An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 1

On Kropina change of m-th root Finsler metrics with special curvature properties

Bankteshwar Tiwari · Ghanashyam Kr. Prajapati · Ranadip Gangopadhyay

Abstract In the present paper, we consider Kropina change of m-th root Finsler metrics and prove that every Kropina change of m-th root Finsler metrics with isotropic Berwald curvature, isotropic mean Berwald curvature, isotropic Landsberg curvature, isotropic mean Landsberg curvature reduces to Berwald metric, weakly Berwald metric, Landsberg metric and weakly Landsberg metric respectively. Also we show that every Kropina change of m-th root Finsler metrics with almost vanishing H-curvature has vanishing H-curvature.

Keywords Finsler space · Kropina metrics · m-th root metrics · Berwald curvature · Landsberg curvature · H-curvature

Mathematics Subject Classification (2010) 53B40 · 53C60

1 Introduction

The theory of m-th root Finsler metrics has been developed by Shimada [9] in 1979, applied to ecology by Antonelli [12] and studied by several authors ([1], [2], [4], [19]). The m-th root Finsler metric is regarded as a generalization of Riemannian metric in the sense that for $m = 2, 3$ and 4, it is called Riemannian metric, cubic metric and quartic metric [4] respectively. In the four-dimension, a special fourth root metric in the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moór metric [14], which is considered by physicists as an important subject for a possible model of space time. Recent studies show that m-th root Finsler metrics plays a very important role in physics, space-time and general relativity as well as in unified gauge field theory ([7], [8], [13]). Z. Shen and B. Li have studied the geometric properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ and generalized fourth root metrics in the form $F = \sqrt{\sqrt{a_{ijkl}(x)y^iy^jy^ky^l} + b_{ij}(x)y^iy^j}$ [4].

Recently, B. Tiwari and M. Kumar [5] have studied Randers change of a Finsler space with m-th root metric and conformally related Randers transformed m-th root metric. Also A. Tayebi ([1], [2], [3]) has studied several properties of m-th root metrics.

Let $(M, F) = F^n$ be an n-dimensional Finsler manifold. For a 1-form $\beta(x, y) = b_i(x)y^i$ on M, define a Finsler change as follows
$F(x, y) \to \overline{F}(x, y) = f(F, \beta)$, where $f(F, \beta)$ is a positively homogeneous function of F and β [6]. A Finsler change is called Kropina change if $f(F, \beta) = \frac{F^2}{\beta}$. The purpose of this paper is to characterize several curvature properties of Kropina change of m-th root metrics, defined by

$$\overline{F} = \frac{F^2}{\beta}, \tag{1.1}$$

where $F = \sqrt[m]{A}$ is an m-th root metric on the manifold M, for which we shall restrict our consideration to the domain where $\beta = b_i(x)y^i > 0$.

In [21] a geodesic curve $c = c(t)$ of a Finsler metric $F = F(x, y)$ on a smooth manifold M is given by $\ddot{c}^i(t) + 2G^i\,(c(t), \dot{c}(t)) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients given by

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}. \tag{1.2}$$

For a non zero tangent vector $y \in T_xM$, in local coordinate system, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u, v, w) = B^i_{jkl}(x, y)u^j v^k w^l \frac{\partial}{\partial x^i}$ and $E_y(u, v) = E_{jk}(x, y)u^j v^k$ respectively, where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$, $E_{jk} = \frac{1}{2} B^m_{mjk} = \frac{1}{2} \frac{\partial^2}{\partial y^j \partial y^k}[\frac{\partial G^m}{\partial y^m}]$, $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$. Then $B = \{B_y | y \in TM_0\}$ and $E = \{E_y | y \in TM_0\}$ are called Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called Berwald metric and mean Berwald metric if $B = 0$ and $E = 0$ respectively [22].

A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = c(F_{y^j y^k} \delta^i_l + F_{y^k y^l} \delta^i_j + F_{y^l y^j} \delta^i_k + F_{y^j y^k y^l} y^i), \tag{1.3}$$

where $c = c(x)$ is a scalar function on M [16].

A Finsler metric F is said to be isotropic mean Berwald metric if its mean curvature is in the following form $E_{ij} = \frac{n+1}{2F} c h_{ij}$, where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric tensor.

