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A unified generalization of perturbed mid-point and trapezoid inequalities and asymptotic expressions for its error term

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Abstract In this paper, we first establish a new unified proof of perturbed mid-point and trapezoid inequalities and give an application of it in numerical integration. This result in special cases yield the known results. We then derive some asymptotic expressions for error terms of this unified inequality, which not only unify the known results, but also give some new asymptotic expressions for remainder terms of other perturbed quadrature rules as special cases. Finally, corresponding formulas with finite sums are given.

Keywords unified generalizations · perturbed mid-point and trapezoid inequalities · numerical integration · remainder terms · asymptotic expressions

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1 Introduction

Error analysis for known and new quadrature rules has been extensively studied in recent years. The approach from an inequalities point of view to estimate the error terms has been used in these studies (see [1]-[25] and the references therein). In [26], Ujević and Billć considered the above mentioned topic in a way of deriving asymptotic expressions for error terms of the mid-point, trapezoid and Simpson's rules. Precisely, based on the "*Assumption 1*: Let $f \in C^{\infty}[a,b]$ and $\sup_{n\in N} |\frac{f^{(n)}(c)}{f^{(n-1)}(c)}| \le M < \infty$ for some arbitrary but fixed $c \in [a,b]$", they proved the following theorems:

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Theorem 1.1 *Let Assumption* 1 *holds with* $c = a$, *we have*

$$\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right)(b-a) + \sum_{k=3}^{\infty} \frac{2^{k-1}-k}{k!2^{k-1}} f^{(k-1)}(a)(b-a)^k. \tag{1.1}$$

$$\int_a^b f(t)dt = \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{2}\sum_{k=3}^{\infty} \frac{k-2}{k!} f^{(k-1)}(a)(b-a)^k, \tag{1.2}$$

$$\int_a^b f(t)dt = \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a) - \frac{1}{3}\sum_{k=5}^{\infty} \frac{k+(k-6)2^{k-3}}{k!2^{k-2}} f^{(k-1)}(a)(b-a)^k. \tag{1.3}$$

Theorem 1.2 *Let Assumption* 1 *holds with* $c = b$, *we have*

$$\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right)(b-a) - \sum_{k=3}^{\infty} \frac{2^{k-1}-k}{k!2^{k-1}} f^{(k-1)}(b)(a-b)^k, \tag{1.4}$$

$$\int_a^b f(t)dt = \frac{f(a)+f(b)}{2}(b-a) + \frac{1}{2}\sum_{k=3}^{\infty} \frac{k-2}{k!} f^{(k-1)}(b)(a-b)^k, \tag{1.5}$$

$$\int_a^b f(t)dt = \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a) + \frac{1}{3}\sum_{k=5}^{\infty} \frac{k+(k-6)2^{k-3}}{k!2^{k-2}} f^{(k-1)}(b)(a-b)^k. \tag{1.6}$$

In [2–6, 18, 8, 19, 25], the perturbed mid-point and trapezoid inequalities are considered. In [25], Ujević obtained the perturbed mid-point and trapezoid inequalities

$$\left|\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)(b-a) - \frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \le \frac{1}{12}(S-\gamma)(b-a)^3, \tag{1.7}$$

$$\left|\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{1}{12}(S-\gamma)(b-a)^3, \tag{1.8}$$

where $f : [a,b] \to R$ is a twice differentiable function and there exist constants $\gamma, \Gamma \in \mathbf{R}$, with $\gamma \le f''(t) \le \Gamma, t \in [a,b]$, $S = \frac{f'(b)-f'(a)}{b-a}$. In [17], Liu et al. derived asymptotic expressions for error terms of these perturbed mid-point and trapezoid rules.

Theorem 1.3 *Let Assumption* 1 *holds with* $c = a$*, we have*

$$\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right)(b-a) + \frac{(b-a)^2}{24}[f'(b) - f'(a)] + \sum_{k=5}^{\infty} \frac{1}{k!}\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right] f^{(k-1)}(a)(b-a)^k, \tag{1.9}$$

$$\int_a^b f(t)dt = \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{12}[f'(b) - f'(a)] + \frac{1}{12}\sum_{k=5}^{\infty} \frac{(k-3)(k-4)}{k!} f^{(k-1)}(a)(b-a)^k. \tag{1.10}$$

Theorem 1.4 *Let Assumption* 1 *holds with* $c = b$*, we have*

$$\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right)(b-a) + \frac{(b-a)^2}{24}[f'(b) - f'(a)] - \sum_{k=5}^{\infty} \frac{1}{k!}\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right] f^{(k-1)}(b)(a-b)^k, \tag{1.11}$$

$$\int_a^b f(t)dt = \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{12}[f'(b) - f'(a)] - \frac{1}{12}\sum_{k=5}^{\infty} \frac{(k-3)(k-4)}{k!} f^{(k-1)}(b)(a-b)^k. \tag{1.12}$$

In [7], Chen et al. obtained a unified generalization of perturbed trapezoid and mid-point inequalities.

