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Some types of ideals in $BZMV^{dm}$-algebra

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Abstract In this paper, we introduce (normal, implicative, maximal, involutive) ideals of $BZMV^{dm}$-algebras. Also the ideal generated by a set in $BZMV^{dm}$-algebra are introduced and the relationships between them are studied.

Keywords MV-algebra · $BZMV^{dm}$-algebra · (normal, implicative, maximal, involutive) ideal

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1 Introduction

Cattaneo, Giuntini and Pilla [4] introduced the concept of $BZMV^{dm}$-algebra and the main properties of this system were studied. A $BZMV^{dm}$-algebra is a system endowed with a commutative and associative binary operator $\oplus$ and two orthocomplementation: a Kleene orthocomplementations $\neg$ and a Brouwerian one $\sim$ and every $BZMV^{dm}$ algebra is both an MV-algebra and a distributive demorgan BZ$_-$ lattice. In a $BZMV^{dm}$-algebra $\nu(a) =\sim\sim a$ is called the necessity and $\mu(a) = \neg\sim a$ is called the possibility. Thus $\nu(a) \leq \mu(a)$ and the set of

$$A_{e,m} = \{e \in A : \nu(e) = e\} = \{e \in A : \mu(e) = e\}$$

is called modal sharp. So the structure $(A, A_{e,m}, \nu, \mu)$ is a rough approximation space. For any element $a \in A$ its rough approximation is defined as the pair of M-sharp elements: $r(a) :=< \nu(a), \mu(a) >$ (see [5,6]).

Let X be a universe set. Then a shadowed set on X is any mapping $s : X \to \{0, 1, (0, 1)\}$. We denote the collection of shadowed sets on X as $S = \{0, 1, (0, 1)\}^X$ (see [3,8,9]).

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Let S be the collection of shadowed sets on the universe X. Once defined the operators $(f \oplus g)(x) := min\{1, f(x) + g(x)\}$, $\neg f(x) := 1 - f(x)$ and

$$\sim a = \begin{cases} 1, & \text{if } f(x) = 0 \\ 0, & \text{otherwise} \end{cases}$$

and the identically zero shadowed set $O(x) := 0$. Then the structure $< S, \oplus, \neg, \sim, 0 >$ is a $BZMV^{dm}$ algebra.

Cattaneo, Ciucci, Giuntini and Konig proved that several algebra structures (namely HW, Stonian MV and MV_Δ) are equivalent to $BZMV^{dm}$-algebra (see [2]). Cattaneo, Giuntini and Pilla defined ideal, $\sim$-ideal, prime ideal in $BZMV^{dm}$- algebra (see [4]). The Ideal theory of the logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various ideals correspond to various sets of provable formulas.

In this paper we define normal, implicative, involutive, maximal and generated ideals. We study some relationship between these ideals in $BZMV^{dm}$-algebra.

2 Preliminary

Definition 2.1 ([7]) An MV-algebra is a system $\langle A, \oplus, \neg, 0\rangle$ where A is a non empty set, 0 is a constant element of A, $\oplus$ is a binary operation on A, $\neg$ is a unary operator, obeying the following axioms:
[MV1] $(x \oplus y) \oplus z = (y \oplus z) \oplus x$, [MV2] $x \oplus 0 = x$,
[MV3] $x \oplus \neg 0 = \neg 0$, [MV4] $\neg(\neg 0) = 0$,
[MV5] $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x)$.

Definition 2.2 ([1]) A distributive Brouwer Zadeh(BZ) lattice is a structure $\langle B, \vee, \wedge, \neg, \sim, 0\rangle$, where

(i) $\langle B, \vee, \wedge, 0\rangle$ is a (nonempty) distributive lattice with minimum element 0.
(ii) the mapping $\neg : B \to B$ is a Kleene orthocomplementation, that is

- $\neg(\neg a) = a$
- $\neg(a \vee b) = \neg a \wedge \neg b$, $(a \wedge \neg a < b \vee \neg b)$

(iii) the mapping $\sim: B \to B$ is a Brouwer orthocomplementation, that is $a \wedge \sim\sim a = a$, $\sim (a \vee b) = \sim a \wedge \sim b$, $a \wedge \sim a = 0$.
(iv) The two orthocomplementation are linked by the following interconnection rule $\neg \sim a = \sim\sim a$.