For a non zero tangent vector $y \in T_xM$, in local coordinate system, define $L_y(u,v,w) = L_{ijk}(x,y)u^i v^j w^k$, where $L_{ijk} = -\frac{1}{2}FF_{y^s}[G^s]_{y^i y^j y^k}$, $u = u^i\frac{\partial}{\partial x^i}, v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. Then $L = \{L_y | y \in TM_0\}$ is called Landsberg curvature and a Finsler metric is called Landsberg metric if $L = 0$.

For a non-zero vector $y \in T_pM$, the Cartan torsion $C_y = C_{ijk}dx^i \otimes dx^j \otimes dx^k :$ $T_pM \otimes T_pM \otimes T_pM \to \mathbb{R}$ is defined by $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and the mean Cartan torsion $I_y = I_i(x,y)dx^i : T_pM \to \mathbb{R}$ is defined by

$$I_i = g^{jk}C_{ijk}. \tag{1.4}$$

The horizontal covariant derivative of I along a vector $u \in T_xM$ gives rise to the mean Landsberg curvature $J_y(u) := J_i(y)u^i$, where $J_i = I_{i|s}y^s$ [22].

A Finsler metric with $J = cFI$ and $J = 0$ is called isotropic mean Landsberg and weakly Landsberg metric respectively, where $c = c(x)$ is a scalar function on M [3].

In [10], Akbar-Zadeh has introduced the non-Riemannian quantity H which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity $H = H_{ij}dx^i \otimes dx^j$ is defined by $H_{ij} := E_{ij|s}y^s$. It has been investigated in [10], that for a Finsler manifold of scalar flag curvature K with dimension $n \geq 3, K = \textit{constant}$ if and only if $H = 0$. A Finsler metric is called of almost vanishing H-curvature if $H_{ij} = \frac{n+1}{2F}\theta h_{ij}$, for some 1-form θ on M, where h_{ij} is the angular metric tensor.

In this paper, we have considered Kropina change of m-th root Finsler metric and established the following theorems

Theorem 1.1 *Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric on the manifold of dimension $n \geq 2$. Suppose that $\overline{F}$ has isotropic Berwald curvature. Then $\overline{F}$ reduces to Berwald metric.*

Theorem 1.2 *Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric on the manifold of dimension $n \geq 2$. Suppose that $\overline{F}$ is of isotropic mean Berwald curvature. Then $\overline{F}$ reduces to weakly Berwald metric.*

Theorem 1.3 *Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric on the manifold of dimension $n \geq 2$. Suppose that $\overline{F}$ is of isotropic Landsberg curvature. Then $\overline{F}$ reduces to Landsberg metric.*

Theorem 1.4 *Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric on the manifold of dimension $n \geq 2$. Suppose that $\overline{F}$ is of isotropic mean Landsberg curvature. Then $\overline{F}$ reduces to weakly Landsberg metric.*

Theorem 1.5 *Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric on the manifold of dimension $n \geq 2$. Suppose that $\overline{F}$ has almost vanishing $\overline{H}$-curvature. Then $\overline{H} = 0$.*

Throughout the paper we call the Finsler metric $\overline{F}$ as Kropina change of m-th root metric and $\overline{F}^n := (M, \overline{F})$ as Kropina transformed Finsler space. For $m = 2$ the Kropina change of m-th root metric is nothing but the Kropina metric which has been studied by several authors ([11], [15], [17], [18], [20]). We restrict ourselves for $m \geq 3$ throughout the paper and also the quantities corresponding to the Kropina transformed Finsler space $\overline{F}^n$ will be denoted by putting bar on the top of that quantity.

2 Preliminaries

Let M be an n-dimensional C^∞-manifold, T_xM denotes the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M and $y \in T_xM$. We denote $TM_0 = TM \setminus \{0\}$.

Definition 2.1 *A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:*
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and
(iii) the Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair $F^n = (M, F)$ is called a Finsler space of dimension n. The metric F is called fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

The normalized supporting element l_i and angular metric tensor h_{ij} of F are defined, respectively as

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j} \quad . \tag{2.1}$$

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given by $A = a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ with $a_{i_1\ldots i_m}$ symmetric in all its indices [9]. Then F is called an m-th root Finsler metric. Clearly, A is homogeneous of degree m in y. Let

$$A_i = \frac{\partial A}{\partial y^i}, A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, A_{x^i} = \frac{\partial A}{\partial x^i}, A_0 = A_{x^i} y^i. \tag{2.2}$$

Then the following holds.

