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On warped product super generalized recurrent manifolds

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Abstract The object of the present paper is to obtain the characterization of a warped product semi-Riemannian manifold with a special type of recurrent like structure, called super generalized recurrent. As consequence of this result we also find out the necessary and sufficient conditions for a warped product manifold to satisfy some other recurrent like structures such as weakly generalized recurrent manifold, hyper generalized recurrent manifold etc. Finally as a support of the main result, we present an example of warped product super generalized recurrent manifold.

Keywords recurrent manifold · weakly generalized recurrent manifold · hyper generalized recurrent manifold · super generalized recurrent manifold · warped product manifold

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1 Introduction

To generalize the notion of a manifold of constant curvature, Cartan [4] first introduced the notion of local symmetry which can be presented as the curvature restriction $\nabla R = 0$ (i.e., the Riemann-Christoffel curvature tensor R is covariantly constant). But there are many manifolds which does not bear local symmetry and hence to investigate the type of symmetry of such manifolds it is necessary to generalize the notion of local symmetry. During the last eight decades various researchers are working on this area to generalize or to extend the notion of local symmetry by weakening its curvature restriction in different directions.

Cartan [5] first gave a proper generalization of local symmetry and introduced the notion of semisymmetry, which was later classified by Szabó ([42], [43], [44]). Then in 1983 Adamów and Deszcz [1] generalized the notion of semisymmetry and introduced the notion of pseudosymmetry, also known as Deszcz pseudosymmetry (see [7], [8], [13], [27], [46]). On the other hand as a generalization of local symmetry, Chaki [6] introduced the notion of pseudosymmetry. We note that the interrelation between two types of pseudosymmetry is studied by Shaikh et. al. [27]. In 1989 Tamássy and Binh [45] generalized the Chaki's notion of pseudosymmetry and introduced the concept of weakly symmetric manifold. We note that Shaikh and his co-authors ([18], [28]-[32]) studied this notion of weak symmetry with various generalized curvature tensors. Again generalizing the results of Binh [2], recently, Shaikh and Kundu [33] obtained the characterization of warped product weakly symmetric manifolds.

Again recurrent manifold (briefly, K_n) ([23], [24], [25]) is another type of generalization of local symmetry. In 1979 Dubey [15] defined generalized recurrent manifold (briefly, GK_n). Recently Olszak and Olszak [21] showed that every GK_n is concircularly recurrent and every concircularly recurrent manifold is again a K_n and hence every GK_n is a K_n. Again as a generalization of K_n, recently, Shaikh and his coauthors introduced three new type of generalized recurrent structures together with their proper existence, namely, quasi generalized recurrent manifold (briefly, QGK_n) [38], hyper-generalized recurrent manifold (briefly, HGK_n) [37] and weakly generalized recurrent manifold (briefly, WGK_n) [39]. For the existence of such structures we refer the reader to see [26], [41]. Very recently, Shaikh et al. ([34], [40]) introduced another generalization of recurrent manifold, called, super generalized recurrent manifold (briefly, SGK_n) which also generalizes the notion of HGK_n as well as WGK_n. These kinds of generalization of recurrent structures may be called as recurrent like structures.

The main object of the present paper is to investigate the characterization of a warped product manifold with some recurrent like structures, such as, K_n, HGK_n, WGK_n and SGK_n. For this purpose we first determine the necessary and sufficient condition for a warped product manifold to be SGK_n and as consequence of this result we find out the corresponding results for HGK_n, WGK_n and K_n. We know that decomposable or product manifold is a special case of warped product manifold when the warping function is identically 1. Thus we can present the characterization of a decomposable manifold with various recurrent like structures.

The paper is organized as follows: Section 2 deals with rudimentary facts of various recurrent like structures. Section 3 is concerned with basic curvature relations of a warped product manifold. In section 4 we present our main result and, finally, in section 5 a proper example of a warped product SGK_n is presented.

2 Preliminaries

Let M be a non-flat n-dimensional ($n \geq 3$) smooth manifold with semi-Riemannian metric g, Levi-Civita connection ∇, Riemann-Christoffel curvature tensor R, Ricci tensor S and scalar curvature κ. In this section we define various necessary terms and curvature restricted geometric structures and for this purpose at first we consider some notations:

$C^\infty(M)$ = the algebra of all smooth functions on M,
$\chi(M)$ = the Lie algebra of all smooth vector fields on M,
$\chi^*(M)$ = the Lie algebra of all smooth 1-forms on M and
$\mathcal{T}^r_k(M)$ = the space of all smooth tensor fields of type (r,k) on M.
For $A, E \in \mathcal{T}^0_2(M)$, the Kulkarni-Nomizu product [34] $A \wedge E \in \mathcal{T}^0_4(M)$ is given by

$$(A \wedge E)_{ijkl} = A_{il}E_{jk} + A_{jk}E_{il} - A_{ik}E_{jl} - A_{jl}E_{ik}. \tag{2.1}$$

In particular we can get $g \wedge g$, $g \wedge S$ and $S \wedge S$ as follows:

$$(g \wedge g)_{ijkl} = 2(g_{il}g_{jk} - g_{ik}g_{jl}),$$

$$(g \wedge S)_{ijkl} = g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl} \text{ and}$$

$$(S \wedge S)_{ijkl} = 2(S_{il}S_{jk} - S_{ik}S_{jl}).$$

Definition 2.1 *The manifold M is said to be recurrent [47] if*

$$\nabla R = \Pi \otimes R \quad (\otimes \text{ denotes the tensor product}) \tag{2.2}$$

$$(\text{or locally, } R_{ijkl,m} = \Pi_m R_{ijkl}, \quad \text{where `,' denotes the covariant derivative})$$

holds on $\{x \in M : (R)_x \neq 0\}$ for an 1-form Π, called the associated 1-form of the recurrent structure. Such an n-dimensional manifold is denoted by K_n.

Again M is said to be concircularly recurrent if its concircular curvature tensor $W = R - \frac{\kappa}{2n(n-1)}(g \wedge g)$ satisfies the condition

$$\nabla W = \Pi \otimes W \tag{2.3}$$

on $\{x \in M : (W)_x \neq 0\}$ for an 1-form Π, called the associated 1-form. It may be mentioned that concircularly recurrent manifolds were investigated in [20].
The manifold (M, g) is said to be generalized recurrent [15] if it satisfies

$$\nabla R = \Pi \otimes R + \Theta \otimes (g \wedge g) \tag{2.4}$$

$$(\text{or locally, } R_{ijkl,m} = \Pi_m R_{ijkl} + 2\Theta_m(g_{il}g_{jk} - g_{ik}g_{jl}))$$

on $\{x \in M : (R)_x \neq 0\}$ for some 1-forms Π and Θ. The 1-forms Π and Θ are called the associated 1-forms of this structure. Such an n-dimensional manifold is denoted by GK_n. In [21] Olszak and Olszak showed that every GK_n satisfying (2.4) is concircularly recurrent with (2.3) and every concircularly recurrent manifold is again a K_n with same associated 1-form and thus $\Theta = 0$. Hence the structure GK_n reduces to K_n. Consequently the notion of GK_n does not exist.