The mapping $\neg$ is also called the Lukasiewicz [fuzzy (Zadeh)] orthocomplementation while the mapping $\sim$ is an intuitionistic-like orthocomplementation. The element $1 := \sim 0 = \neg 0$ is the greatest element of B.

Definition 2.3 A distributive de Morgan BZ-lattice (BZ^{dm}-lattice) is a distributive BZ-lattice for which the following hold $\sim (a \wedge b) = \sim a \vee \sim b$.

Definition 2.4 **([4])** A $BZMV^{dM}$-algebra is a system $\langle A, \oplus, \neg, \sim, 0\rangle$ where A is a non-empty set of elements, 0 is a constant element of A, $\neg$ and $\sim$ are unary operations on A, $\oplus$ is a binary operation on A, obeying the following axioms:

$BZMV^{dm}1$	$(x \oplus y) \oplus z = (y \oplus z) \oplus x.$
$BZMV^{dm}2$	$x \oplus 0 = x.$
$BZMV^{dm}3$	$\neg(\neg x) = x.$
$BZMV^{dm}4$	$\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x.$
$BZMV^{dm}5$	$\sim x \oplus \sim\sim x = \neg 0.$
$BZMV^{dm}6$	$x \oplus \sim\sim x = \sim\sim x.$
$BZMV^{dm}7$	$\sim \neg[\neg(x \oplus \neg y) \oplus \neg y] = \neg(\sim\sim x \oplus \neg \sim\sim y) \oplus \neg \sim\sim y.$

Theorem 2.5 **([4])** *If A is a $BZMV^{dm}$-algebra, then the following are true for all $x, y \in A$:*

- $x \oplus y = y \oplus x, \quad x \oplus 1 = 1,$
- $x \oplus \neg x = 1, \quad x \wedge \sim x = 0,$
- $\sim x \oplus \sim x = \sim x, \quad \sim x = \sim\sim\sim x,$
- $\neg 0 = \sim 0, \quad \neg \sim x = \sim\sim x,$
- $x \wedge \sim\sim x = x, \quad \neg x \oplus \sim\sim x = 1,$
- $(x \oplus y) \oplus z = x \oplus (y \oplus z), \quad \neg(x \oplus \sim\sim x) \oplus \sim\sim x = 1,$
- $\sim (x \wedge y) = \sim x \vee \sim y, \quad \sim (x \vee y) = \sim x \wedge \sim y$ (*if $x \leq y$, then $\sim y \leq \sim x$*).

Remark 2.1 In any $BZMV^{dm}$-algebra we have $\sim \neg x \leq x \leq \sim\sim x$.

Theorem 2.6 **([4])** *In a $BZMV^{dm}$-algebra the following holds $\sim\sim x = x$ iff $\sim x \oplus x = 1$ iff $x \oplus x = x$.*

Remark 2.2 If $\sim\sim x = x$, then $\sim x = \neg x$.

Proof. If $\sim\sim x = x$, thus we have $\sim\sim x \leq x$ and $\neg x \leq \neg \sim\sim x = \sim\sim\sim x = \sim x$ so $\neg x \leq \sim x$. On the other hand $\sim x \leq \neg x$, hence $\neg x = \sim x$. Now we consider $\neg x = \sim x$ thus $\neg x \leq \sim x$ and $\sim\sim x \leq \sim \neg x$ from Remark 2.1 $\sim \neg x \leq x \leq \sim\sim x$ so $\sim\sim x = x = \sim \neg x$. □

Remark 2.3 Let $\langle A, \oplus, \neg, \sim, 0, 1\rangle$ be a $BZMV^{dm}$-algebra. Then $\langle A, \oplus, \neg, 0, 1\rangle$ is an MV-algebra and $\langle B, \vee, \wedge, \neg, \sim, 0, 1\rangle$ is a BZ-lattice such that: $\neg 0 = 1$, $a \vee b = \neg(\neg a \oplus b) \oplus b = a \oplus (\neg a \odot b)$, $a \wedge b = \neg[\neg(\neg a \oplus b) \oplus \neg a] = b \odot (\neg b \oplus a)$, $a \odot b = \neg(\neg a \oplus \neg b)$.

Definition 2.7 **([4])** Let A and B be $BZMV^{dm}$-algebra.We say that the function $\varphi : A \to B$ is a homomorphism of A onto B if φ is such that: $\varphi(0) = 0$, $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$, $\varphi(\neg x) = \neg\varphi(x)$, $\varphi(\sim x) = \sim \varphi(x)$, for all $x, y \in A$.