Theorem 1.5 *Let* $I \subset \mathbf{R}$ *be an open interval,* $a, b \in I, a < b$*. If* $f : I \to \mathbf{R}$ *is a twice differentiable function such that* f'' *is integrable and there exist constants* $\gamma, \Gamma \in \mathbf{R}$*, with* $\gamma \le f''(t) \le \Gamma$*,* $t \in [a, b]$*,* $0 \le \lambda \le 1$*. Then*

$$\left|\int_a^b f(t)dt - \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right](b-a) - \frac{1-3\lambda}{24}(b-a)^2[f'(b) - f'(a)]\right|$$

$$\le \begin{cases} \dfrac{2\left(\lambda^2 - \lambda + \frac{1}{3}\right)^{\frac{3}{2}} + \lambda(1-\lambda)(1-2\lambda)}{24}(\Gamma - \gamma)(b-a)^3, & \lambda \in \left[0, \frac{1}{3}\right], \\ \dfrac{\left(\lambda^2 - \lambda + \frac{1}{3}\right)^{\frac{3}{2}}}{6}(\Gamma - \gamma)(b-a)^3, & \lambda \in \left(\frac{1}{3}, \frac{2}{3}\right], \\ \dfrac{2\left(\lambda^2 - \lambda + \frac{1}{3}\right)^{\frac{3}{2}} + \lambda(1-\lambda)(2\lambda-1)}{24}(\Gamma - \gamma)(b-a)^3, & \lambda \in \left(\frac{2}{3}, 1\right]. \end{cases}$$

In this paper, we first establish a new unified proof of perturbed mid-point inequality (1.7) and perturbed trapezoid inequality (1.8) by using a unified $p(t)$ as in (2.2) below and give an application of it in numerical integration (Section 2). This result in special cases yield Theorem 4 and Corollary 2 in [25]. We then derive some asymptotic expressions for error terms of this unified inequality (Section 3), which not only unify the above Theorems 1.3 and 1.4, but also give some new asymptotic expressions for remainder terms of other perturbed quadrature rules as special cases. Corresponding formulas with finite sums will also be given.

2 A new unified proof of perturbed mid-point and trapezoid inequalities and application

In this section, we first establish a new unified proof of perturbed mid-point and trapezoid inequalities.

Theorem 2.1 *Let $I \subset R$ be an open interval, $a, b \in I, a < b$. If $f : I \to R$ is a twice differentiable function such that f'' is integrable and there exist constants $\gamma, \Gamma \in \mathbf{R}$, with $\gamma \le f''(t) \le \Gamma, t \in [a, b], 0 \le \lambda \le 1$. Then*

$$\left| \int_a^b f(t)dt - \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right](b-a) \right.$$
$$\left. - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)] \right| \tag{2.1}$$
$$\le \begin{cases} \frac{2-3\lambda}{24}(S-\gamma)(b-a)^3, \ 0 \le \lambda \le \frac{1}{2}, \\ \frac{3\lambda-1}{24}(S-\gamma)(b-a)^3, \ \frac{1}{2} < \lambda \le 1, \end{cases}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. Let $p : [a, b] \to \mathbb{R}$ be given by

$$p(t) = \begin{cases} \frac{1}{2}(t-a)[t-(1-\lambda)a-\lambda b], \ t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(b-t)[\lambda a+(1-\lambda)b-t], \ t \in (\frac{a+b}{2}, b]. \end{cases} \tag{2.2}$$

Integrating by parts, we have

$$\int_a^b p(t)f''(t)dt = \int_a^b f(t)dt - (b-a) \cdot$$
$$\cdot \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right]. \tag{2.3}$$

If C is a constant, then

$$\int_a^b \left[p(t) - \frac{1}{b-a}\int_a^b p(s)ds\right][f''(t) - C]dt$$
$$= \int_a^b \left[p(t) - \frac{1}{b-a}\int_a^b p(s)ds\right] f''(t)dt. \tag{2.4}$$

