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Special elements in bounded integral residuated lattices

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Abstract Residuation is a fundamental concept of ordered structures and the residuated lattices, obtained by adding a residuated monoid operation to lattices, have been applied in several branches of mathematics. The aim of this paper is to give new characterizations for boolean elements in bounded integral residuated lattices using the notions of idempotent element, involutive element and introducing the notions of regular element and strong element in a bounded integral residuated lattice. Also, we characterize the bounded integral residuated lattices which are Boolean algebras.

Keywords Bounded integral residuated lattice · residuated lattice · FL_w-algebra · FL_{ew}-algebra · pseudo MTL-algebra · RL-monoid · pseudo BL-algebra · Boolean algebra · Tarski algebra · Hilbert algebra · G-algebra · boolean element · idempotent element · involutive element · regular element · strong element · dense element

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1 Introduction

Residuation is a fundamental concept of ordered structures and the residuated lattices, obtained by adding a residuated monoid operation to lattices, have been applied in several branches of mathematics. Residuated lattices originated in Mathematical Logic without contraction. The commutative residuated lattices have been investigated by Krull ([15]), Dilworth ([8]), Ward and Dilworth ([23]), Ward ([22]), Balbes and Dwinger ([1]) and Pavelka ([19]).

In [13], Idziak proved that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [12], *full BCK- algebras* in [15], FL_{ew}*- algebras* in [18], and *integral, residuated, commutative l-monoids* in [2], [3].

Apart from their logical interest, commutative residuated lattices have interesting algebraic properties (see [4], [11], [20], [21]) and include two important classes of algebras : *BL-algebras* (introduced by Hajek as the algebraic counterpart of his Basic Logic, see [11]) and *MV-algebras* (introduced by Chang in [6] to prove the completeness theorem for Łukasiewicz calculus).

Non-commutative residuated lattices, sometimes called *pseudo-residuated lattices*, are the algebraic counterparts of sub-structural logics, i.e. logics which lack at least one of the three structural rules, namely contraction, weakening and exchange. Complete studies on non-commutative residuated lattices were developed by Ono, Jipsen, Galatos, Tsinakis and Kowalski ([9], [16]). Particular classes of non-commutative residuated lattices are full Lambek algebras (*FL- algebras)* and *bounded integral residuated lattices* (*FL_w- algebras*).

In this paper we work in the general setting of a non-commutative residuated lattice (more precisely, we work with *bounded integral residuated lattices*, or *FL_w- algebras,* for short) and extend to the non-commutative case some results from [4].

An important construction for a bounded integral residuated lattice A is the boolean center $B(A)$, which contains all complemented elements of A.

For a bounded integral residuated lattice A we denote by $Id(A)$ the set of idempotent elements of A, by $Inv(A)$ the set of involutive elements of A, by $R(A)$ the set of regular elements of A, by $S(A)$ the set of strong elements of A and by $D(A)$ the set of dense elements of A. We present new characterization for these elements.

We prove that in general, $B(A) \subsetneqq R(A) \subsetneqq Inv(A)$, $B(A) \subsetneqq Id(A) \cap Inv(A) \cap R(A)$.

If A is a bounded integral residuated lattice, we prove that $x \in R(A)$ iff $x \in Inv(A)$ and $x^- \odot (x^- \rightsquigarrow x) = (x^\sim \to x) \odot x^\sim = 0$, see Theorem 3.11. Also, $x \in B(A)$ iff $x \in R(A) \cap Id(A) \cap S(A)$ and $(x \rightsquigarrow x^-) \vee (x^- \rightsquigarrow x) = (x \to x^\sim) \vee (x^\sim \to x) = 1$, see Theorem 3.16. In particular, if A is a pseudo $MTL-$ algebra, then $B(A) = R(A) \cap Id(A) \cap S(A)$, see Corollary 3.17. Also, if A is an RL-monoid, then $x \in B(A)$ iff $x \in Id(A) \cap Inv(A)$ and $(x \rightsquigarrow x^-) \vee (x^- \rightsquigarrow x) = 1$ iff $x \in Id(A) \cap Inv(A)$ and $(x \to x^\sim) \vee (x^\sim \to x) = 1$, see Corollary 3.9.

Theorem 3.19 characterizes the bounded integral residuated lattices which are Boolean algebras.

2 Preliminaries

In this section we recall some basic notions and results regarding non-commutative residuated lattices and bounded integral residuated lattices, which will be used in the following sections of the paper.