Trivially, if φ is homomorphism we have $\varphi(1) = 1$, $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$. If the function φ is one-to-one and onto, then φ is called an isomorphism from A onto B.

Theorem 2.8 **([4])** *Let A be a linear $BZMV^{dm}$-algebra. Then the Brouwer orthocomplementation $\sim$ is uniquely defined in the following way for all $x \in A$*

$$\sim a = \begin{cases} 1, & \textit{if } \ a = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

3 Ideal theory in $BZMV^{dm}$-algebras

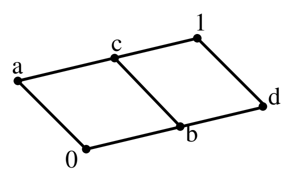
From now on A is a $BZMV^{dm}$-algebra.

Definition 3.1 **([4])** An ideal of A is any subset I of A such that the following conditions are satisfied:

I1: $0 \in I$.
I2: if $x, y \in I$, then $x \oplus y \in I$.
I3: if $x \in I$ and $y \in A$, then $x \odot y \in I$.

We denote the set of ideals of A, by $Id(A)$.

Example 3.1 Let $A = \{0, a, b, c, d, 1\}$ be a set with following diagram and operations $\sim, \neg, \oplus, \odot$ come in the following tables



	$\neg$	$\sim$
0	1	1
a	d	d
b	c	a
c	b	0
d	a	a
1	0	0

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

then $\langle A, \oplus, \neg, \sim, 0\rangle$ is a $BZMV^{dm}$-algebra and $Id(A) = \{\{0, a\}, \{0, b, d\}, \{0\}, \{0, a, b, c, d, 1\}\}$.

Remark 3.1 Condition $(I3)$ is equivalent to the following:
$(I3')$ if $x \in I$ and $y \leq x$, then $y \in I$.

If A is a linear $BZMV^{dm}$-algebra, then the set of ideals of A is totally ordered by inclusion. Indeed if I, J are two ideals of A such that $I \nsubseteq J$ and $J \nsubseteq I$ then there would be elements $a, b \in A$ such that $a \in J \backslash I$ and $b \in I \backslash J$ thus $a \nleqslant b$ and $b \nleqslant a$ (if $a \leqslant b$, then a is in I or if $b \leqslant a$, hence b is in J) which is impossible because A is a linear $BZMV^{dm}$-algebra.

Definition 3.2 **([4])** Let I be an ideal of A. I is called a $\sim$-ideal of A, which satisfies the following condition: $\forall x, y \in A$, if $x \odot y \in I$, then $\sim\sim x \odot \sim \neg y \in I$.

In Example 3.1 set of $\sim$-ideals is $\{\{0\}, \{0, a\}, \{0, b, d\}, \{0, a, b, c, d, 1\}\}$.

Theorem 3.3 **([4])** *Let I be a $\sim$-ideal of A. The relation $x \equiv_I y$ iff $(x \odot \neg y) \oplus (\neg x \odot y) \in I$ is a congruence relation on A.*

If I is $\sim$-ideal of A, then we shall denote the equivalence class of $x \in A$ with respect to $\equiv_I$ by x/I and the quotient set $A/\equiv_I$ by A/I. We remark that A/I becomes a $BZMV^{dm}$-algebra with the natural operations $(x/I) \oplus (y/I) = (x \oplus y)/I$; $\neg(x/I) = (\neg x)/I$; $\sim (x/I) = (\sim x)/I$. The $BZMV^{dm}$-algebra $(A/I, \oplus, \neg, \sim, 0/I = I, 1/I)$ is called the quotient algebra of A by the $\sim$-ideal I.

Definition 3.4 **([4])** An ideal I of A is called a prime ideal iff $\forall x, y \in A$, $x \odot \neg y \in I$ or $\neg x \odot y \in I$.

In Example 3.1 set of prime ideals is $\{\{0, a\}, \{0, b, d\}, \{0, a, b, c, d, 1\}\}$.