We also have

$$\int_a^b f''(t)dt = f'(b) - f'(a) \tag{2.5}$$
$$\int_a^b p(t)dt = \frac{1-3\lambda}{24}(b-a)^3. \tag{2.6}$$

From (2.3)-(2.6) it follows

$$\begin{aligned} &\int_a^b \left[p(t) - \frac{1}{b-a}\int_a^b p(s)ds\right][f''(t) - C]dt \\ &= \int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] \\ &- \frac{1-3\lambda}{24}(b-a)^2[f'(b) - f'(a)]. \end{aligned} \tag{2.7}$$

On the other hand, if we set $C = \gamma$, then we have

$$\int_a^b \left[p(t) - \frac{1}{b-a}\int_a^b p(s)ds\right][f''(t) - \gamma]dt$$

$$\begin{aligned} &\le \max_{t\in[a,b]}\left|p(t) - \frac{1}{b-a}\int_a^b p(s)ds\right|\int_a^b |f''(t) - \gamma|dt \\ &= \frac{1}{2}(S-\gamma)(b-a)\max_{t\in[a,\frac{a+b}{2}]}\left|(t-a)^2 - \lambda(b-a)(t-a) - \frac{1-3\lambda}{12}(b-a)^2\right| \\ &= \begin{cases} \frac{2-3\lambda}{24}(S-\gamma)(b-a)^3, & 0 \le \lambda \le \frac{1}{2}, \\ \frac{3\lambda-1}{24}(S-\gamma)(b-a)^3, & \frac{1}{2} < \lambda \le 1. \end{cases} \end{aligned} \tag{2.8}$$

From (2.7) and (2.8) we see that (2.1) holds. $\square$

Remark 2.1 We note that in the special cases, if we take $\lambda = 0$ and $\lambda = 1$ in Theorem 2.1 respectively, we get Theorem 4 and Corollary 2 in [25] respectively.

To verify the correctness of Theorem 2.1, we give several specific examples shown as the following Table 1, in which we set $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{2}{3}$, $G_1(\lambda) = \frac{2-3\lambda}{24}(S-\gamma)(b-a)^3$, $G_2(\lambda) = \frac{3\lambda-1}{24}(S-\gamma)(b-a)^3$, and

$$F(\lambda) := \left| \int_a^b f(t)dt - \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2} \right](b-a) \right.$$
$$\left. - \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)] \right|.$$

We find that $F(\lambda_1) \le G_1(\lambda_1)$ and $F(\lambda_2) \le G_1(\lambda_2)$.

$f(x)$	$[a,b]$	$F(\lambda_1)$	G_1	$F(\lambda_2)$	G_2
$\cos x - x$	$[0, \frac{\pi}{2}]$	0.038551	0.220174	0.065358	0.220174
e^x	$[0,1]$	5.793240×10^{-4}	0.029929	8.758710×10^{-4}	0.029929
$\frac{1}{x}$	$[1,3]$	0.012499	0.123457	0.024538	0.123457
$e^x \sin x$	$[1,3]$	0.267058	9.788669	3.528357	9.788669

Corollary 2.2 *Under the assumptions of Theorem* 2.1 *and with* $\lambda = \frac{1}{2}$*, we have the perturbed averaged mid-point-trapezoid type inequality*

$$\left| \int_a^b f(t)dt - \frac{1}{2} f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{2}\frac{f(a)+f(b)}{2}(b-a) \right.$$
$$\left. + \frac{1}{48}(b-a)^2[f'(b)-f'(a)] \right| \le \frac{1}{48}(S-\gamma)(b-a)^3. \tag{2.9}$$

Corollary 2.3 *Under the assumptions of Theorem* 2.1 *and with* $\lambda = \frac{1}{3}$*, we have the Simpson inequality*

$$\left| \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right|$$
$$\le \frac{1}{24}(S-\gamma)(b-a)^3. \tag{2.10}$$

Now, we give an application of Theorem 2.1 in numerical integration.