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (2-m)A_iA_j\right],$$

$$y^iA_i = mA, y^iA_{ij} = (m-1)A_j, y_i = \tfrac{1}{m}A^{\frac{2}{m}-1}A_i,$$

$$A^{ij}A_{jk} = \delta^i_k, A^{ij}A_i = \tfrac{1}{m-1}y^j, A_iA_jA^{ij} = \tfrac{m}{m-1}A.$$

3 Fundamental metric tensors and geodesic sprays of Kropina *changed* m-th root metrics

The differentiation of equation (1.1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = \overline{F}\left[\frac{2A_i}{mA} - \frac{b_i}{\beta}\right] \tag{3.1}$$

and angular metric tensor h_{ij} given by

$$\overline{h}_{ij} = 2\overline{F}^2\left\{\frac{1}{mA}A_{ij} + \frac{(2-m)}{m^2A^2}A_iA_j - \frac{1}{mA\beta}(A_ib_j + A_jb_i) + \frac{1}{\beta^2}b_ib_j\right\}. \tag{3.2}$$

Put $\overline{\eta}_{ij} = \frac{1}{mA}A_{ij} + \frac{(2-m)}{m^2A^2}A_iA_j - \frac{1}{mA\beta}(A_ib_j + A_jb_i) + \frac{1}{\beta^2}b_ib_j$, which are rational functions in y. Then the angular metric tensor $\overline{h}_{ij}$ rewritten as

$$\overline{h}_{ij} = 2\overline{F}^2\overline{\eta}_{ij}. \tag{3.3}$$

Also the fundamental metric tensor $\overline{g}_{ij}$ of Finsler space $\overline{F}^n$ is given by

$$\overline{g}_{ij} = \overline{h}_{ij} + \bar{l}_i\bar{l}_j. \tag{3.4}$$

Therefore using equations (3.1), (3.2) and (3.4), we obtain metric tensor $\overline{g}_{ij}$ as

$$\overline{g}_{ij} = \overline{F}^2\left[\frac{2}{mA}A_{ij} + \frac{2(4-m)}{m^2A^2}A_iA_j - \frac{4}{mA\beta}(A_ib_j + A_jb_i) + \frac{3}{\beta^2}b_ib_j\right]. \tag{3.5}$$

Above equation can be rewritten as

$$\overline{g}_{ij} = \overline{F}^2\overline{\gamma}_{ij}, \tag{3.6}$$

where $\overline{\gamma}_{ij} = \left[\frac{2}{mA}A_{ij} + \frac{2(4-m)}{m^2A^2}A_iA_j - \frac{4}{mA\beta}(A_ib_j + A_jb_i) + \frac{3}{\beta^2}b_ib_j\right]$ are rational functions in y. Therefore the components of inverse metric tensor $\overline{g}^{ij}$ are given as

$$\overline{g}^{ij} = (\overline{F})^{-2}\overline{\gamma}^{ij}, \tag{3.7}$$

where $\overline{\gamma}^{ij} = (\overline{\gamma}_{ij})^{-1}$ are also rational functions in y.
To calculate spray coefficients $\overline{G}^i$ of Finsler space $\overline{F}^n$, we need the following

$$\left[\overline{F}^2\right]_{x^k} = \overline{F}^2 \left[\frac{4A_{x^k}}{mA} - \frac{2\beta_{x^k}}{\beta}\right] \tag{3.8}$$

and

$$\left[\overline{F}^2\right]_{x^k y^l} y^k = \overline{F}^2 \left[\frac{4(A_l)_0}{mA} + \frac{(16-4m)A_l A_0}{m^2 A^2} - \frac{2(b_l)_0}{\beta} + \frac{6b_l\beta_0}{\beta^2} - \frac{8(A_l\beta_0 + A_0 b_l)}{mA\beta}\right]. \tag{3.9}$$

In view of equations (1.2), (3.8) and (3.9), we have

$$\overline{G}^i = \frac{1}{4}\overline{\gamma}^{il} \left[\frac{4mA\beta(A_l)_0 + (16-4m)\beta A_l A_0 - 8mA(A_l\beta_0 + A_0 b_l) - 2mA\beta A_{x^l}}{m^2A^2\beta} + \frac{6b_l\beta_0 + 2\beta\beta_{x^l} - 2\beta(b_l)_0}{\beta^2}\right]. \tag{3.10}$$

Proposition 3.1 Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric F on a manifold of dimension $n \geq 2$. Then the spray coefficients $\overline{G}^i$ of $\overline{F}^n$ are given by equation (3.10).