Theorem 3.5 *I is a prime ideal of A iff $x \wedge y \in I$ implies $x \in I$ or $y \in I$.*

Proof. If I is a prime ideal, then $\neg x \odot y \in I$ or $x \odot \neg y \in I$ now without lose of generality we consider $\neg x \odot y \in I$ and

$$x \wedge y = y \odot (\neg y \oplus x) = \neg[\neg y \oplus \neg(\neg y \oplus x)] = \neg[\neg y \oplus (y \odot \neg x)] \in I. \tag{3.1}$$

From $\neg x \odot y \in I$ and (3.1) we have

$$\neg[\neg y \oplus (y \odot \neg x)] \oplus (y \odot \neg x) \in I. \tag{3.2}$$

Hence $y \vee (y \odot \neg x) \in I$ whereas $y \odot \neg x \leq y$ so $y \vee (y \odot \neg x) = y \in I$ and if $\neg y \odot x \in I$, then $s \in I$. Conversely, we have $\neg x \odot y \wedge x \odot \neg y = 0 \in I$, then $\neg x \odot y \in I$ or $x \odot \neg y \in I$. □

Theorem 3.6 **(Extension property)** *Let P and Q be ideals of A and $P \subseteq Q$. If P is a prime ideal, then Q is a prime ideal too.*

Theorem 3.7 *Let I and J are ideals of A and $P = I \cap J$ is prime ideal. Then $P = I$ or $P = J$.*

Proof. We have $P = I \cap J \subseteq I$ and $P = I \cap J \subseteq J$ and P is prime ideal thus by Theorem 3.6, J and I are prime ideal. Now, consider that $P \neq I$ and $P \neq J$, thus there exists $x \in I \backslash J$ $(x \in I \backslash (I \cap J))$ and $y \in J \backslash I$ $(y \in J \backslash (I \cap J))$. P is prime so $\neg x \odot y \in P$ or $x \odot \neg y \in P$, if $\neg x \odot y \in P \subseteq I$, therefore $x \oplus (\neg x \odot y) = x \vee y \in I$ and $y \leq (x \vee y)$ so $y \in I$ that is a contradiction and if $x \odot \neg y \in P$ by same proof $x \in J$ that is a contradiction too, so $P = I$ or $P = J$. □

Definition 3.8 Let I be an ideal of A. I is called normal ideal whenever,

$$\forall x, y \in A, \neg x \odot y \in I \quad \text{iff} \quad \sim x \odot y \in I. \tag{3.3}$$

In Example 3.1, set of normal ideals is $\{\{0, a, b, c, d, 1\}, \{0, b, d\}\}$.

Theorem 3.9 *Intersection of every family of normal ideals ($\sim$-ideals) is a normal ideal ($\sim$-ideal).*

In Example 3.1, consider the ideals $\{0, a\}$ and $\{0, b, d\}$. $\{0, a\} \cup \{0, b, d\} = \{0, a, b, d\}$ isn't an ideal so union of two ideal isn't always an ideal. The above theorem isn't true for prime ideals in Example 3.1 the ideals $\{0, a\}$ and $\{0, b, d\}$ are prime ideals but $\{0, a\} \cap \{0, b, d\} = \{0\}$ isn't a prime ideal.

Theorem 3.10 *If I is an ideal of B and $\phi : A \to B$ is a homomorphism, then $\phi^{-1}(I)$ is an ideal of A, and if $\phi : A \to B$ is an epimorphism and J is an ideal of A, then $\phi(J)$ is an ideal of B.*

Proof. Let I be an ideal of A. Thus $0 \in \phi^{-1}(I)$ and

$$\begin{aligned} x, y \in \phi^{-1}(I) &\Rightarrow \phi(x), \phi(y) \in I \Rightarrow \phi(x) \oplus \phi(y) \in I \text{ (I is an ideal)} \\ &\Rightarrow \phi(x \oplus y) \in I \Rightarrow x \oplus y \in \phi^{-1}(I); \\ x \in \phi^{-1}(I), y \in A &\Rightarrow \phi(x) \in I, \phi(y) \in B \\ &\Rightarrow \phi(x) \odot \phi(y) \in I \text{ (I is an ideal)} \\ &\Rightarrow \phi(x \odot y) \in I \Rightarrow (x \odot y) \in \phi^{-1}(I). \end{aligned}$$