Definition 2.2 *The manifold M is said to be quasi generalized recurrent [38], hyper generalized recurrent [37], weakly generalized recurrent [39] and super generalized recurrent respectively if the condition*

$$\nabla R = \Pi \otimes R + \Psi \otimes [g \wedge (g + \eta \otimes \eta)], \tag{2.5}$$

$$\nabla R = \Pi \otimes R + \Psi \otimes (g \wedge S), \tag{2.6}$$

$$\nabla R = \Pi \otimes R + \Phi \otimes (S \wedge S) \quad and \tag{2.7}$$

$$\nabla R = \Pi \otimes R + \Phi \otimes (S \wedge S) + \Psi \otimes (g \wedge S) + \Theta \otimes (g \wedge g) \tag{2.8}$$

holds on $\{x \in M : (R)_x \neq 0$ and $[g \wedge (g + \eta \otimes \eta)]_x \neq 0\}$, $\{x \in M : (R)_x \neq 0$ and $(g \wedge S)_x \neq 0\}$, $\{x \in M : (R)_x \neq 0$ and $(S \wedge S)_x \neq 0\}$ and $\{x \in M : (R)_x \neq 0$ and either $(S \wedge S)_x \neq 0$ or $(g \wedge S)_x \neq 0\}$ respectively for some Π, Φ, Ψ, Θ and $\eta \in \chi^*(M)$, called the associated 1-forms.

An n-dimensional manifold satisfying (2.5), (2.6), (2.7) and (2.8) respectively is denoted by QGK_n with (Π, Θ, η) or simply QGK_n, HGK_n with (Π, Ψ) or simply HGK_n, WGK_n with (Π, Φ) or simply WGK_n and SGK_n with $(\Pi, \Phi, \Psi, \Theta)$ or simply SGK_n.
In terms of local coordinates (2.5)-(2.8) can respectively be written as:

$$\begin{aligned} R_{ijkl,m} = \Pi_m R_{ijkl} + \Theta_m[2(g_{il}g_{jk} - g_{ik}g_{jl}) \\ + g_{il}\eta_j\eta_k + g_{jk}\eta_i\eta_l - g_{ik}\eta_j\eta_l - g_{jl}\eta_i\eta_k], \end{aligned} \tag{2.9}$$

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Psi_m[g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl}], \tag{2.10}$$

$$R_{ijkl,m} = \Pi_m R_{ijkl} + 2\Psi_m[S_{il}S_{jk} - S_{ik}S_{jl}] \quad \text{and} \tag{2.11}$$

$$\begin{aligned} R_{ijkl,m} = \Pi_m R_{ijkl} + 2\Phi_m[S_{il}S_{jk} - S_{ik}S_{jl}] \\ + \Psi_m[g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl}] + 2\Theta[g_{il}g_{jk} - g_{ik}g_{jl}]. \end{aligned} \tag{2.12}$$

It is obvious that the above structures QGK_n, WGK_n and HGK_n are all generalization of K_n but each of three exists independently (see [26], [41]). We also mention that SGK_n is a proper generalization of WGK_n and HGK_n (see Section 5).

Definition 2.3 *Let $D \in \mathcal{T}^0_4(M)$ and $A, E, F \in \mathcal{T}^0_2(M)$. Then M is said to be Roter type manifold (briefly, RT_n) with $(D; A, E)$ ([9], [10] and see also [12], [14]) and generalized Roter type manifold (briefly, GRT_n) with $(D; A, E, F)$ ([11], [27], [35], [36]) respectively if*

$$D = N_1(A \wedge A) - N_2(A \wedge E) - N_3(E \wedge E) \quad and$$

$$D = L_1(A \wedge A) - L_2(A \wedge E) - L_3(E \wedge E) - L_4(A \wedge F) - L_5(E \wedge F) - L_6(F \wedge F)$$

respectively, for some $N_i, L_j \in C^\infty(M)$, $1 \leq i \leq 3$ and $1 \leq j \leq 6$.

3 Warped product manifolds

In 1957 Kručkovič [19] initiated the study of semi-decomposable manifolds which were latter named as warped product manifolds by Bishop and O'Neill [3]. Let $(\overline{M}, \overline{g})$ and $(\widetilde{M}, \widetilde{g})$ be two semi-Riemannian manifolds of dimension p and $(n-p)$ respectively $(1 \leq p < n)$, and f is a positive smooth function on $\overline{M}$. Then the warped product $M = \overline{M} \times_f \widetilde{M}$ is the product manifold $\overline{M} \times \widetilde{M}$ of dimension n endowed with the metric

$$g = \pi^*(\overline{g}) + (f \circ \pi)\sigma^*(\widetilde{g}), \tag{3.1}$$

where $\pi : M \to \overline{M}$ and $\sigma : M \to \widetilde{M}$ are the natural projections. Then the local components of the metric g are given by

$$g_{ij} = \begin{cases} \overline{g}_{ij} & \text{for } i = a \text{ and } j = b, \\ f\widetilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Here $a, b \in \{1, 2, ..., p\}$ and $\alpha, \beta \in \{p+1, p+2, ..., n\}$. We note that throughout the paper we consider $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$. Here $\overline{M}$ is called the base, $\widetilde{M}$ is called the fiber and f is called warping function of the warped product manifold $M = \overline{M} \times_f \widetilde{M}$. It may be mentioned that the warped product metric g can also be taken as (see [17], [19], [48])

$$g = \pi^*(\overline{g}) + (f \circ \pi)^2\sigma^*(\widetilde{g}).$$

However throughout the paper we will consider the warped product metric given in (3.1). Again we assume that, when Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ and $\widetilde{\Omega}$, the similar quantities formed with respect to $\overline{g}$ and $\widetilde{g}$ respectively. By straightforward calculation one can easily calculate the local components of Ω in terms of $\overline{\Omega}$ and $\widetilde{\Omega}$.
The non-zero local components of the Levi-Civita connection ∇ of M are given by