Definition 2.1 *([5], [9]) A* residuated lattice *is an algebra* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, e)$ *of type* $(2, 2, 2, 2, 2, 0)$ *satisfying the following axioms:*

(LR_1) $(A, \wedge, \vee)$ *is a lattice;*

(LR_2) $(A, \odot, e)$ *is a monoid;*

(LR_3) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$ *for every* $x, y, z \in A$ *(*pseudo-residuation*).*

Our notation differs slightly from that of [9]; we write $x \to y$ and $x \rightsquigarrow y$ instead of y/x and $x \backslash y$, respectively.

Note that generally, e is not the top element of A.

Proposition 2.2 *([5]) If a residuated lattice* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, e)$ *has a bottom element* 0, *then it also has a top element* 1.

A residuated lattice A with a constant 0 (which can denote any element) is called a *full Lambek algebra* (FL- algebra, for short). If $x \leq e$ for all $x \in A$, then A is called an *integral residuated lattice.* An FL- algebra A which satisfies the condition $0 \leq x \leq e$ for all $x \in A$, is called an *bounded integral residuated lattice* (or, FL_w- algebra, for short).

According to Proposition 2.2, in this case we have $e = 1$, so a bounded integral residuated lattice will be denote by $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$.

Definition 2.3 *([5]) A* bounded integral residuated lattice *is an algebra* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 2, 0, 0)$ *satisfying the following axioms:*

(bLR_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
(bLR_2) $(A, \odot, 1)$ *is a monoid;*
(bLR_3) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$ *for every* $x, y, z \in A$.

A bounded integral residuated lattice is *commutative* if the operation $\odot$ is commutative and we shall call such algebras commutative bounded integral residuated lattices, or FL_{ew}- algebras, for short; clearly, in this case operations $\to$ and $\rightsquigarrow$ coincides.

Example 2.1 ([14]) We give an example of commutative bounded integral residuated lattice. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A is a commutative bounded integral residuated lattice relative to the following operations:

$\to = \rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Example 2.2 ([5]) We give another example of bounded integral residuated lattice. Let $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$. A is a bounded integral residuated lattice relative to the following operations:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

In what follows, by A we denote the universe of a bounded integral residuated lattice.

For $x \in A$ we denote $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$.

The following rules of calculus in A can be found in [5], [9], [17]:

(c_1) $x \odot 0 = 0 \odot x = 0, 0 \to x = 0 \rightsquigarrow x = 1, x \to 1 = x \rightsquigarrow 1 = 1, x \to x = x \rightsquigarrow x = 1, 1 \to x = 1 \rightsquigarrow x = x$;

(c_2) $x \le y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$;

(c_3) If $x \le y$, then $x \odot z \le y \odot z$ and $z \odot x \le z \odot y$ so $x \to y \le (x \odot z) \to (y \odot z)$ and $x \rightsquigarrow y \le (z \odot x) \rightsquigarrow (z \odot y)$;

(c_4) $x \to y \le (y \to z) \rightsquigarrow (x \to z)$ and $x \rightsquigarrow y \le (y \rightsquigarrow z) \to (x \rightsquigarrow z)$, so $x \le y$ implies $y \to z \le x \to z$ and $y \rightsquigarrow z \le x \rightsquigarrow z$;

(c_5) $x \to y \le (z \to x) \to (z \to y)$ and $x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$ so $x \le y$ implies $z \to x \le z \to y$ and $z \rightsquigarrow x \le z \rightsquigarrow y$;

(c_6) $x \odot (x \rightsquigarrow y) \le y$ and $(x \to y) \odot x \le y$;

(c_7) $x \to (y \to z) = (x \odot y) \to z$ and $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;

(c_8) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$, so $x \to x^\sim = x \rightsquigarrow x^-$;

(c_9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $(y \vee z) \odot x = (y \odot x) \vee (z \odot x)$ and $x \odot (y \wedge z) \le (x \odot y) \wedge (x \odot z)$, $(y \wedge z) \odot x \le (y \odot x) \wedge (z \odot x)$;

(c_{10}) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ and $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$;

(c_{11}) $(y \vee z) \to x = (y \to x) \wedge (z \to x)$ and $(y \vee z) \rightsquigarrow x = (y \rightsquigarrow x) \wedge (z \rightsquigarrow x)$;

(c_{12}) $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$ and $x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim = x^{\sim -} \rightsquigarrow y^{\sim -}$;

(c_{13}) $1^\sim = 1^- = 0, 0^\sim = 0^- = 1$;