Thus $\phi^{-1}(I)$ is an ideal of A, now if J is an ideal of A, then $0 \in \phi(J)$ and $x, y \in \phi(J)$ so there exist $a, b \in J$ such that $\phi(a) = x$ and $\phi(b) = y$ now J is an ideal and $a, b \in J$, thus $a \oplus b \in J$ and $\phi(a \oplus b) \in \phi(J), \phi$ is a homomorphism so $\phi(a) \oplus \phi(b) \in \phi(J)$ at last $x \oplus y \in \phi(J)$; $x \in \phi(J), y \in B$ so there exists $a \in J$ and ϕ is an epimorphism so there exists $b \in A$ such that $\phi(a) = x, \phi(b) = y$, where $a \in J$ and $b \in A$ and J is an ideal thus $a \odot b \in J$ and $\phi(a \odot b) \in \phi(J)$ ϕ is homomorphism thus $\phi(a) \odot \phi(b) \in \phi(J)$ so $x \odot y \in \phi(J)$ and $\phi(J)$ is an ideal of A. □

Theorem 3.11 *If I is a normal ideal of B and $\phi : A \to B$ is a homomorphism then $\phi^{-1}(I)$ is a normal ideal of A, and if $\phi : A \to B$ is a epimorphism and J is a normal ideal of A, then $\phi(J)$ is a normal ideal of B.*

Proof. If I is a normal ideal of B, then by Theorem 3.10, $\phi^{-1}(I)$ is an ideal and $\forall a, b \in \phi^{-1}(I), \neg a \odot b \in \phi^{-1}(I) \Leftrightarrow \phi(\neg a \odot b) \in I \Leftrightarrow \neg\phi(a) \odot \phi(b) \in I \Leftrightarrow \sim \phi(a) \odot \phi(b) \in I$ (I is a normal ideal) $\Leftrightarrow \phi(\sim a \odot b) \in I$ (ϕ is homomorphism) $\Leftrightarrow (\sim a \odot b) \in \phi^{-1}(I)$. If J is a normal ideal, we can see that $\phi(J)$ is a normal ideal. □

In Example 3.1, {0} and {0,a} are $\sim$-ideal but they are not a normal ideal; the following theorem states that the notions of normal ideal and $\sim$-ideal are equivalent.

Theorem 3.12 *Let I be an ideal of A and $\sim\sim x = x$, for all $x \in A$. Then the following conditions are equivalent*:

1 : *I is a normal ideal.*
2 : *I is a $\sim$-ideal.*
3 : *$\sim x \in I$ iff $\neg x \in I$.*

Proof. $(1 \to 2)$: Let I be a normal ideal. Then $\forall x, y \in A, x \odot y \in I \Rightarrow \neg\neg x \odot y \in I \Rightarrow \sim \neg x \odot y \in I \Rightarrow \sim \neg x \odot \sim\sim y \in I$ so I is a $\sim$-ideal of A.

$(2 \to 1)$: Let I be a $\sim$-ideal. Then $\neg x \odot y \in I \Rightarrow \sim \neg(\neg x) \odot \sim\sim y \in I \Rightarrow \sim (\neg\neg x) \odot \sim\sim y \in I \Rightarrow \sim x \odot y \in I$, $\sim x \odot y \in I \Rightarrow \sim \neg(\sim x) \odot \sim\sim y \in I \Rightarrow \sim (\neg \sim x) \odot \sim\sim y \in I \Rightarrow \sim\sim (\sim x) \odot \sim\sim y \in I \Rightarrow \neg(\sim\sim x) \odot \sim\sim y \in I \Rightarrow \neg x \odot y \in I$, so I is a normal ideal of A.

$(1 \to 3)$: Let I be a normal ideal of A and $y = 1$. Then $\sim x \odot y = \sim x \odot 1 = \sim x \in I$ iff $\neg x \odot y = \neg x \odot 1 = \neg x \in I$.

$(3 \to 1)$: $\sim x \odot y \in I \Leftrightarrow \sim x \odot \sim\sim y \in I \Leftrightarrow \sim (x \oplus \sim y) \in I \Leftrightarrow \neg(x \oplus \sim y) \in I \Leftrightarrow \neg x \odot \neg \sim y \in I \Leftrightarrow \neg x \odot \sim\sim y \in I \Leftrightarrow \neg x \odot y \in I$. □

By Theorem 2.6, if I is a normal ideal, then $I \cap B(A)$ is a $\sim$-ideal and if I is a $\sim$-ideal, then $I \cap B(A)$ is a normal ideal so that $B(A)$ is Boolean center of A. A $BZMV^{dm}$-algebra is simple if it has no proper ideals.