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\beta\gamma} = -\frac{1}{2}\overline{g}^{ab} f_b \widetilde{g}_{\beta\gamma}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{3.3}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$.
The local components of the Riemann-Christoffel curvature tensor R and the Ricci tensor S of M which may not vanish identically are the following:

$$R_{abcd} = \overline{R}_{abcd}, \quad R_{a\alpha b\beta} = f T_{ab} \widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f\widetilde{R}_{\alpha\beta\gamma\delta} - f^2 P \widetilde{G}_{\alpha\beta\gamma\delta}, \tag{3.4}$$

$$S_{ab} = \overline{S}_{ab} - (n-p)T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}, \tag{3.5}$$

where $G_{ijkl} = \frac{1}{2}(g \wedge g)_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ are the components of the Gaussian curvature and

$$T_{ab} = -\frac{1}{2f}(\nabla_b f_a - \frac{1}{2f} f_a f_b), \quad tr(T) = g^{ab}T_{ab},$$

$$Q = f((n-p-1)P - tr(T)), \quad P = \frac{1}{4f^2} g^{ab} f_a f_b.$$

The scalar curvature κ of M is given by

$$\kappa = \bar{\kappa} + \frac{\tilde{\kappa}}{f} - (n-p)[(n-p-1)P - 2\, tr(T)]. \tag{3.6}$$

Again the non-zero local components of ∇R are given by [16]:

$$\left.\begin{array}{l}
(i) R_{abcd,e} = \overline{R}_{abcd,e}, \\
(ii) R_{a\alpha b\beta,e} = f T_{ab,e} \widetilde{g}_{\alpha\beta}, \\
(iii) R_{\alpha\beta\gamma\delta,e} = -f_e \widetilde{R}_{\alpha\beta\gamma\delta} + f^2 P_e \widetilde{G}_{\alpha\beta\gamma\delta}, \\
(iv) R_{\alpha\beta\gamma\delta,\epsilon} = f \widetilde{R}_{\alpha\beta\gamma\delta,\epsilon}, \\
(v) R_{\alpha\beta\gamma d,\epsilon} = -\frac{f_d}{2} \widetilde{R}_{\alpha\beta\gamma\epsilon} + \frac{f^2}{2} P_d \widetilde{G}_{\alpha\beta\gamma\epsilon}, \\
(vi) R_{abc\delta,\epsilon} = \frac{1}{2}\widetilde{g}_{\epsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \frac{1}{2} f^d \overline{R}_{abcd} \widetilde{g}_{\epsilon\delta}.
\end{array}\right\} \tag{3.7}$$

The non-zero components of $(g \wedge g)$, $(g \wedge S)$ and $(S \wedge S)$ are given by

$$\left.\begin{array}{l}
(i)(g \wedge g)_{abcd} = (\overline{g} \wedge \overline{g})_{abcd}, \\
(ii)(g \wedge g)_{a\alpha b\beta} = -2f \overline{g}_{ab} \widetilde{g}_{\alpha\beta}, \\
(iii)(g \wedge g)_{\alpha\beta\gamma\delta} = f^2 (\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta},
\end{array}\right\} \tag{3.8}$$

$$\left.\begin{array}{l}
(i)(g \wedge S)_{abcd} = (\overline{g} \wedge \overline{S})_{abcd} - (n-p)(\overline{g} \wedge T)_{abcd}, \\
(ii)(g \wedge S)_{a\alpha b\beta} = -\overline{g}_{ab}(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}) - f\widetilde{g}_{\alpha\beta}[\overline{S}_{ab} - (n-p)T_{ab}], \\
(iii)(g \wedge S)_{\alpha\beta\gamma\delta} = f(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + fQ(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta},
\end{array}\right\} \tag{3.9}$$

$$\left.\begin{array}{l}
(i)(S \wedge S)_{abcd} = (\overline{S} \wedge \overline{S})_{abcd} - (n-p)(\overline{S} \wedge T)_{abcd} \\
\qquad\qquad\qquad +(n-p)^2 (T \wedge T)_{abcd}, \\
(ii)(S \wedge S)_{a\alpha b\beta} = -2(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta})[\overline{S}_{ab} - (n-p)T_{ab}], \\
(iii)(S \wedge S)_{\alpha\beta\gamma\delta} = (\widetilde{S} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + Q(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + Q^2 (\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}.
\end{array}\right\} \tag{3.10}$$

For detailed information about the local components of various tensors on a warped product manifold we refer the reader to see [22], [33], [36] and also references therein.

4 Warped product SGK_n

Theorem 4.1 *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a SGK_n with $(\Pi, \Phi, \Psi, \Theta)$ if and only if the following conditions hold simultaneously*

$$\left.\begin{array}{l}
(i)\overline{\nabla}\overline{R} = \overline{\Pi}\otimes\overline{R} + \overline{\Phi}\otimes(\overline{S}\wedge\overline{S}) + \overline{\Psi}\otimes(\overline{g}\wedge\overline{S}) + \overline{\Theta}\otimes(\overline{g}\wedge\overline{g}) \\
\quad -2(n-p)\overline{\Phi}\otimes(\overline{S}\wedge T) + (n-p)^2\overline{\Phi}\otimes(T\wedge T) \\
\quad -(n-p)\overline{\Psi}\otimes(\overline{g}\wedge T), \\
(ii) - \widetilde{\Pi}\otimes\overline{R} = \widetilde{\Phi}\otimes(\overline{S}\wedge\overline{S}) + \widetilde{\Psi}\otimes(\overline{g}\wedge\overline{S}) + \widetilde{\Theta}\otimes(\overline{g}\wedge\overline{g}) \\
\qquad -2(n-p)\widetilde{\Phi}\otimes(\overline{S}\wedge T) + (n-p)^2\widetilde{\Phi}\otimes(T\wedge T) \\
\qquad -(n-p)\widetilde{\Psi}\otimes(\overline{g}\wedge T),
\end{array}\right\} \tag{4.1}$$