Theorem 2.4 *Let the assumptions of Theorem* 2.1 *hold. If* $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *is a given division of the interval* $[a,b]$ *then we have* $\int_a^b f(t)dt = A_{MT}(f,D) + R_{MT}(f,D)$*, where*

$$A_{MT}(f,D) = \sum_{i=0}^{n-1} h_i \left[(1-\lambda) f\left(\frac{x_i + x_{i+1}}{2}\right) + \lambda \frac{f(x_i)+f(x_{i+1})}{2} \right]$$
$$+ \frac{1-3\lambda}{24} \sum_{i=0}^{n-1} h_i^3 [f'(x_{i+1}) - f'(x_i)],$$

$$|R_{MT}(f,D)| \le \begin{cases} \dfrac{2-3\lambda}{24} \displaystyle\sum_{i=0}^{n-1} (S_i - \gamma) h_i^3, & 0 \le \lambda \le \dfrac{1}{2}, \\ \dfrac{3\lambda-1}{24} \displaystyle\sum_{i=0}^{n-1} (S_i - \gamma) h_i^3, & \dfrac{1}{2} < \lambda \le 1, \end{cases}$$

and $h_i = x_{i+1} - x_i$, $S_i = \frac{f'(x_{i+1}) - f'(x_i)}{h_i}$, $i = 0, 1, 2, \cdots, n-1$.

Proof. Apply Theorem 2.1 to the interval $[x_i, x_{i+1}], i = 0, 1, 2, \cdots, n-1$ and sum. Then use the triangle inequality to obtain the desired result. □

3 Some asymptotic expressions for error term of the unified inequality

In this section, we derive some asymptotic expressions for error term of the above unified inequality (2.1).

Theorem 3.1 *Let Assumption* 1 *holds with* $c = a$*, we have*

$$\int_a^b f(t)dt = \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right](b-a)$$

$$\begin{aligned} &+ \frac{1-3\lambda}{24}(b-a)^2[f'(b) - f'(a)] \\ &+ \sum_{k=5}^{\infty} \frac{1}{k!}\left\{(1-\lambda)\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right]\right. \\ &\left. +\lambda \frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(b-a)^k, \ \forall \lambda \in [0,1]. \end{aligned} \tag{3.1}$$

Proof. We define the function

$$\begin{aligned} R(x) = &\int_a^x f(t)dt - \left[(1-\lambda) f\left(\frac{a+x}{2}\right) + \lambda \frac{f(a)+f(x)}{2}\right](x-a) \\ &- \frac{1-3\lambda}{24}(x-a)^2[f'(x) - f'(a)], \end{aligned}$$

for all $\lambda \in [0,1]$. Obviously, $R(a) = 0$.

We have

$$\begin{aligned} R'(x) = f(x) &- (1-\lambda)\left[f\left(\frac{a+x}{2}\right) + \frac{1}{2}(x-a)f'\left(\frac{a+x}{2}\right)\right] \\ &- \lambda\left[\frac{f(a)+f(x)}{2} + \frac{1}{2}(x-a)f'(x)\right] \\ &- \frac{1-3\lambda}{12}(x-a)[f'(x) - f'(a)] - \frac{1-3\lambda}{24}(x-a)^2 f''(x) \end{aligned}$$

such that $R'(a) = 0$. We also have

$$\begin{aligned} R''(x) = (1-\lambda)&\left[f'(x) - f'\left(\frac{a+x}{2}\right)\right] - \frac{1-3\lambda}{12}[f'(x) - f'(a)] - \frac{1}{6}(x-a)f''(x) \\ &- \frac{1-\lambda}{4}(x-a)f''\left(\frac{a+x}{2}\right) - \frac{1-3\lambda}{24}(x-a)^2 f'''(x) \end{aligned}$$

and $R''(a) = 0$. Further,

$$\begin{aligned}
R'''(x) = &(1-\lambda)\left[\frac{3}{4}f''(x) - \frac{3}{4}f''\left(\frac{a+x}{2}\right) - \frac{1}{4}(x-a)f'''(x)\right.\\
&\left.-\frac{1}{8}(x-a)f'''\left(\frac{a+x}{2}\right) - \frac{1}{24}(x-a)^2 f^{(4)}(x)\right] + \lambda\frac{1}{12}(x-a)^2 f^{(4)}(x),\\
R'''(a) = &\,0,\\
R^{(4)}(x) = &(1-\lambda)\left[\frac{1}{2}f'''(x) - \frac{1}{2}f'''\left(\frac{a+x}{2}\right) - \frac{1}{3}(x-a)f^{(4)}(x)\right.\\
&\left.-\frac{1}{16}(x-a)f^{(4)}\left(\frac{a+x}{2}\right) - \frac{1}{24}(x-a)^2 f^{(5)}(x)\right]\\
&+\lambda\left[\frac{1}{6}(x-a)f^{(4)}(x) + \frac{1}{12}(x-a)^2 f^{(5)}(x)\right],\\
R^{(4)}(a) = &\,0,\\
R^{(5)}(x) = &(1-\lambda)\left[\frac{1}{6}f^{(4)}(x) - \frac{5}{16}f^{(4)}\left(\frac{a+x}{2}\right) - \frac{5}{12}(x-a)f^{(5)}(x)\right.\\
&\left.-\frac{1}{32}(x-a)f^{(5)}\left(\frac{a+x}{2}\right) - \frac{1}{24}(x-a)^2 f^{(6)}(x)\right]\\
&+\lambda\left[\frac{1}{6}f^{(4)}(x) + \frac{1}{3}(x-a)f^{(5)}(x) + \frac{1}{12}(x-a)^2 f^{(6)}(x)\right],\\
R^{(5)}(a) = &\,\frac{15\lambda-7}{48}f^{(4)}(a),
\end{aligned}$$