Theorem 3.13 *If A is a finite linear $BZMV^{dm}$-algebra, then A is a simple $BZMV^{dm}$-algebra.*

Proof. Let I be a finite linear $BZMV^{dm}$-algebra and I be an ideal of A. If $x \in I$ and $x \neq 0$, then $x \leq x \oplus x$. By Theorem 2.6, we have $x = x \oplus x = \sim\sim x = 1$ thus $I = A$. So if $x \leq 2x \leq 4x \leq 8x \leq ... \leq 2^n x \leq ...$ such that $x, 2x, 4x, ..., 2^n x, ... \in I$. A is finite thus there exists $n \in N$ such that $2^n x = 2^{n+1} x$ and $2^n x \oplus 2^{n+1} x = 2^n x \oplus 2^n x = \sim\sim (2^n x) = 1 \in I$, so $I = A$. □

Theorem 3.14 *Let A be a linear $BZMV^{dm}$-algebra. Then A doesn't have a proper normal ideal nor a $\sim$- ideal.*

Proof. A is a chain by Theorem 2.8, for all $a \in A$,

$$\sim a = \begin{cases} 1, & \text{if } \ a = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that

$$\forall a, b \in A, \neg a \odot b \in I \quad \text{iff} \quad \sim a \odot b \in I. \tag{3.4}$$

If $a = 0$, then $\neg a = \neg 0 = \sim 0 = \sim a = 1$ thus $\forall b \in A \ 1 \odot b = b \in I$ iff $1 \odot b = b \in I$ now if $a \neq 0$, then $\sim a = 0$ and $\sim a \odot b = 0 \in I$ so (3.4) changed to $\forall a, b \in A, \neg a \odot b \in I$ iff $0 \in I$.

I is a normal ideal if and only if $\forall a, b \in A, a \neq 0 \neg a \odot b \in I$.

Now, if $b = 1$ and $a = x \neq 0$, then $\neg a \odot b = \neg x \odot 1 = \neg x \in I$ and if $b = 1$ and $a = \neg x \neq 0$, then $\neg a \odot b = \neg(\neg x) \odot 1 = x \odot 1 = x \in I$ so $x \in I$ and $\neg x \in I$ and I is an ideal hence $1 = x \oplus \neg x \in I$ so $I = A$. Now, let I be a $\sim$-ideal and $0 \neq x \in I$. Then $\sim\sim x = 1$ and $x \odot 1 \in I$, I is a $\sim$-ideal so $\sim\sim x \odot \sim \neg 1 = 1 \odot 1 = 1 \in I$ so $I = A$. □

Definition 3.15 A is a $BZMV^{dm}$-algebra, for every subset $M \subseteq A$, the smallest ideal of A which contains M (i.e., the intersection of all ideals $I \subseteq M$)is said to be the ideal generated by M, and we denote by $(M]$. If $M = \{a\}$, $a \in A$ we denote it by $(a]$ and called it principal ideal of A.

We can show that $(M] = \{x \in A : x \leq x_1 \oplus x_2 \oplus \ldots \oplus x_n \text{ for some } x_1, x_2 \ldots x_n \in M\}$ and $(a] = \{x \in A : x \leq na, 0 \leq n\}$. In Example 3.1 we have $(b] = \{0, b, d\}$ and $(a] = \{0, a\}$.

Theorem 3.16 $c \wedge (a \oplus b) \leq (c \wedge a) \oplus (c \wedge b)$*, for all $a, b, c \in A$.*

Proof. $c\wedge(a\oplus b) = c\odot(\neg c\oplus(a\oplus b)), (c\wedge a)\oplus(c\wedge b) = (c\odot(\neg c\oplus a))\oplus(c\odot(\neg c\oplus b))$ so we have $\neg[(c\wedge(a\oplus b)]\oplus[(c\wedge a)\oplus(c\wedge b)] = \neg[c\odot(\neg c\oplus(a\oplus b))]\oplus[(c\odot(\neg c\oplus a))\oplus(c\odot(\neg c\oplus b))] = (\neg c)\oplus(c\odot\neg(a\oplus b))\oplus[c\odot(\neg c\oplus a)]\oplus[c\odot(\neg c\oplus b)] = (\neg c)\oplus[c\odot(\neg c\oplus a)]\oplus[c\odot(\neg c\oplus b)]\oplus(c\odot\neg(a\oplus b)) = [\neg c\vee(\neg c\oplus a)]\oplus[c\odot(\neg c\oplus b)]\oplus(c\odot\neg(a\oplus b)) = (a\oplus\neg c)\oplus[c\odot(\neg c\oplus b)]\oplus(c\odot\neg(a\oplus b)) = a\oplus(\neg c\vee(\neg c\oplus b))\oplus(c\odot\neg(a\oplus b)) = a\oplus b\oplus\neg c\oplus(c\odot\neg(a\oplus b)) = (a\oplus b\oplus\neg c)\oplus\neg(\neg c\oplus a\oplus b) = 1.$ □

By induction we have $c\wedge n(a)\leq n(c\wedge a)$.