$$\left.\begin{array}{l}
(i) - (df + f\overline{\Pi})\otimes\widetilde{R} = \overline{\Phi}\otimes(\widetilde{S}\wedge\widetilde{S}) + (2Q\overline{\Phi} + f\overline{\Psi})\otimes(\widetilde{g}\wedge\widetilde{S}) \\
\qquad +\left[-\frac{1}{2}f^2(P\overline{\Pi} + dP) + Q^2\overline{\Phi} + fQ\overline{\Psi} + f^2\overline{\Theta}\right]\otimes(\widetilde{g}\wedge\widetilde{g}), \\
(ii) f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi}\otimes\widetilde{R} + \widetilde{\Phi}\otimes(\widetilde{S}\wedge\widetilde{S}) + (2Q\widetilde{\Phi} + f\widetilde{\Psi})\otimes(\widetilde{g}\wedge\widetilde{S}) \\
\qquad +(-\frac{1}{2}f^2P\widetilde{\Pi} + Q^2\widetilde{\Phi} + fQ\widetilde{\Psi} + f^2\widetilde{\Theta})\otimes(\widetilde{g}\wedge\widetilde{g}),
\end{array}\right\} \tag{4.2}$$

$$\left.\begin{array}{l}
(i)\ \left[2\overline{\Phi}\otimes(\overline{S} - (n-p)T) + \overline{\Psi}\otimes\overline{g}\right]\otimes\widetilde{S} = -f(\overline{\nabla}T - \overline{\Pi}\otimes T)\otimes\widetilde{g} \\
\qquad -(2Q\overline{\Phi} + f\overline{\Psi})\otimes(\overline{S} - (n-p)T)\otimes\widetilde{g} \\
\qquad -(Q\overline{\Psi} + 2f\overline{\Theta})\otimes\overline{g}\otimes\widetilde{g}, \\
(ii)\ \left[2\widetilde{\Phi}\otimes(\overline{S} - (n-p)T) + \widetilde{\Psi}\otimes\overline{g}\right]\otimes\widetilde{S} = f\widetilde{\Pi}\otimes T\otimes\widetilde{g} \\
\qquad -(2Q\widetilde{\Phi} + f\widetilde{\Psi})\otimes(\overline{S} - (n-p)T)\otimes\widetilde{g} \\
\qquad -(Q\widetilde{\Psi} + 2f\widetilde{\Theta})\otimes\overline{g}\otimes\widetilde{g},
\end{array}\right\} \tag{4.3}$$

$$\left.\begin{array}{l}
(i) f^d\overline{R}_{abcd} = -(f_aT_{bc} - f_bT_{ac}) \quad \textit{and} \\
(ii) df\otimes\widetilde{R} = f^2\ dP\otimes\widetilde{G}.
\end{array}\right\} \tag{4.4}$$

Proof. First suppose that M is SGK_n. Then in terms of local coordinates the defining condition can be written as

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Phi_m(S\wedge S)_{ijkl} + \Psi_m(g\wedge S)_{ijkl} + \Theta_m(g\wedge g)_{ijkl}. \tag{4.5}$$

Putting

$$\begin{cases}
(i)\ i = a, j = b, k = c, l = d, m = e \ \text{ and} \\
(ii)\ i = a, j = b, k = c, l = d, m = \epsilon
\end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.10) it is easy to check that (4.1) holds. Similarly putting

$$\begin{cases}
(iii)\ i = \alpha, j = \beta, k = \gamma, l = \delta, m = e; \\
(iv)\ i = \alpha, j = \beta, k = \gamma, l = \delta, m = \epsilon; \\
(v)\ i = a, j = \alpha, k = b, l = \beta, m = e; \\
(vi)\ i = a, j = \alpha, k = b, l = \beta, m = \epsilon; \\
(vii)\ i = a, j = b, k = c, l = \alpha, m = \epsilon \ \text{ and} \\
(viii)\ i = \alpha, j = \beta, k = \gamma, l = a, m = \epsilon
\end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.10) we get (4.2)-(4.4) respectively. The converse part is obvious. This proves the theorem. □

From Theorem 4.1, it follows that the nature of base and fiber of a warped product SGK_n is given by the following:

Corollary 4.2 *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product $SGK_n$with $(\Pi, \Phi, \Psi, \Theta)$. Then*
(i) $\overline{M}$ is a SGK_p if T can be expressed as a linear combination of $\overline{S}$ and $\overline{g}$.
(ii) $\overline{M}$ is a GRT_p with $(\overline{R}; \overline{g}, \overline{S}, T)$ on $\{x \in M : \widetilde{\Pi}_x \neq 0\}$.
(iii) $\widetilde{M}$ is a SGK_{n-p} with $(\widetilde{\Pi}, \frac{1}{f}\widetilde{\Phi}, \frac{2Q}{f}\widetilde{\Phi} + \widetilde{\Psi}, -\frac{1}{2}fP\widetilde{\Pi} + \frac{Q^2}{f}\widetilde{\Phi} + Q\widetilde{\Psi} + f\widetilde{\Theta})$.
(iv) $\widetilde{M}$ is a RT_{n-p} with $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(v) $\widetilde{M}$ satisfies Einstein metric condition on $\{x \in M : [2(\overline{\kappa} - (n-p)tr(T))\overline{\Phi} + p\overline{\Psi}]_x \neq 0\} \cup \{x \in M : [2(\overline{\kappa} - (n-p)tr(T))\widetilde{\Phi} + p\widetilde{\Psi}]_x \neq 0\} = \{x \in M : [2(\overline{\kappa} - (n-p)tr(T))\Phi + p\Psi]_x \neq 0\}$.
(vi) $\widetilde{M}$ is of constant curvature on $\{x \in M : df_x \neq 0\}$.

From Theorem 4.1, the characterization of a decomposable or product SGK_n is given by the following:

Corollary 4.3 *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product manifold. Then M is a SGK_n with $(\Pi, \Phi, \Psi, \Theta)$ if and only if*
1.(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes (\overline{S} \wedge \overline{S}) + \overline{\Psi} \otimes (\overline{g} \wedge \overline{S}) + \overline{\Theta} \otimes (\overline{g} \wedge \overline{g})$,
(ii) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Phi} \otimes (\overline{S} \wedge \overline{S}) + \widetilde{\Psi} \otimes (\overline{g} \wedge \overline{S}) + \widetilde{\Theta} \otimes (\overline{g} \wedge \overline{g})$,
2.(i) $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S}) + \widetilde{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S}) + \widetilde{\Theta} \otimes (\widetilde{g} \wedge \widetilde{g})$,
(ii) $-\overline{\Pi} \otimes \widetilde{R} = \overline{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S}) + \overline{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S}) + \overline{\Theta} \otimes (\widetilde{g} \wedge \widetilde{g})$,
3.(i) $\left[2\overline{\Phi} \otimes \overline{S} + \overline{\Psi} \otimes \overline{g}\right] \otimes \widetilde{S} = -[\overline{\Psi} \otimes \overline{S} + 2\overline{\Theta} \otimes \overline{g}] \otimes \widetilde{g}$,
(ii) $\left[2\widetilde{\Phi} \otimes \widetilde{S} + \widetilde{\Psi} \otimes \widetilde{g}\right] \otimes \overline{S} = -[\widetilde{\Psi} \otimes \widetilde{S} + 2\widetilde{\Theta} \otimes \widetilde{g}] \otimes \overline{g}$.