Generally, by induction, we can get

$$\begin{aligned}
R^{(k)}(x) = &(1-\lambda)\left\{\left[1 - \frac{k(k-1)}{24}\right]f^{(k-1)}(x)\right.\\
&-\frac{k}{2^{k-1}}f^{(k-1)}\left(\frac{a+x}{2}\right) - \frac{k}{12}(x-a)f^{(k)}(x)\\
&\left.-\frac{1}{2^k}(x-a)f^{(k)}\left(\frac{a+x}{2}\right) - \frac{1}{24}(x-a)^2 f^{(k+1)}(x)\right\}\\
&+\lambda\left\{\frac{(k-3)(k-4)}{12}f^{(k-1)}(x) + \frac{k-3}{6}(x-a)f^{(k)}(x)\right.\\
&\left.+\frac{1}{12}(x-a)^2 f^{(k+1)}(x)\right\}, \quad k \geq 5,
\end{aligned} \tag{3.2}$$

and so

$$R^{(k)}(a) = \left\{(1-\lambda)\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right] + \lambda\frac{(k-3)(k-4)}{12}\right\}f^{(k-1)}(a), k \geq 5.$$

In fact, suppose that (3.2) holds for $k = m$ $(m \geq 5)$, then we have

$$R^{(m+1)}(x) = (1-\lambda)\left\{\left[1 - \frac{m(m-1)}{24}\right] f^{(m)}(x) - \frac{m}{2^m} f^{(m)}\left(\frac{a+x}{2}\right) - \frac{m}{12} f^{(m)}(x) - \frac{m}{12}(x-a) f^{(m+1)}(x)\right.$$

$$- \frac{1}{2^m} f^{(m)}\left(\frac{a+x}{2}\right) - \frac{1}{2^{m+1}}(x-a) f^{(m+1)}\left(\frac{a+x}{2}\right) - \frac{1}{12}(x-a) f^{(m+1)}(x)$$

$$\left. - \frac{1}{24}(x-a)^2 f^{(m+2)}(x)\right\} + \lambda\left\{\frac{(m-3)(m-4)}{12} f^{(m)}(x) + \frac{m-3}{6} f^{(m)}(x)\right.$$

$$\left. + \frac{m-3}{6}(x-a) f^{(m+1)}(x) + \frac{1}{6}(x-a) f^{(m+1)}(x) + \frac{1}{12}(x-a)^2 f^{(m+2)}(x)\right\}$$

$$= (1-\lambda)\left\{\left[1 - \frac{m(m+1)}{24}\right] f^{(m)}(x) - \frac{m+1}{2^m} f^{(m)}\left(\frac{a+x}{2}\right)\right.$$

$$- \frac{m+1}{12}(x-a) f^{(m+1)}(x)$$

$$\left. - \frac{1}{2^{m+1}}(x-a) f^{(m+1)}\left(\frac{a+x}{2}\right) - \frac{1}{24}(x-a)^2 f^{(m+2)}(x)\right\}$$

$$+ \lambda\left\{\frac{(m-2)(m-3)}{12} f^{(m)}(x) + \frac{m-2}{6}(x-a) f^{(m+1)}(x)\right.$$

$$\left. + \frac{1}{12}(x-a)^2 f^{(m+2)}(x)\right\}, \quad m \geq 5,$$

which implies that (3.2) holds for $k = m+1$. By using of the Taylor series $R(x) = \sum_{k=0}^{\infty} \frac{R^{(k)}(a)}{k!}(x-a)^k$ with the above data, we have

$$R(x) = \sum_{k=5}^{\infty} \frac{1}{k!}\left\{(1-\lambda)\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right] + \lambda \frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(x-a)^k.$$