Theorem 3.17 *If $g,k\in A$, then $(g\wedge k] = (g]\cap(k]$.*

Proof. $g\in(g]$ and $k\in(k]$ since $g\wedge k\leq g,k$ we get that $g\wedge k\in(g]$ and $g\wedge k\in(k]$, then $g\wedge k\in(g]\cap(k]$ is an ideal since $(g\wedge k] = \cap\{D\in Id(A) : g\wedge k\in D\}$, thus $(g\wedge k]\leq(g]\cap(k]$.

Conversely, suppose that $h\in(g]\cap(k]$, then $h\leq ng$ and $h\leq mk$ for $m,n\geq 1$ hence $h\leq ng\wedge mk\leq n(g\wedge mk)\leq nm(g\wedge k)$ (by Theorem 3.16) thus $h\in(g\wedge k]$. □

Definition 3.18 An ideal M of A is called maximal whenever J is an ideal such that $M\subseteq J\subseteq A$, then either $J = M$ or $J = A$.

In Example 3.1 the ideals $\{0,b,d\}$ and $\{0,a\}$ are maximal ideals of A.

Theorem 3.19 *If I is a prime and normal ideal of A, then I is a maximal ideal of A.*

Proof. Let J be an ideal of A. If $I\subseteq J\subseteq A$ and $I\neq J$, then there exists an element $x\in J\setminus I$. I is a prime ideal and consider $\sim\sim x\in A$ and $\sim x\in A$ we have $\neg(\sim\sim x)\odot\sim x\in I$ or $\sim\sim x\odot\neg\sim x\in I$. If $\sim\sim x\odot\neg\sim x\in I$, then whereas $\neg\sim x=\sim\sim x$, thus $\sim\sim x\odot\sim\sim x\in I$ and from Proposition 4.4 [4] $\sim(\sim x\oplus\sim x)\in I$ and from Theorem 2.5 $\sim\sim x\in I$ hence $x\leq\sim\sim x\in I$ so $x\in I$ and it is contradiction. So $\neg\sim\sim x\odot\sim x\in I$ and from Theorem 2.5 $\sim x\odot\sim x\in I$ so $\sim(x\oplus x)\in I$ and I is normal ideal thus $\neg(x\oplus x)\in I$ and $I\subset J$, then $\neg(x\oplus x)\in J$ and $x\in j$ so $x\oplus x\in J$ and J is ideal thus $\neg(x\oplus x)\oplus(x\oplus x) = 1\in J$ hence $J = A$. □

Theorem 3.20 *If J is a $\sim$-ideal and $I\subseteq J$ where I is a prime ideal, then $J = A$ or $J = I$.*

Proof. If I is a prime ideal and $I\subseteq J\subseteq A$, $J\neq I$ and J is a $\sim$-ideal, then there exists $x\in J\setminus I$. I is a prime ideal and we consider $\sim x$ and $\sim\sim x$ in A, thus we have $\neg\sim x\odot\sim\sim x\in I$ therefore $x\in I$ and it is contradiction. So $\sim x\odot\neg(\sim\sim x)\in I$ hence $\sim x\odot\sim x\in I$ and $\sim(x\oplus x)\in I\subseteq J$. J is a $\sim$-ideal and $x\in J$ so $x\oplus x\in J$ and J is a $\sim$-ideal thus $\sim\sim(x\oplus x)\in J$ and $\sim(x\oplus x)\in J$ and I is an ideal $\sim\sim(x\oplus x)\oplus\sim(x\oplus x) = 1\in J$ $(BZMV^{dm}5)$ thus $I = A$. □

Definition 3.21 An ideal I of A is called involutive ideal whenever $x\in I$, implies $\sim\sim x\in I$.

In Example 3.1, set of involutive ideals is $\{\{0\},\{0,a\},\{0,b,d\},\{0,a,b,c,d,1\}\}$.

