An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 1

Almost Ricci solitons and concircular vector fields

Dilek Açıkgöz Kaya · Leyla Onat

Abstract In this paper, we study an n-dimensional compact almost Ricci soliton (M^n, g, v, λ) on a Riemannian manifold (M, g). We show that if the soliton vector field is concircular then M^n is isometric to an Euclidean sphere. We also give a characterization of an almost Ricci soliton for a totally umbilical submanifold whose potential field is tangential.

Keywords Almost Ricci soliton · concircular vector field · submanifolds · concircular potential field

Mathematics Subject Classification (2010) 53C25 · 53C40

1 Introduction

A Riemannian manifold (M^n, g) is said to be a Ricci soliton if there exists a vector field v on M satisfying the following equation

$$Ric + \frac{1}{2}L_v g = \lambda g \tag{1.1}$$

where L_v is the Lie derivative with respect to v, Ric is the Ricci tensor of (M^n, g) and λ is constant. Recently, a generalization of Eq.(1.1) has been considered in [7], allowing λ to be a smooth function on M and (M^n, g, v, λ) is said to be an almost Ricci soliton if the vector field v satisfies the Eq.(1.1) for some function λ. The almost Ricci soliton is said to be proper, shrinking, steady or expanding according to the fact that λ is non-constant, positive, zero or negative, respectively.

When the vector field v is the gradient of some smooth function f, Eq.(1.1) becomes

$$Ric + Hessf = \lambda g \tag{1.2}$$

where $Hessf$ denotes the Hessian tensor and (M, g, f, λ) is called a gradient almost Ricci soliton. Moreover, when either the vector field v is trivial or

the function f is constant, the almost Ricci soliton will be called trivial. We notice that when $n \geq 3$ and v is a Killing vector field, an almost Ricci soliton will be simply a Ricci soliton, since in this case we have an Einstein manifold, which implies that λ is a constant. We just refer to [2],[3], [4] for further references on almost Ricci solitons.

A. Fialkow introduced in [6] the notion of concircular vector fields on a Riemannian manifold M as vector fields which satisfy

$$\nabla_X v = \mu X, \tag{1.3}$$

where $X \in TM$, and ∇ denotes the Levi-Civita connection, TM the tangent bundle of M and μ a non-trivial function on M. Concircular vector fields are also known as geodesic fields because their integral curves are geodesics [6].

Using the Koszul's formula we immediately obtain the following formula

$$2g(\nabla_Y v, Z) = (L_v g)(Y, Z) + d\theta(Y, Z)$$

for every $Y, Z \in TM$, where $\theta = g(v, \cdot)$ is the covariant form of v with respect to g as defined in [9]. If $L_v g = 2\mu g$ and $d\theta = 0$, where μ is a function on M, then the vector field v is called a closed conformal vector field. Hence, we have the Eq.(1.3).

Barros and Ribeiro Jr. proved in [1] that a compact almost Ricci soliton with non-trivial conformal vector field is isometric to an Euclidean sphere. This is our motivation for studying the concircular vector fields or closed conformal vector fields on almost Ricci solitons. In this spirit, we have the following theorem.

Theorem 1.1 *Let (M^n, g, v, λ) be a compact almost Ricci soliton with $n \geq 3$. If v is a concircular vector field, then M^n is isometric to an Euclidean sphere S^n.*

As a consequence of this theorem, we get the following corollary.

Corollary 1.2 *Let (M^n, g, v, λ) be a compact almost Ricci soliton with $n \geq 3$. If v is a closed conformal vector field, then M^n is isometric to an Euclidean sphere S^n.*

In a recent paper [5], Chen studied Ricci solitons on submanifolds M^n of a Riemannian manifold N^m equipped with a concircular vector field. We have the analogous result to Theorem 6.1 for compact almost Ricci solitons.

In this paper, the Riemannian m-manifold $(N^m, \widetilde{g})$ is assumed to be endowed with a concircular vector field v.

Theorem 1.3 *A submanifold M^n in N^m admits an almost Ricci soliton (M^n, g, v^T, λ) if and only if the Ricci tensor of M^n satisfies*

$$Ric(X, Y) = (\lambda - \mu)g(X, Y) - \widetilde{g}(h(X, Y), v^N) \tag{1.4}$$

for any X, Y tangent to M^n.

Remark 1.1 A Riemannian submanifold M^n is called ξ- umbilical with respect to a normal vector field ξ if its shape operator satisfies $A_\xi = \varphi I$, where φ is a function on M^n and I is the identity map.

So, we have the following corollaries.