Remark 3.1 It is remarkable to note that the metric tensors $\overline{g}_{ij}$ and $\overline{g}^{ij}$ of $\overline{F}^n$ are not necessarily rational functions in y but spray coefficients $\overline{G}^i$ of $\overline{F}^n$ are rational functions in y.

4 Berwald curvature

Proof of Theorem 1.1

Let $\overline{F} = \frac{F^2}{\beta}$ be a Kropina change of m-th root Finsler metric. Suppose that $\overline{F}$ has isotropic Berwald curvature given by equation (1.3). By Remark 3.1, the left hand side of equation (1.3) are rational functions in y, while the right hand side are irrational functions in y, which implies that $c = 0$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2

Let $\overline{F} = \frac{F^2}{\beta}$ be of isotropic mean Berwald curvature, that is,

$$\overline{E}_{ij} = \frac{n+1}{2\overline{F}} c\overline{h}_{ij}, \tag{4.1}$$

where $c = c(x)$ is a scalar function on M and $\overline{h}_{ij}$ is the angular metric tensor. Plugging the angular metric $\overline{h}_{ij}$ given by the equation (3.3) in (4.1), we have

$$\overline{E}_{ij} = (n+1)c\overline{F}\overline{\eta}_{ij}. \tag{4.2}$$

But the left hand side of above expression are rational functions in y while right hand side are irrational due to presence of $\overline{F}$. Thus equation (4.2) implies either $c = 0$ or $\overline{\eta}_{ij} = 0$. If $\overline{\eta}_{ij} = 0$, then $\overline{h}_{ij} = 0$ which is not possible. Hence $c = 0$ and $\overline{E}_{ij} = 0$. This completes the proof of Theorem 1.2.

5 Landsberg Curvature

Proof of Theorem 1.3

Let $\overline{F} = \frac{F^2}{\beta}$ be isotropic Landsberg metric, that is,

$$\overline{L}_{ijk} = c\overline{F}\overline{C}_{ijk}, \tag{5.1}$$

where $c = c(x)$ is a scalar function on M. The Cartan torsion $\overline{C}_{ijk}$ of $\overline{F}$ is given by the following equation

$$\overline{C}_{ijk} = \overline{F}^2\overline{\nu}_{ijk}, \tag{5.2}$$

where $\overline{\nu}_{ijk} = \Bigg[\frac{1}{mA}A_{ijk} + \frac{(4-m)(4-2m)}{m^3A^3}A_iA_jA_k + \frac{4-m}{m^2A^2}\sum\limits_{(i,j,k)}A_{ij}A_k$
$-\frac{2(4-m)}{m^2A^2\beta}\sum\limits_{(i,j,k)}A_iA_jb_k + \frac{6}{mA\beta^2}\sum\limits_{(i,j,k)}A_ib_jb_k - \frac{2}{mA\beta}\sum\limits_{(i,j,k)}A_{ij}b_k - \frac{6}{\beta^3}b_ib_jb_k\Bigg]$
are rational functions in y and $\sum\limits_{(i,j,k)}$ denotes the cyclic interchange of indices i, j, k and summation, for instance $\sum\limits_{(i,j,k)}A_{ij}A_k = A_{ij}A_k + A_{jk}A_i + A_{ki}A_j$. Also we know that

$$\overline{L}_{ijk} = -\frac{1}{2}\overline{F}\overline{F}_{y^s}\overline{G}^s_{y^iy^jy^k}. \tag{5.3}$$

In our case

$$\overline{L}_{ijk} = -\frac{\overline{F}^2}{2}\left[\frac{2A_s}{mA} - \frac{b_s}{\beta}\right]\overline{G}^s_{y^iy^jy^k}. \tag{5.4}$$

Using equations (5.1), (5.2) and (5.4), we obtain

$$\left(\frac{2A_s}{mA} - \frac{b_s}{\beta}\right)G^s_{y^iy^jy^k} = -2c\overline{F}\overline{\nu}_{ijk}.$$

Again expression on left hand side are rational functions in y while on right hand side are irrational functions in y. Therefore, either $c = 0$ or $\overline{\nu}_{ijk} = 0$. If $\overline{\nu}_{ijk} = 0$, then $\overline{C}_{ijk} = 0$. Hence, $\overline{F}$ is Riemannian metric, which contradicts our assumption. Therefore $c = 0$. This completes the proof of Theorem 1.3.