From the above, the nature of each factor of a product SGK_n is given by the following:

Corollary 4.4 *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product SGK_n with $(\Pi, \Phi, \Psi, \Theta)$. Then*
(i) base and fiber both are super generalized recurrent manifolds.
(ii) $\overline{M}$ is a RT_p with $(\overline{R}; \overline{g}, \overline{S})$ on $\{x \in M : \widetilde{\Pi}_x \neq 0\}$.
(iii) $\widetilde{M}$ is a RT_{n-p} with $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ on $\{x \in M : \overline{\Pi}_x \neq 0\}$.
(iv) $\overline{M}$ satisfies Einstein metric condition on $\{x \in M : (2\widetilde{\kappa}\widetilde{\Phi} + (n-p)\widetilde{\Psi})_x \neq 0\}$.
(v) $\widetilde{M}$ satisfies Einstein metric condition on $\{x \in M : (2\overline{\kappa}\overline{\Phi} + p\overline{\Psi})_x \neq 0\}$.

Corollary 4.5 *[33] Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a recurrent manifold with*

$$\nabla R = \Pi \otimes R$$

if and only if the following conditions hold simultaneously
1.(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R}$, (ii) $\widetilde{\Pi} \otimes \overline{R} = 0$,
2.(i) $-(df + f\overline{\Pi}) \otimes \widetilde{R} = \frac{1}{2} f^2 (P\overline{\Pi} - dP) \otimes (\widetilde{g} \wedge \widetilde{g})$,
(ii) $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$ *and* $P\widetilde{\Pi} = 0$,
3.(i) $\overline{\nabla} T = \overline{\Pi} \otimes T$, (ii) $\widetilde{\Pi} \otimes T = 0$,
4.(i) $f^d \overline{R}_{abcd} = -(f_a T_{bc} - f_b T_{ac})$ *and* (ii) $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.6 *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product recurrent manifold satisfying* $\nabla R = \Pi \otimes R$. *Then*
(i) $\overline{M}$ *and* $\widetilde{M}$ *both are recurrent. Also* T *is recurrent with associated 1-form* Π.
(ii) $\overline{R} = 0$, $T = 0$ *and* $P = 0$ *on the set* $\{x \in M : \widetilde{\Pi} \neq 0\}$.
(iii) $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\} \cup \{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.

Corollary 4.7 *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product manifold. Then* M *is recurrent satisfying*

$$\nabla R = \Pi \otimes R$$

if and only if the following conditions hold simultaneously
(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R}$, $\widetilde{\Pi} \otimes \overline{R} = 0$, (ii) $\overline{\Pi} \otimes \widetilde{R} = 0$, $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$.

Corollary 4.8 *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product manifold. Then* M *is a* HGK_n *with* (Π, Ψ) *if and only if the following conditions hold simultaneously*
1.(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Psi} \otimes (\overline{g} \wedge \overline{S}) - (n-p)\overline{\Psi} \otimes (\overline{g} \wedge T)$,
(ii) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Psi} \otimes (\overline{g} \wedge \overline{S}) - (n-p)\widetilde{\Psi} \otimes (\overline{g} \wedge T)$,
2.(i) $-(df + f\overline{\Pi}) \otimes \widetilde{R} = f\overline{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S}) + [-\frac{1}{2} f^2 (P\overline{\Pi} + dP) + fQ\overline{\Psi}] \otimes (\widetilde{g} \wedge \widetilde{g})$,
(ii) $f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi} \otimes \widetilde{R} + f\widetilde{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S}) + (-\frac{1}{2} f^2 P\widetilde{\Pi} + fQ\widetilde{\Psi}) \otimes (\widetilde{g} \wedge \widetilde{g})$,
3.(i) $\overline{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -[f(\overline{\nabla}T - \overline{\Pi} \otimes T) + f\overline{\Psi} \otimes (\overline{S} - (n-p)T) + Q\overline{\Psi} \otimes \overline{g}] \otimes \widetilde{g}$,
(ii) $\widetilde{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -[-f\widetilde{\Pi} \otimes T + f\widetilde{\Psi} \otimes (\overline{S} - (n-p)T) + Q\widetilde{\Psi} \otimes \overline{g}] \otimes \widetilde{g}$,
4.(i) $f^d \overline{R}_{abcd} = -(f_a T_{bc} - f_b T_{ac})$ *and*
(ii) $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.9 *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product* HGK_n *with* (Π, Ψ). *Then*
(i) $\overline{M}$ *is hyper generalized recurrent manifold if* T *and* S *are linearly dependent.*
(ii) $\widetilde{M}$ *is a* SGK_{n-p} *with* $(\widetilde{\Pi}, 0, \widetilde{\Psi}, -\frac{1}{2} fP\widetilde{\Pi} + Q\widetilde{\Psi})$.
(iii) $\widetilde{M}$ *is of vanishing conformal curvature tensor on* $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(iv) $\widetilde{M}$ *satisfies Einstein metric condition on* $\{x \in M : \Psi_x \neq 0\}$ *and* $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\}$.

Proof. Results of (i), (ii) and (iv) are obvious from Corollary 4.8. By virtue of 2.(i) of Corollary 4.8, on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$, $\widetilde{R}$ can be expressed as a linear combination of $\widetilde{g} \wedge \widetilde{S}$ and $\widetilde{g} \wedge \widetilde{g}$. Hence in view of Corollary 6.1 of [34], the conformal curvature tensor of $\widetilde{M}$ vanishes on this set, which proves (iii). □

Corollary 4.10 *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product manifold. Then M is a HGK_n with (Π, Ψ) if and only if the following conditions hold simultaneously*
1.(*i*) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Psi} \otimes (\overline{g} \wedge \overline{S})$, (*ii*) $\widetilde{\Pi} \otimes \overline{R} + \widetilde{\Psi} \otimes (\overline{g} \wedge \overline{S}) = 0$,
2.(*i*) $\overline{\Pi} \otimes \widetilde{R} + \overline{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S}) = 0$, (*ii*) $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Psi} \otimes (\widetilde{g} \wedge \widetilde{S})$,
3.(*i*) $\overline{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -\overline{\Psi} \otimes \overline{S} \otimes \widetilde{g}$, (*ii*) $\widetilde{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -\widetilde{\Psi} \otimes \overline{S} \otimes \widetilde{g}$.