Corollary 1.4 *An almost Ricci soliton* (M^n, g, v^T, λ) $(n \geq 3)$ *on a submanifold* M^n *in* N^m *is trivial if and only if* M^n *is* v^N*- umbilical.*

Corollary 1.5 *Every almost Ricci soliton* (M^n, g, v^T, λ) *on a totally umbilical submanifold* M^n *in* N^m *is a trivial Ricci soliton.*

We already know by [8] that if (M^n, g, v, λ) is an almost Ricci soliton with parallel Ricci tensor, then v is an affine conformal vector field. Hence, we have the following corollary for a compact manifold M^n.

Corollary 1.6 *Let* (M^n, g, v^T, λ) *be a compact almost Ricci soliton on a submanifold* M^n *in* N^m *with parallel Ricci tensor. Then* M^n *is isometric to a sphere.*

We shall observe that (cf. Lemma 2.1) if an almost Ricci soliton (M^n, g, v^T, λ) is totally umbilical submanifold of $(N^m, \widetilde{g})$, then for any concircular vector field v on N^m, the tangential part v^T of v on M^n reduces to a concircular vector field on M^n. More precisely, we have the following theorem.

Theorem 1.7 *Let* (M^n, g, v^T, λ) $(n \geq 3)$ *be totally umbilical almost Ricci soliton on a submanifold* M^n *in* N^m*. Then* M^n *is isometric to a sphere.*

2 Preliminaries

Let $(N^m, \widetilde{g})$ denote an m-dimensional Riemannian manifold and $\phi : M^n \to N^m$ be an isometric immersion of n-dimensional Riemannian manifold (M^n, g) into $(N^m, \widetilde{g})$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections on (M^n, g) and $(N^m, \widetilde{g})$, respectively.

Given tangent vector fields X, Y over M^n and ξ normal to M^n, we have the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

where $\nabla_X Y$ and $h(X, Y)$ stand for the tangential and the normal components of $\widetilde{\nabla}_X Y$. Similarly $-A_\xi X$ and $D_X \xi$ are the tangential and normal components of $\widetilde{\nabla}_X \xi$. These two formulas define the second fundamental form h, the shape operator A and the normal connection D of M^n in the ambient space N^m.

For a normal vector $\xi \in T_p^{\perp}M$ at $p \in M$, A_ξ is self-adjoint endomorphism. The shape operator and the second fundamental form are related by

$$\widetilde{g}(h(X,Y),\xi) = g(A_\xi X, Y).$$

For a concircular vector field on N^m, we denote by v^T and v^N the tangential and normal components on M^n. By using the Gauss formula as well as Weingarten formula, we get the following lemma.

Lemma 2.1 *A submanifold M^n in N^m is flat or totally umbilical if and only if v^T is a concircular vector field on M^n.*

Proof. Assume that a submanifold M^n in N^m is flat or totally umbilical. Since v is a concircular vector field on N^m, then there is a function μ on N^m such that $\widetilde{\nabla}_X v = \mu X$. Using Equations (2.1) and (2.2) we get

$$\mu X = \nabla_X v^T + h(X, v^T) - A_{v^N} X + D_X v^N \tag{2.3}$$

for any vector field X tangent to M^n. By comparing the tangential and normal components in Eq.(2.3), we have

$$\nabla_X v^T = A_{v^N} X + \mu X, \tag{2.4}$$

which shows that v^T is concircular vector field on M^n.

Conversely, if v^T is a concircular vector field on submanifold M^n, then there is a non-trivial function η on M such that

$$\nabla_X v^T = \eta X, \quad X \in TM.$$

By combining the above equation and Eq.(2.4), we get

$$A_{v^N} X = (\eta - \mu) X.$$

This shows that when $\eta = \mu$, M is flat. Otherwise, M is a totally umbilical. □

Now, we are ready to prove the main results of this paper.

3 Proofs of the main results

3.1 Proof of Theorem 1.1

Proof. Assume that (M^n, g, v, λ) is a compact almost Ricci soliton which admits a non-zero concircular vector field v satisfying condition (1.3). We then have

$$\frac{1}{2} L_v g = \mu g. \tag{3.1}$$

Therefore, we combine Equations (3.1) and (1.1) to obtain

$$Ric = (\lambda - \mu)g. \tag{3.2}$$

By taking the contraction of (3.2) we also find $R = n(\lambda - \mu)$. It is well known that the scalar curvature R is constant for $n \geq 3$. Thus after taking the Lie derivation of (3.2) with respect to the vector field v, we have

$$L_v Ric = (\lambda - \mu)L_v g = 2(\lambda - \mu)\mu g.$$

We may apply Theorem 4.2 (p. 54 of [10]) to conclude that M^n is isometric to an Euclidean sphere. □