If we substitute $x = b$ in the above series then we get formula (3.1). We use Lemma 2.1 of [26] to show that the series in (3.1) converges. □

Theorem 3.2 *Let Assumption* 1 *holds with* $c = b$*, we have*

$$\int_a^b f(t)dt = \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right](b-a) + \frac{1-3\lambda}{24}(b-a)^2[f'(b) - f'(a)] - \sum_{k=5}^{\infty} \frac{1}{k!}\left\{(1-\lambda)\left[1 - \frac{k}{2^{k-1}} - \frac{k(k-1)}{24}\right]\right. \quad (3.3)$$

$$+\lambda\frac{(k-3)(k-4)}{12}\Bigg\} f^{(k-1)}(b)(a-b)^k.$$

Proof. We define the function

$$\begin{aligned}R(x) &= \int_x^b f(t)dt - \left[(1-\lambda)f\left(\frac{x+b}{2}\right) + \lambda\frac{f(x)+f(b)}{2}\right](b-x)\\ &- \frac{1-3\lambda}{24}(b-x)^2[f'(b)-f'(x)]\\ &= -\left\{\int_b^x f(t)dt - \left[(1-\lambda)f\left(\frac{b+x}{2}\right) + \lambda\frac{f(b)+f(x)}{2}\right](x-b)\right.\\ &\left.-\frac{1-3\lambda}{24}(x-b)^2[f'(x)-f'(b)]\right\}.\end{aligned}$$

Now we can use the results of Theorem 3.1. We simply substitute $b \to a$ in the above relation and get

$$\begin{aligned}R(x) = -\sum_{k=5}^{\infty}\frac{1}{k!}&\left\{(1-\lambda)\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right.\\ &\left.+\lambda\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(b)(x-b)^k.\end{aligned}$$

The relation (3.3) follows if we now set $x = a$. We use Lemma 2.1 of [26] to show that the series in (3.3) converges. □

Corollary 3.3 *Under the assumptions of Theorems* 3.1 *and* 3.2 *we have*

$$\begin{aligned}\int_a^b f(t)dt &= \left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right](b-a)\\ &+ \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)]\\ &+ \frac{1}{2}\sum_{k=5}^{\infty}\frac{1}{k!}\left\{(1-\lambda)\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]+\lambda\frac{(k-3)(k-4)}{12}\right\}\\ &\times\left[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)\right](b-a)^k.\end{aligned}$$

Proof. We sum (3.1) and (3.3). □

Remark 3.1 We note that in the special cases, if we take $\lambda = 0$ and 1 in Theorem 3.1 and 3.2, respectively, we get Theorem 1.3 and 1.4, respectively. If we take $\lambda = \frac{1}{3}$ in Theorem 3.1 and 3.2, respectively, we get (1.3) in Theorem 1.1 and (1.6) in Theorem 1.2, respectively.

Now, we can give some new asymptotic expressions for remainder terms of other perturbed quadrature rules as special cases.

Corollary 3.4 *Under the assumptions of Theorem* 3.1 *and* 3.2 *with* $\lambda = \frac{1}{2}$, *we have the asymptotic expressions for remainder terms of perturbed averaged mid-point-trapezoid type rule*

$$\begin{aligned}\int_a^b f(t)dt =& \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right](b-a) - \frac{1}{48}(b-a)^2[f'(b)-f'(a)] \\ &+ \frac{1}{2}\sum_{k=5}^{\infty}\frac{1}{k!}\left\{\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ &\left.+\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(b-a)^k, \\ \int_a^b f(t)dt =& \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right](b-a) - \frac{1}{48}(b-a)^2[f'(b)-f'(a)] \\ &- \frac{1}{2}\sum_{k=5}^{\infty}\frac{1}{k!}\left\{\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ &\left.+\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(b)(a-b)^k.\end{aligned}$$

Corollary 3.5 *Under the assumptions of Theorem* 3.1 *and* 3.2 *with* $\lambda = \frac{2}{3}$, *we have the asymptotic expressions for remainder terms of perturbed averaged 3-point type rule*

$$\begin{aligned}\int_a^b f(t)dt =& \frac{1}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right](b-a) - \frac{1}{24}(b-a)^2[f'(b)-f'(a)] \\ &+ \frac{1}{3}\sum_{k=5}^{\infty}\frac{1}{k!}\left\{\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ &\left.+\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(b-a)^k, \\ \int_a^b f(t)dt =& \frac{1}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right](b-a) - \frac{1}{24}(b-a)^2[f'(b)-f'(a)] \\ &- \frac{1}{3}\sum_{k=5}^{\infty}\frac{1}{k!}\left\{\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ &\left.+\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(b)(a-b)^k.\end{aligned}$$

Finally, we derive the corresponding formulas with finite sums.