Corollary 4.11 *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a WGK_n with (Π, Φ) if and only if the following conditions hold simultaneously*
1.(*i*) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes (\overline{S} \wedge \overline{S}) - 2(n-p)\overline{\Phi} \otimes (\overline{S} \wedge T) + (n-p)^2\overline{\Phi} \otimes (T \wedge T)$,
(*ii*) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Phi} \otimes (\overline{S} \wedge \overline{S}) - 2(n-p)\widetilde{\Phi} \otimes (\overline{S} \wedge T) + (n-p)^2\widetilde{\Phi} \otimes (T \wedge T)$,
2.(*i*) $-(df + f\overline{\Pi}) \otimes \widetilde{R} = \overline{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S}) + 2Q\overline{\Phi} \otimes (\widetilde{g} \wedge \widetilde{S}) + (-\frac{1}{2}f^2(P\overline{\Pi} - dP) + Q^2\overline{\Phi}) \otimes (\widetilde{g} \wedge \widetilde{g})$,
(*ii*) $f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S}) + 2Q\widetilde{\Phi} \otimes (\widetilde{g} \wedge \widetilde{S}) + (-\frac{1}{2}f^2P\widetilde{\Pi} + Q^2\widetilde{\Phi}) \otimes (\widetilde{g} \wedge \widetilde{g})$,
3.(*i*) $2\overline{\Phi} \otimes (\overline{S} - (n-p)T) \otimes \widetilde{S} = -[f(\overline{\nabla}T - \overline{\Pi} \otimes T) + 2Q\overline{\Phi} \otimes (\overline{S} - (n-p)T)] \otimes \widetilde{g}$,
(*ii*) $2\widetilde{\Phi} \otimes (\overline{S} - (n-p)T) \otimes \widetilde{S} = -[-f\widetilde{\Pi} \otimes T + 2Q\widetilde{\Phi} \otimes (\widetilde{S} - (n-p)T)] \otimes \widetilde{g}$,
4.(*i*) $f^d\overline{R}_{abcd} = -(f_aT_{bc} - f_bT_{ac})$ *and*
(*ii*) $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.12 *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product WGK_n with (Π, Φ). Then*
(*i*) *$\overline{M}$ is a WGK_p if T and S are linearly dependent.*
(*ii*) *$\widetilde{M}$ is a SGK_{n-p} with $(\widetilde{\Pi}, \frac{1}{f}\widetilde{\Phi}, \frac{2Q}{f}\widetilde{\Psi}, -\frac{1}{2}fP\widetilde{\Pi} + \frac{Q^2}{f}\widetilde{\Phi})$.*
(*iii*) *$\widetilde{M}$ is a RT_{n-p} with $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.*
(*iv*) *$\widetilde{M}$ satisfies Einstein metric condition on*

$$\{x \in M : ((\overline{\kappa} - (n-p)tr(T))\Phi)_x \neq 0\}$$

and is of constant curvature on $\{x \in M : df_x \neq 0\}$.

Corollary 4.13 *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product manifold. Then M is a WGK_n with (Π, Φ) if and only if the following conditions hold simultaneously*
1.(*i*) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes (\overline{S} \wedge \overline{S})$, (*ii*) $\widetilde{\Pi} \otimes \overline{R} + \widetilde{\Phi} \otimes (\overline{S} \wedge \overline{S}) = 0$,
2.(*i*) $\overline{\Pi} \otimes \widetilde{R} + \overline{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S}) = 0$, (*ii*) $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes (\widetilde{S} \wedge \widetilde{S})$,
3.(*i*) $\overline{\Phi} \otimes \overline{S} \otimes \widetilde{S} = 0$, (*ii*) $\widetilde{\Phi} \otimes \overline{S} \otimes \widetilde{S} = 0$.

5 An example of warped product SGK_4

Example 1: Consider the warped product $M = \overline{M} \times_f \widetilde{M}$, where $\overline{M}$ is a 3-dimensional manifold equipped with the metric

$$\overline{ds}^2 = e^{x^2}(dx^1)^2 + e^{x^1}(dx^2)^2 + (dx^3)^2$$

in local coordinates (x^1, x^2, x^3) and $\widetilde{M}$ is an open interval of $\mathbb{R}$ with local coordinate x^4 and the warping function $f = e^{x^3}$. The non-zero components of the Riemann-Christoffel curvature tensor $\overline{R}$ and Ricci tensor $\overline{S}$ of $\overline{M}$ are given by

$$\overline{R}_{1212} = -\frac{1}{4}\left(e^{x^1} + e^{x^2}\right), \quad \overline{S}_{11} = \frac{1}{4}\left(1 + e^{x^2 - x^2}\right), \; \overline{S}_{22} = \frac{1}{4}\left(e^{x^1 - x^2}\right).$$

Again the non-zero components of $\overline{\nabla R}$ are

$$\overline{R}_{1212,1} = \frac{e^{x^2}}{4}, \quad \overline{R}_{1212,2} = \frac{e^{x^1}}{4}.$$

If we consider the 1-form $\overline{\Pi} = \left(-\frac{e^{x^2}}{e^{x^1}+e^{x^2}}, -\frac{e^{x^1}}{e^{x^1}+e^{x^2}}, 0\right)$, then we can easily check that the manifold $\overline{M}$ is recurrent satisfying $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R}$.
Now the metric of $M = \overline{M} \times_f \widetilde{M}$ is given by

$$ds^2 = e^{x^2}(dx^1)^2 + e^{x^1}(dx^2)^2 + (dx^3)^2 + e^{x^3}(dx^4)^2. \tag{5.1}$$

The non-zero local components of the Riemann-Christoffel curvature tensor R and Ricci tensor S of M are given by

$$R_{1212} = -\frac{1}{4}\left(e^{x^1} + e^{x^2}\right), \quad R_{3434} = -\frac{e^{x^3}}{4},$$

$$S_{11} = \frac{1}{4}\left(e^{x^2 - x^1} + 1\right), \quad S_{22} = \frac{1}{4}\left(e^{x^1 - x^2} + 1\right), \quad S_{33} = \frac{1}{4}, \quad S_{44} = \frac{e^{x^3}}{4}.$$