3.2 Proof of Theorem 1.3

Proof. Since v is a concircular vector field on the ambient space N^m, we have

$$v = v^T + v^N. \tag{3.3}$$

It follows from (1.3), (3.3) and the formulas of Gauss and Weingarten that

$$\mu X = \nabla_X v^T + h(X, v^T) - A_{v^N} X + D_X v^N \tag{3.4}$$

for any X tangent to M^n. By comparing the tangential and normal components from (3.4) we get

$$\nabla_X v^T = -A_{v^N} X + \mu X. \tag{3.5}$$

From the definition of Lie derivative and Eq.(3.5), we find

$$\begin{aligned} (L_{v^T} g)(X, Y) &= g(\nabla_X v^T, Y) + g(\nabla_Y v^T, X) \\ &= 2\mu g(X, Y) + 2g(A_{v^N} X, Y) \\ &= 2\mu g(X, Y) + 2\widetilde{g}(h(X, Y), v^N) \end{aligned} \tag{3.6}$$

for X, Y tangent to M^n. Finally, we compare the above expression with Eq.(1.1) to conclude that (M^n, g, v^T, λ) is a Ricci soliton if and only if

$$Ric(X, Y) + \mu g(X, Y) + \widetilde{g}(h(X, Y), v^N) = \lambda g(X, Y)$$

which implies (1.4). □

3.3 Proof of Theorem 1.7

Proof. Assume that an almost Ricci soliton (M^n, g, v^T, λ) on a submanifold M^n in N^m is totally umbilical. Then, Lemma 2.1 implies that v^T is a concircular vector field, i.e. $\nabla_X v^T = \mu X$ and also from the definition of Lie derivative, we obtain

$$\begin{aligned}(L_{v^T} g)(X,Y) &= g(\nabla_X v^T, Y) + g(\nabla_Y v^T, X)\\ &= 2\mu g(X,Y)\end{aligned}$$

for X, Y tangent to M^n. By using the above equation into (1.1), we get

$$Ric = (\lambda - \mu) g(X,Y)$$

and also have $R = n(\lambda - \mu)$. Here, it suffices to invoke Theorem 2 in [1]. □

References

1. Barros, A.; Ribeiro, Jr.E. – *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc., **140** (2012), no. 3, 1033-1040.
2. Barros, A.; Batista, R.; Ribeiro, Jr.E. – *Rigidity of gradient almost Ricci solitons*, Illinois J. Math., **56** (2012), no. 4, 1267-1279.
3. Barros, A.; Gomes, J.N.; Ribeiro, Jr.E. – *A note on rigidity of the almost Ricci soliton*, Arch. Math. (Basel), **100** (2013), no. 5, 481-490.
4. Barros, A.; Batista, R.; Ribeiro, Jr.E. – *Compact almost Ricci solitons with constant scalar curvature are gradient*, Monatsh. Math., **174** (2014), no. 1, 29-39.
5. Chen, B.Y. – *Some results on concircular vector fields and their applications to Ricci solitons*, Bull. Korean Math. Soc., **52** (2015), no. 5, 1535-1547.
6. Fialkow, A. – *Conformal geodesics*, Trans. Amer. Math. Soc.,**45** (1939), no. 3, 443-473.
7. Pigola, S.; Rigoli, M.; Rimoldi, M.; Setti A.G. – *Ricci almost solitons*, Ann. Sc. Norm. Super Pisa Cl. Sci. (5), **10** (2011) , no. 4, 757-799.
8. Sharma, R.; Duggal, K.L. – *Characterization of an affine conformal vector field*, C. R. Math. Rep. Acad. Sci. Canada, **7** (1985), no. 3, 201-205.
9. Tanno, S.; Weber, W.C. – *Closed conformal vector fields*, J. Differential Geometry, **3** (1969), 361-366.
10. Yano, K. – *Integral formulas in Riemannian geometry*, Pure and Applied Mathematics, no. 1 Marcel Dekker, Inc., New York (1970) ix+156 pp.

Received: 12.VII.2016 / Revised: 27.IX.2016 / Accepted: 22.XI.2016

Authors

Dilek Açıkgöz Kaya (Corresponding author),
Adnan Menderes University,
Faculty of Arts and Sciences,
Department of Mathematics,
Aydın, Turkey
E-mail: dilek.acikgoz@adu.edu.tr

Leyla Onat,
Adnan Menderes University,
Faculty of Arts and Sciences,
Department of Mathematics,
Aydın, Turkey
E-mail: lonat@adu.edu.tr