Theorem 3.6 *Let $f \in C^{n+1}[a,b]$. Then*

$$\begin{aligned}\int_a^b f(t)dt = & \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right](b-a) \\ & + \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)] \\ & + \sum_{k=5}^{n} \frac{1}{k!}\left\{(1-\lambda)\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ & \left. +\lambda \frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(b-a)^k \\ & + \frac{1}{n!}\int_a^b R^{(n+1)}(t)(b-t)^n dt,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}R^{(m)}(t) = & (1-\lambda)\left\{\left[1-\frac{m(m-1)}{24}\right] f^{(m-1)}(t)\right. \\ & -\frac{m}{2^{m-1}} f^{(m-1)}\left(\frac{a+t}{2}\right) - \frac{m}{12}(t-a) f^{(m)}(t) \\ & \left. -\frac{1}{2^m}(x-a) f^{(m)}\left(\frac{a+t}{2}\right) - \frac{1}{24}(t-a)^2 f^{(m+1)}(t)\right\}\end{aligned} \tag{3.5}$$

$$\begin{aligned}& + \lambda\left\{\frac{(m-3)(m-4)}{12} f^{(m-1)}(t) + \frac{m-3}{6}(t-a) f^{(m)}(t)\right. \\ & \left. + \frac{1}{12}(t-a)^2 f^{(m+1)}(t)\right\}\end{aligned}$$

and

$$\begin{aligned}\int_a^b f(t)dt = & \left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right](b-a) \\ & + \frac{1-3\lambda}{24}(b-a)^2[f'(b)-f'(a)]\end{aligned} \tag{3.6}$$

$$\begin{aligned}& - \sum_{k=5}^{n} \frac{1}{k!}\left\{(1-\lambda)\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]\right. \\ & \left. +\lambda \frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(b)(a-b)^k \\ & + \frac{1}{n!}\int_a^b R^{(n+1)}(t)(a-t)^n dt,\end{aligned}$$

where in this case the derivatives $R^{(m)}(t)$ are equal to (3.5) with the substitution $a = b$.

Proof. Let $R(x)$ be defined in the proof of Theorem 3.1. From Lemma 2.2 of [26] with $g{=}R, c{=}a$ we get $R(x){=}\sum_{k=0}^{n} \frac{R^{(k)}(a)}{k!}(x-a)^k{+}\frac{1}{n!}\int_a^x R^{(n+1)}(t)(x-t)^n dt$. If we substitute the values from the above mentioned proof in the above relation then we obtain

$$\int_a^x f(t)dt - \left[(1-\lambda)f\left(\frac{a+x}{2}\right) + \lambda\frac{f(a)+f(x)}{2}\right](x-a)$$
$$- \frac{1-3\lambda}{24}(x-a)^2[f'(x)-f'(a)]$$
$$= \sum_{k=5}^{n} \frac{1}{k!}\left\{(1-\lambda)\left[1-\frac{k}{2^{k-1}}-\frac{k(k-1)}{24}\right]+\lambda\frac{(k-3)(k-4)}{12}\right\} f^{(k-1)}(a)(x-a)^k$$
$$+ \frac{1}{n!}\int_a^x R^{(n+1)}(t)(x-t)^n dt$$

and this is equivalent to (3.4) with the substitution $x = b$. The formula (3.5) can be proved by induction.

If $R(x)$ is defined in the proof of Theorem 3.2 and we substitution $x = a$ then, in a similar way as above, we get (3.6). □

Remark 3.2 We note that in the special cases, if we take $\lambda = 0$ and 1 in Theorem 3.6, we get Theorems 5 and 8 of [17], respectively. If we take $\lambda = \frac{1}{3}$ in Theorem 3.6, we get Theorem 2.9 of [26]. We can also give some corresponding formulas with finite sums for remainder terms of other quadrature rules as special cases. For examples, we can set $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{3}$ to get corresponding formulas with finite sums for remainder terms of perturbed averaged mid-point-trapezoid type rule and perturbed averaged 3-point type rule, respectively.