(c_{14}) $x \odot x^\sim = x^- \odot x = 0$;

(c_{15}) $y \le x^\sim$ iff $x \odot y = 0$ and $y \le x^-$ iff $y \odot x = 0$;

(c_{16}) $x \le x^- \rightsquigarrow y, x \le x^\sim \to y$;

(c_{17}) $x \le (x \rightsquigarrow y) \to y, x \le (x \to y) \rightsquigarrow y$, hence $x \le (x^\sim)^-, x \le (x^-)^\sim$;

(c_{18}) $x \rightsquigarrow y \le y^\sim \to x^\sim, x \to y \le y^- \rightsquigarrow x^-$;

(c_{19}) $x \to y^\sim = y \rightsquigarrow x^-, x \rightsquigarrow y^- = y \to x^\sim$;

(c_{20}) $x \le y$ implies $y^\sim \le x^\sim$ and $y^- \le x^-$;

(c_{21}) $((x^\sim)^-)^\sim = x^\sim, ((x^-)^\sim)^- = x^-$;

(c_{22}) $(x \odot y)^- = x \to y^-, (x \odot y)^\sim = y \rightsquigarrow x^\sim$;

(c_{23}) $(x \wedge y)^\sim \ge x^\sim \vee y^\sim, (x \vee y)^\sim = x^\sim \wedge y^\sim$ and $(x \wedge y)^- \ge x^- \vee y^-, (x \vee y)^- = x^- \wedge y^-$.

We recall that (see [5]):

A *pseudo MTL-algebra* is a bounded integral residuated lattice A satisfying the *pseudo-prelinearity condition* $(ps-lin)$: $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, for every $x, y \in A$.

A *bounded RL-monoid* or *divisible residuated lattice* is a bounded integral residuated lattice A satisfying the *pseudo-divisibility condition* $(ps-div)$: $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$, for every $x, y \in A$.

A *pseudo BL-algebra* is a pseudo MTL-algebra A satisfying the pseudo-divisibility condition.

A *pseudo MV-algebra* is a pseudo BL-algebra A satisfying the condition (pDN) (*pseudo Double-Negation condition*): $(x^-)^\sim = (x^\sim)^- = x$, for every $x \in A$.

Using [5] (Theorem 4.4 and Propositions 2.1, 2.2) we deduce the following result:

Theorem 2.4 *A bounded integral residuated lattice* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ *is a pseudo MV-algebra iff it satisfies two additional conditions:*
(W) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$ *and* $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$,
(H) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$, *for any* $x, y \in A$.

Proposition 2.5 *For a bounded integral residuated lattice A the following conditions are equivalent:*
(i) *A is a bounded RL-monoid;*
(ii) $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \odot [(x \wedge y) \rightsquigarrow z]$ *and* $x \to (y \wedge z) = [(x \wedge y) \to z] \odot (x \to y)$, *for every* $x, y, z \in A$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y, z \in A$. We have $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z) = (x \rightsquigarrow y) \odot [(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)] = (x \rightsquigarrow y) \odot [(x \odot (x \rightsquigarrow y)) \rightsquigarrow z] = (x \rightsquigarrow y) \odot [(x \wedge y) \rightsquigarrow z]$ and $x \to (y \wedge z) = (x \to y) \wedge (x \to z) = [(x \to y) \to (x \to z)] \odot (x \to y) = [((x \to y) \odot x) \to z] \odot (x \to y) = [(x \wedge y) \to z] \odot (x \to y)$.

$(ii) \Rightarrow (i)$. Conversely, in (ii), for $x = 1$, we obtain for every $y, z \in A$: $1 \rightsquigarrow (y \wedge z) = (1 \rightsquigarrow y) \odot [(1 \wedge y) \rightsquigarrow z] \Leftrightarrow y \wedge z = y \odot (y \rightsquigarrow z)$ and $1 \to (y \wedge z) = [(1 \wedge y) \to z] \odot (1 \to y) \Leftrightarrow y \wedge z = (y \to z) \odot y$.

Finally, $y \wedge z = y \odot (y \rightsquigarrow z) = (y \to z) \odot y$, so A is a bounded RL-monoid. $\square$

Proposition 2.6 *In a bounded integral residuated lattice A the following assertions are equivalent :*
(i) $x^2 = x$ *for every* $x \in A$;
(ii) $x \odot (x \rightsquigarrow y) = (x \to y) \odot x = x \odot y = y \odot x = x \wedge y$ *for every* $x, y \in A$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$. From c_6, $x \odot (x \rightsquigarrow y) \leq y$, thus $x \odot (x \odot (x \rightsquigarrow y)) = x^2 \odot (x \odot y) = x \odot (x \rightsquigarrow y) \leq x \odot y$.