Again the non-zero local components of ∇R are given by

$$R_{1212,1} = \frac{e^{x^2}}{4}, \quad R_{1212,2} = \frac{e^{x^1}}{4}.$$

Again the non-zero components of $g \wedge g$, $g \wedge S$ and $S \wedge S$ are given as follows:

$$(g \wedge g)_{1212} = -2e^{x^1 + x^2}, \quad (g \wedge g)_{1313} = -2e^{x^2}, \quad (g \wedge g)_{1414} = -2e^{x^2 + x^3},$$

$$(g \wedge g)_{2323} = -2e^{x^1}, (g \wedge g)_{2424} = -2e^{x^1 + x^3}, \quad (g \wedge g)_{3434} = -2e^{x^3},$$

$$(g \wedge S)_{1212} = \frac{1}{2}\left(-e^{x^1} - e^{x^2}\right), \quad (g \wedge S)_{1313} = \frac{1}{4}\left(-e^{x^2 - x^1}\left(e^{x^1} + 1\right) - 1\right),$$

$$(g \wedge S)_{1414} = -\frac{1}{4}e^{x^3-x^1}\left(e^{x^2}\left(e^{x^1}+1\right)+e^{x^1}\right),$$

$$(g \wedge S)_{2323} = \frac{1}{4}\left(-e^{x^1-x^2}\left(e^{x^2}+1\right)-1\right),$$

$$(g \wedge S)_{2424} = -\frac{1}{4}e^{x^3-x^2}\left(e^{x^2}\left(e^{x^1}+1\right)+e^{x^1}\right), \quad (g \wedge S)_{3434} = -\frac{e^{x^3}}{2},$$

$$(S \wedge S)_{1212} = \frac{1}{4}\left(-\cosh\left(x^1-x^2\right)-1\right), \quad (S \wedge S)_{1313} = \frac{1}{8}\left(-e^{x^2-x^1}-1\right),$$

$$(S \wedge S)_{1414} = -\frac{1}{8}e^{x^3}\left(e^{x^2-x^1}+1\right), \quad (S \wedge S)_{2323} = \frac{1}{8}\left(-e^{x^1-x^2}-1\right),$$

$$(S \wedge S)_{2424} = -\frac{1}{8}e^{x^3}\left(e^{x^1-x^2}+1\right), \quad (S \wedge S)_{3434} = -\frac{e^{x^3}}{8}.$$

If we consider the 1-forms Π, Φ, Ψ and Θ as:

$$\Pi_i = \begin{cases} \frac{\Psi_1 e^{x^1}\left(e^{x^2}-2\right)+2\Psi_1\cosh(x^1-x^2)-(2\Psi_1+1)e^{x^2}+2\Psi_1}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=1 \\ \frac{\Psi_2\left(e^{x^1}-2\right)e^{x^2}+2\Psi_2\cosh(x^1-x^2)-(2\Psi_2+1)e^{x^1}+2\Psi_2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=2 \\ \frac{\Psi_3 e^{-x^1-x^2}\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)^2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=3 \\ \frac{\Psi_4 e^{-x^1-x^2}\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)^2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=4, \end{cases}$$

$$\Phi_i = \begin{cases} \frac{\left(\Psi_1\left(-e^{x^1+x^2}\right)+2\Psi_1\cosh(x^1-x^2)+2\Psi_1+e^{x^2}\right)}{-2\cosh(x^1-x^2)+\sinh(x^1)+\cosh(x^1)+\sinh(x^2)+\cosh(x^2)-2} & \text{for } i=1 \\ e^{x^1}\left(\frac{\Psi_2 e^{x^1}+1}{e^{x^1}+e^{x^2}}-\Psi_2+\frac{1}{\left(e^{x^1}-1\right)e^{x^2}-e^{x^1}}\right)-\Psi_2 & \text{for } i=2 \\ -\frac{\Psi_3\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)}{e^{x^1}+e^{x^2}} & \text{for } i=3 \\ -\frac{\Psi_4\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)}{e^{x^1}+e^{x^2}} & \text{for } i=4, \end{cases}$$

$$\Theta_i = \begin{cases} \frac{-\Psi_1 e^{x^1-x^2}+2\Psi_1 e^{x^2}\sinh(x^1)-2\Psi_1+e^{x^2}}{16\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)} & \text{for } i=1 \\ \frac{1}{16}\left(-\Psi_2 e^{-x^1}-\Psi_2 e^{-x^2}-\Psi_2+\frac{e^{x^1}}{-e^{x^1+x^2}+e^{x^1}+e^{x^2}}\right) & \text{for } i=2 \\ -\frac{1}{16}\Psi_3 e^{-x^1-x^2}\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right) & \text{for } i=3 \\ -\frac{1}{16}\Psi_4 e^{-x^1-x^2}\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right) & \text{for } i=4 \end{cases}$$

then we can check that M is a SGK_4 with $(\Pi, \Phi, \Psi, \Theta)$, which is neither a HGK_4 nor a WGK_4. Thus we can state the following:

Theorem 5.1 *Let $M = \overline{M} \times_f \widetilde{M}$ be a warped product manifold equipped with the metric* (5.1)*. Then M is a SGK_4 which is neither a HGK_4 nor a WGK_4.*