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References

1. Barnett, N.S.; Dragomir, S.S. – *Applications of Ostrowski's version of the Grüss inequality for trapezoid type rules*, Tamkang J. Math., 37 (2006), 163–173.
2. Cerone, P. – *On perturbed trapezoidal and midpoint rules*, Korean J. Comput. Appl. Math., 9 (2002), 423–435.
3. Cerone, P. – *Perturbed rules in numerical integration from product branched Peano kernels*, Nonlinear Anal. Forum, 9 (2004), 1–23.
4. Cerone, P.; Dragomir, S.S. – *Trapezoidal-type rules from an inequalities point of view*, Handbook of analytic-computational methods in applied mathematics, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.
5. Cerone, P.; Dragomir, S.S. – *Midpoint-type rules from an inequalities point of view*, Handbook of analytic-computational methods in applied mathematics, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.
6. Cerone, P.; Dragomir, S. S.; Roumeliotis, J. – *An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration*, Kyungpook Math. J., 39 (1999), 333–341.
7. Chen, W.B.; Chen, Q.; Liu, W.J. – *A unified generalization of perturbed trapezoid and midpoint inequalities and applications in numerical integration*, Advances in Applied Mathematical Analysis, 3 (2008), 1–5.

8. Cheng, X.-L.; Sun, J. – *A note on the perturbed trapezoid inequality*, JIPAM. J. Inequal. Pure Appl. Math., 3 (2002), Article 29, 7 pp. (electronic).
9. Dragomir, S.S. – *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. (Basel), 91 (2008),450–460.
10. Dragomir, S.S.; Cerone, P.; Sofo, A. – *Some remarks on the trapezoid rule in numerical integration*, Indian J. Pure Appl. Math., 31 (2000), 475–494.
11. Dragomir, S.S.; Rassias, T.M. – *Ostrowski type inequalities and applications in numerical integration*, School and Communications and Informatics, Victoria University of Technology, Victoria, Australia.
12. Huy, V.N.; Ngô, Q.-A. – *New inequalities of Simpson-like type involving n knots and the mth derivative*, Math. Comput. Modelling, 52 (2010), 522–528.
13. Kikianty, E.; Dragomir, S.S.; Cerone, P. – *Ostrowski type inequality for absolutely continuous functions on segments in linear spaces*, Bull. Korean Math. Soc., 45 (2008), 763–780.
14. Liu, W.J. – *Some weighted integral inequalities with a parameter and applications*, Acta Appl. Math., 109 (2010), 389–400.
15. Liu, W. – *Several error inequalities for a quadrature formula with a parameter and applications*, Comput. Math. Appl., 56 (2008), 1766–1772.
16. Liu, W.J.; Xue, Q.L.; Wang, S.F. – *Several new perturbed Ostrowski-like type inequalities*, JIPAM. J. Inequal. Pure Appl. Math., 8 (2007), Article 110, 6 pp.
17. Liu, W.J.; Zhu, J.; Fu, M.F. – *Asymptotic expressions for error terms of the perturbed mid-point and trapezoid rules*, J. Interdiscip. Math., 15 (2012), 449–460.
18. Liu, Z. – *On sharp perturbed midpoint inequalities*, Tamkang J. Math., 36 (2005), 131–136.
19. Liu, Z. – *Error estimates for some composite corrected quadrature rules*, Appl. Math. Lett., 22 (2009), 771–775.
20. Matić, M.; Pečarić, J.; Ujević, N. – *Improvement and further generalization of inequalities of Ostrowski-Grüss type*, Comput. Math. Appl., 39 (2000), 161–175.
21. Rafiq, A.; Mir, N.A.; Zafar, F. – *A generalized Ostrowski-Grüss type inequality for twice differentiable mappings and applications*, JIPAM. J. Inequal. Pure Appl. Math., 7 (2006), Article 124, 7 pp. (electronic).
22. Sarikaya, M.Z. – *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenian. (N.S.), 79 (2010), 129–134.
23. Sarikaya, M.Z.; Set, E.; Ozdemir, M.E. – *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., 60 (2010), 2191–2199.
24. Tuna, A.; Daghan, D.– *Generalization of Ostrowski and Ostrowski-Grüss type inequalities on time scales*, Comput. Math. Appl., 60 (2010), 803–811.
25. Ujević, N. – *On perturbed mid-point and trapezoid inequalities and applications*, Kyungpook Math. J., 43 (2003), 327–334.
26. Ujević, N.; Bilić, N. – *Asymptotic expressions for remainder terms of some quadrature rules*, Cent. Eur. J. Math., 6 (2008), 559–567.