Theorem 3.22 *Every $\sim$-ideal is an involutive ideal.*

Proof. Let I be an $\sim$-ideal and $x\in I$. Then $x = x\odot 1\in I$ so $\sim\sim x\odot\sim\neg 1 = \sim\sim x\in I$, hence I is an involutive ideal. □

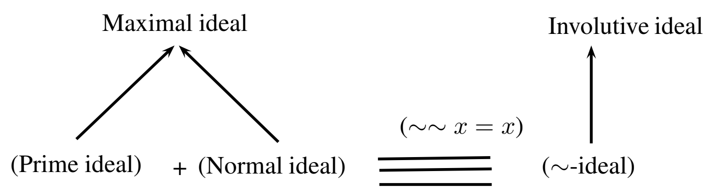
Definition 3.23 An ideal J of A is called implicative if for all $x, y, z \in A$ $x \odot y \odot (\neg z) \in J$ and $z \odot y \in J$ implies $x \odot y \in J$.

In Example 3.1, set of implicative ideals is $\{\{0, a, b, c, d, 1\}, \{0, b, d\}\}$.

Theorem 3.24 *If I is an implicative ideal, then $a \wedge \neg a \in I$, $\forall a \in A$.*

Proof. In Definition 3.23, if $x = \neg(a \odot a)$, $y = a$ and $z = \neg a$, then $x \odot y \odot \neg z = \neg(a \odot a) \odot a \odot \neg(\neg a) = \neg(a \odot a) \odot (a \odot a) = 0 \in I$ and $y \odot z = a \odot \neg a = 0 \in I$ so $x \odot y \in I$ and $x \odot y = \neg(a \odot a) \odot a = a \wedge \neg a \in I$. □

In this diagram we show the relationships between different ideals in $BZMV^{dm}$-algebra.



4 Conclusions and future works

Cattaneo, Giuntini and Pilla [4] introduced the concept of $BZMV^{dm}$-algebras as an abstract environment to describe both shadowed and fuzzy sets. This structure is endowed with two unusual complementations: a fuzzy one $\neg$ and an intuitionistic one $\sim$ and the main properties of this system was studied. For investigating shadowed sets, fuzzy sets and rough sets, $BZMV^{dm}$-algebra are used as an algebraic tool. Ideals in $BZMV^{dm}$-algebra are the subsets of this algebraic structure with special property that are studied for getting more result of it.

In this paper we introduced some ideals in $BZMV^{dm}$-algebra and we studied some relation between them and the other ideals (prime ideal and $\sim$-ideal) of A, that come in above diagram.

For the future works we can use these types of ideals for get classification of $BZMV^{dm}$-algebra and study the quotient $BZMV^{dm}$-algebra via with types of ideals.

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References

1. Birkhoff, G. – *Lattice Theory*, Third edition, American Mathematical Society Colloquium Publications, Vol. XXV American Mathematical Society, Providence, R.I., 1967.
2. Cattaneo, G.; Ciucci, D.; Giuntini, R.; Konig, M. – *Algebraic structures related to many valued logical systems. II. Equivalence among some widespread structures*, Fund. Inform., 63 (2004), 357–373.

3. Cattaneo, G.; Ciucci, D. – *Shadowed sets and related algebraic structures*, Fund. Inform., 55 (2003), 255–284.
4. Cattaneo, G.; Giuntini, R.; Pilla, R. – *$BZMV^{dM}$ algebras and Stonian MV-algebras* (*applications to fuzzy sets and rough approximations*), Fuzzy Sets and Systems, 108 (1999), 201–222.
5. Cattaneo, G. – *Abstract approximation spaces for rough theories*, Rough sets in knowledge discovery, 1, 59–98, Stud. Fuzziness Soft Comput., 18, Physica, Heidelberg, 1998.
6. Cattaneo, G.; Ciucci, D. – *BZW algebras for an abstract approach to roughness and fuzziness,* IPMU 2002, July 1–5 2002, Annecy, France, Proceedings, ESIA-Université de Savoie, 2002.
7. Chang, C.C. – *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc., 88 (1958), 467–490.
8. Pedrycz, W. – *Shadowed Sets: Representing and Processing Fuzzy Sets*, IEEE Transaction on Systems, Man and Cybernetics - Part B: Cybernetics, 28 (1998), 103–109.
9. Pedrycz, W. – *Shadowed sets: bridging fuzzy and rough sets*, Rough fuzzy hybridization, 179–199, Springer, Singapore, 1999.