N, h)$  is defined as the dimension of the largest subspace of  $\Gamma(F^{-1}(TN))$  on which the Hessian  $HessF$  is negative definite. A harmonic map  $F$  is said to be (weakly) stable if the index of  $F$  is zero and otherwise, is said to be unstable. It is known that any harmonic map from a compact Riemannian manifold to a manifold of non-positive sectional curvatures is stable. The following two results were obtained for spaces of positive curvature. (1) Any non-constant harmonic map from a Euclidean sphere of dimension at least three is unstable (see [40]). (2) Any non-constant harmonic map from a compact manifold to a Euclidean sphere of dimension at least three is unstable (see [22]). The stability of smooth maps from various manifolds to another manifolds has been investigated by many authors. For a Riemannian map, the harmonicity conditions were given in Theorem 6.1. It will be an important research problem to find the stability conditions for harmonic Riemannian maps from space forms to manifolds.

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