Using equations (1.4), (3.7) and (5.2), the mean Cartan torsion of $\overline{F}^n$ are given by

$$\overline{I}_i = \overline{\gamma}^{jk}\overline{\nu}_{ijk}.$$

Lemma 5.1 *Let $\overline{F}^n$ be a Kropina change of m-th root Finsler manifold with $n \geq 2$. Then the components $\overline{I}_i$ of mean Cartan torsion of $\overline{F}^n$ are rational functions in y.*

Proof of Theorem 1.4

The mean Landsberg curvature of $\overline{F}^n$ is given by

$$\overline{J}_i = \overline{g}^{jk}\overline{L}_{ijk}. \tag{5.5}$$

In view of equations (3.7), (5.4) and (5.5), we have

$$\overline{J}_i = -\frac{1}{2}\overline{\gamma}^{jk}\left[\frac{2A_s}{mA} - \frac{b_s}{\beta}\right]\overline{G}^s_{y^iy^jy^k}. \tag{5.6}$$

Lemma 5.2 *Let $\overline{F}^n$ be a Kropina change of m-th root Finsler manifold with $n \geq 2$, where $m \geq 3$. Then the components $\overline{J}_i$ of mean Landsberg curvature of $\overline{F}^n$ are rational functions in y.*

In view of equations (1.4), (5.1) and (5.5), we have

$$\overline{J}_i = c\overline{FI}_i. \tag{5.7}$$

Using Lemma 5.1 and Lemma 5.2, left hand side are rational functions in y while right hand side are irrational in y because $\overline{F}$ is irrational. Thus if equation (5.7) holds, then either $c = 0$ or $\overline{I}_i = 0$. If $\overline{I}_i = 0$, $\overline{F}$ reduces to Riemannian metric, which contradicts our assumption. Therefore, $c = 0$. This completes the proof of Theorem 1.4.

6 H-curvature

Proof of Theorem 1.5

Let $\overline{F}=\frac{F^2}{\beta}$ be of almost vanishing $\overline{H}$-curvature, that is,

$$\overline{H}_{ij}=\frac{n+1}{2\overline{F}}\theta\overline{h}_{ij}, \tag{6.1}$$

for some 1-form θ on M, where $\overline{h}_{ij}$ is the angular metric tensor.

Plugging the angular metric tensor $\overline{h}_{ij}$ given by the equation (3.3) in (6.1), we have

$$\overline{H}_{ij}=(n+1)\theta\overline{F}\overline{\eta}_{ij}. \tag{6.2}$$

Again expression on left hand side are rational functions in y while on right hand side are irrational functions in y due to presence of $\overline{F}$. Thus equation (6.2) implies either $\theta=0$ or $\overline{\eta}_{ij}=0$. If $\overline{\eta}_{ij}=0$, then $\overline{h}_{ij}=0$ which is not possible. Hence $\theta=0$ and $\overline{H}_{ij}=0$. This completes the proof of Theorem 1.5.

Acknowledgements The third author express his sincere thanks to the University Grant Commission for funding the joint CSIR-UGC Junior Research Fellowship Reference. No: 23/12/2012 (ii) EU-V, Sr. No.- 2121240958.