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References

1. Adamów, A.; Deszcz, R. – *On totally umbilical submanifolds of some class of Riemannian manifolds*, Demonstratio Math., **16** (1983), no. 1, 39-59.
2. Binh, T.Q. – *On weakly symmetric Riemannian spaces*, Publ. Math. Debrecen, **42** (1993), no. 1-2, 103-107.
3. Bishop, R.L.; O'Neill, B. – *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1-49.
4. Cartan, É. – *Sur une classe remarquable d'espaces de Riemann* (French), Bull. Soc. Math. France, **54** (1926), 214-264.
5. Cartan, É. – *Leçons sur la géométrie des espaces de Riemannian* (French), 2nd Ed., Gauthier-Villars, Paris (1946), viii+378 pp.
6. Chaki, M.C. – *On pseudosymmetric manifolds*, An. Ştiint. Ale Univ., Al. I. Cuza din Iaşi România, Secţ. I a Mat. **33** (1987), no. 1, 53-58.
7. Deszcz, R. – *Notes on totally umbilical submanifolds*, Geometry and topology of submanifolds (Marseille, 1987), 89-97, World Sci. Publ., Teaneck, NJ, 1989.
8. Deszcz, R. – *On pseudosymmetric spaces*, Bull. Belgian Math. Soc., Series A, **44** (1992), no. 1, 1-34.
9. Deszcz, R. – *On Roter type manifolds*, 5-th Conference on Geometry and Topology of Manifolds April 27 - May 3, 2003, Krynica, Poland.
10. Deszcz, R. – *On some Akivis-Goldberg type metrics*, Publ. Inst. Math. (Beograd) (N.S.), **74(88)** (2003), 71-83.
11. Deszcz, R.; Głogowska, M.; Jełowicki J.; Zafindratafa, G. – *Curvature properties of some class of warped product manifolds*, Int. J. Geom. Methods Mod. Phys., 13 (2016), no. 1, 1550135 (36 pages).
12. Deszcz, R.; Głogowska, M.; Petrović-Torgašev, M.; Verstraelen, L. – *On the Roter type of Chen ideal submanifolds*, Results in Math., **59** (2011), no. 3-4, 401-413.
13. Deszcz, R.; Grycak, W. – *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica, **15** (1987), no. 3, 311-322.
14. Deszcz, R.; Kowalczyk, D. – *On some class of pseudosymmetric warped products*, Colloq. Math., **97** (2003), no. 1, 7-22.
15. Dubey, R.S.D. – *Generalized recurrent spaces*, Indian J. Pure Appl. Math., **10** (1979), no. 12, 1508-1513.
16. Hotloś, M. – *On conformally symmetric warped products*, Ann. Academiae Paedagogicae Cracoviensis Stud. Math., **4** (2004), 75-85.
17. Gębarowski, A. – *On Einstein warped products*, Tensor (N. S.), **52** (1993), no. 3, 204-207.
18. Hui, S.K.; Matsuyama, Y.; Shaikh, A.A. – *On decomposable weakly conformally symmetric manifolds*, Acta Math. Hungarica, **128** (2010), no. 1-2, 82-95.
19. Kručkovič, G.I. – *On semireducible Riemannian spaces* (Russian), Dokl. Akad. Nauk SSSR (N.S.), **115** (1957), 862-865.
20. Mantica, C.A.; Molinari L.G. – *A second-order identity for the Riemann tensor and applications*, Colloq. Math., **122** (2011), no. 1, 69-82.
21. Olszak, K.; Olszak, Z. – *On pseudo-Riemannian manifolds with recurrent concircular curvature tensor*, Acta Math. Hungar., **137** (2012), no. 1-2, 64-71.
22. O'Neill, B. – *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, 103. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. xiii+468 pp.

23. Ruse, H.S. – *On simply harmonic spaces*, J. London Math. Soc., **21** (1946), 243-247.
24. Ruse, H.S. – *Three-dimensional spaces of recurrent curvature*, Proc. London Math. Soc. (2), **50** (1949), 438-446.
25. Ruse, H.S. – *On simply harmonic "kappa spaces"of four dimensions*, Proc. London Math. Soc.(2), **50** (1948), 317-329.
26. Shaikh, A.A.; Al-Solamy, F.R.; Roy, I. – *On the existence of a new class of semi-Riemannian manifolds*, Mathematical Sciences, (december 2013), 1-13.
27. Shaikh, A.A.; Deszcz, R.; Hotloś, M.; Jełowicki, J.; Kundu, H. – *On pseudosymmetric manifolds*, Publ. Math. Debrecen, **86** (2015), no. 3-4, 433-456.
28. Shaikh, A.A.; Hui, S.K. – *On weakly conharmonically symmetric manifolds*, Tensor (N. S.), **70** (2008), no. 2, 119-134.
29. Shaikh, A.A.; Hui, S.K. – *On decomposable weakly conharmonically symmetric manifolds*, Lobachevskii J. Math., **29** (2008), no. 4, 206-215.
30. Shaikh, A.A.; Hui, S.K. – *On weakly concircular symmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.), **55** (2009), no. 1, 167-186.
31. Shaikh, A.A.; Jana, S.K. – *On weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen, **71** (2007), no. 1-2, 27-41.
32. Shaikh, A.A.; Jana, S.K.; Eyasmin, S. – *On weakly pseudo quasi-conformally symmetric manifolds*, Indian J. Math., **50** (2008), no. 3, 505-518.
33. Shaikh, A.A.; Kundu, H. – *On weakly symmetric and weakly Ricci symmetric warped product manifolds*, Publ. Math. Debrecen, **81** (2012), no. 3-4, 487-505.
34. Shaikh, A.A.; Kundu, H. – *On equivalency of various geometric structures*, J. Geom., **105** (2014), no. 1, 139-165.
35. Shaikh, A.A.; Kundu, H. – *On generalized Roter type manifolds*, https://arxiv.org /abs/1411.0841, [math.DG] 4 Nov 2014.
36. Shaikh, A.A.; Kundu, H. – *On warped product generalized Roter type manifolds*, Balkan J. Geom. Appl., **21** (2016), no. 2, 82-95.
37. Shaikh, A.A.; Patra, A. – *On a generalized class of recurrent manifolds*, Archivum Mathematicum (Brno), **46** (2010), no. 1, 71-78.
38. Shaikh, A.A.; Roy, I. – *On quasi generalized recurrent manifolds*, Math. Pannonica, **21** (2010), no. 2, 251-263.
39. Shaikh, A.A.; Roy, I. – *On weakly generalized recurrent manifolds*, Annales Univ. Sci. Budapest. Eötvös Sect. Math., **54** (2011), 35-45.
40. Shaikh, A.A.; Roy, I.; Kundu, H. – *On super generalized recurrent manifolds*, https://arxiv.org/abs/1504.02535 [math.DG] 10 Apr 2015.
41. Shaikh, A.A.; Roy, I.; Kundu, H. – *On the existence of a generalized class of recurrent manifolds*, https://arxiv.org/pdf/1504.02534.pdf [math.DG] 7 Sep 2016.
42. Szabó, Z.I. – *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, I, The local version*, J. Diff. Geom., **17** (1982), no. 4, 531-582.
43. Szabó, Z.I. – *Classification and construction of complete hypersurfaces satisfying* $R(X,Y)\cdot R=0$, Acta Sci. Math. (Szeged), **47** (1984), no. 3-4, 321-348.
44. Szabó, Z.I. – *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, II, Global versions*, Geom. Dedicata, **19** (1985), no. 1, 65-108.
45. Tamássy, L.; Binh, T.Q. – *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Differential geometry and its applications (Eger, 1989), 663–670, Colloq. Math. Soc. János Bolyai, 56, North-Holland, Amsterdam, 1992.
46. Verstraelen, L. – *A concise mini history of geometry*, Kragujevac J. Math., **38** (2014), no. 1, 5-21.
47. Walker, A.G. – *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. (2), **52** (1950), 36-64.
48. Zhang, Z. – *Warping functions of some warped products*, J. Math. Anal. Appl., **412** (2014), no. 2, 1019-1024.

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