References

1. Tayebi, A.; Najafi, B. – *On m-th root Finsler metrics*, J. Geom. Phys., **61** (2011), no. 8, 1479-1484.
2. Tayebi, A.; Najafi, B. – *On m-th root metrics with special curvature properties*, C. R. Math. Acad.Sci. Paris, Ser. I, **349** (2011), no. 11-12, 691-693.
3. Tayebi, A.; Nankali, A.; Peyghan, E. – *Some properties of m-th root Finsler metrics*, translated from Izv. Nats. Akad. Nauk Armenii Mat. 49 (2014), no. 4, 61-76 J. Contemp. Math. Anal. **49** (2014), no. 4, 184-193.
4. Li, B.; Shen, Z. – *Projectively flat fourth root Finsler metrics*, Canad. Math. Bull., **55** (2012), no. 1, 138-145.
5. Tiwari, B.; Kumar, M. – *On Randers change of a Finsler space with m-th root metric*, Int. J. Geom. Methods Mod. Phys., **11** (2014), no. 10, 1450087, 1-13.
6. Shibata, C. – *On invariant tensors of β-changes of Finsler metrics*, J. Math. Kyoto Univ., **24** (1984), no. 1, 163-188.
7. Guo, E.; Mo, X.; Zhang, X. – *The explicit construction of Einstein Finsler metrics with non-constant flag curvature*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 045, 7 pp.
8. Asanov, G.S. – *Finslerian extension of general relativity*, Found. Phys. **11** (1981), no. 1-2, 137-154.
9. Shimada, H. – *On Finsler spaces with the metric* $L=\sqrt[m]{a_{i_1\ldots\ldots i_m}y^{i_1}\ldots.y^{i_m}}$, Tensor, N. S., **33** (1979), no. 3, 365-372.
10. Akbar-Zadeh, H. – *Sur les espaces de Finsler à courbures sectionnelles constantes* (French), [[On Finsler spaces with constant sectional curvature]] Acad. Roy. Belg. Bull. Cl. Sci. (5) **74** (1988), no. 10, 281-322.
11. Nagaraja, H.G. – *On Kropina metrics*, Afr. Mat. **25** (2014), no. 3, 519-527.

12. Antonelli, P.L.; Ingarden, R.S.; Matsumoto, M. – *The theory of sprays and Finsler spaces with applications in physics and biology*, Fundamental Theories of Physics, 58, Kluwer Academic Publishers Group, Dordrecht, (1993), xvi+308 pp.
13. Balan, V.; Brinzei, N. – *Einstein equations for (h, v)-Berwald-Moór relativistic models*, Balkan. J. Geom. Appl., **11** (2006), no. 2, 20-27.
14. Balan, V. – *Notable submanifolds in Berwald-Moór spaces*, Proceedings - the International Conference of Differential Geometry and Dynamical Systems (DGDS-2009), 21-30, BSG Proc., 17, Geom. Balkan Press, Bucharest, 2010.
15. Kropina, V.K. – *Projective two-dimensional Finsler spaces with special metric* (Russian), Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277-292.
16. Chen, X.; Shen, Z. – *On Douglas metrics*, Publ. Math. Debrecen **66** (2005), no. 3-4, 503-512.
17. Cheng, X.; Mo, X.; Shen, Z. – *On the flag curvature of Finsler metrics of scalar curvature*, J. London Math. Soc. (2), **68** (2003), no. 3, 762-780.
18. Zhang, X.; Shen, Y.-B. – *On Einstein Kropina Metrics*, arXiv:1207.1944v1 [math.DG] (2012), 17 pp.
19. Yu, Y.; You, Y. – *On Einstein m-th root metrics*, Differential Geom. Appl., **28** (2010), no. 3, 290-294.
20. Shen, Z.; Yu, C. – *On a class of Einstein Finsler metrics*, Internat. J. Math., **25** (2014), 1450030 18 pp.
21. Chern, S.S.; Shen, Z. – *Riemann-Finsler geometry*, Nankai Tracts in Mathematics, 6, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2005), x+192 pp.
22. Shen, Z. – *Differential geometry of spray and Finsler spaces*, Kluwer Academic Publishers, Dordrecht, (2001), xiii+258 pp.

Received: 3.XI.2015 / Revised: 4.IV.2016 / Accepted: 20.IV.2016

Authors

Bankteshwar Tiwari,
DST-Centre for Interdisciplinary Mathematical Sciences,
Banaras Hindu University,
Varanasi-221005, India
E-mail: banktesht@gmail.com

Ghanashyam Kr. Prajapati, (Corresponding author)
DST-Centre for Interdisciplinary Mathematical Sciences,
Banaras Hindu University,
Varanasi-221005, India
E-mail: gspbhu@gmail.com

Ranadip Gangopadhyay,
DST-Centre for Interdisciplinary Mathematical Sciences,
Banaras Hindu University,
Varanasi-221005, India
E-mail: gangulyranadip@gmail.com