Since $y \leq x \rightsquigarrow y$, then $x \odot y \leq x \odot (x \rightsquigarrow y)$, so $x \odot (x \rightsquigarrow y) = x \odot y$.

Clearly, $x \odot y \leq x, y$. To prove $x \odot y = x \wedge y$, let $t \in A$ such that $t \leq x$ and $t \leq y$. Then $t = t^2 \leq x \odot y$, that is, $x \odot y = x \wedge y$.

Similarly, $(x \to y) \odot x \leq y$, thus, $((x \to y) \odot x) \odot x \leq y \odot x$, so $(x \to y) \odot x^2 = (x \to y) \odot x \leq y \odot x$.

Since $y \leq x \to y$, then $y \odot x \leq (x \to y) \odot x$, so $(x \to y) \odot x = y \odot x$.

Clearly, $y \odot x \leq x, y$. To prove $y \odot x = x \wedge y$, let $t \in A$ such that $t \leq x$ and $t \leq y$. Then $t = t^2 \leq y \odot x$, that is, $y \odot x = x \wedge y$.

We conclude that, $x \odot (x \rightsquigarrow y) = (x \to y) \odot x = x \odot y = y \odot x = x \wedge y$ for every $x, y \in A$.

$(ii) \Rightarrow (i)$. In particular for $x = y$ we obtain $x \odot x = x \wedge x = x \Leftrightarrow x^2 = x$. $\square$

We recall that, a commutative bounded integral residuated lattice A in which $x^2 = x$ for all $x \in A$ is called of *Gödel type*.

Remark 2.1 Following Proposition 2.6, in a bounded integral residuated lattice A, the condition $x^2 = x$ for all $x \in A$, is equivalently with $x \odot y = y \odot x = x \wedge y$ for all $x, y \in A$, so A is a commutative bounded integral residuated lattice. We can say that a bounded integral residuated lattice A in which $x^2 = x$ for all $x \in A$ is called of *Gödel type.*

Definition 2.7 *([7]) A* Hilbert algebra *is an algebra* $(H, \rightarrow, 1)$ *of type (2,0) such that the following axioms hold in* H*:*
(H_1) $x \rightarrow (y \rightarrow x) = 1$*;*
(H_2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$;
(H_3) *If* $x \rightarrow y = y \rightarrow x = 1$, *then* $x = y$.

It is easy to see that if H is a Hilbert algebra, then the relation $\leq$ defined on H by $x \leq y$ iff $x \rightarrow y = 1$ is a (partial) order on H (called *natural order* on H) with greatest element 1.

H is called *bounded,* if there is a smallest element $0 \in H$ relative to the natural ordering; in this case for $x \in H$ we denote $x^* = x \rightarrow 0$ and $x^{**} = (x^*)^*$.

Proposition 2.8 *For a bounded integral residuated lattice* $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *the following are equivalent:*
(i) $(A, \rightarrow, 1)$ *is a Hilbert algebra or* $(A, \rightsquigarrow, 1)$ *is a Hilbert algebra;*
(ii) $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *is of Gödel type.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $(A, \rightarrow, 1)$ is a Hilbert algebra, then for every $x, y, z \in A$, $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$. We have $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ and $(x \rightarrow y) \rightarrow (x \rightarrow z) = ((x \rightarrow y) \odot x) \rightarrow z$, so we obtain $(x \odot y) \rightarrow z \leq ((x \rightarrow y) \odot x) \rightarrow z$, hence for $z = x \odot y$, $(x \rightarrow y) \odot x \leq x \odot y$; for $x = y$ we obtain $x \leq x^2$, so, $x^2 = x$, that is, A is of *Gödel type.*

If suppose that $(A, \rightsquigarrow, 1)$ is a Hilbert algebra, then for every $x, y, z \in A$, $x \rightsquigarrow (y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$. We have $(y \odot x) \rightsquigarrow z \leq (x \odot (x \rightsquigarrow y)) \rightsquigarrow z$, hence for $z = y \odot x$ we obtain that $x \odot (x \rightsquigarrow y) \leq y \odot x$; for $x = y$ we obtain $x \leq x^2$, so $x^2 = x$, that is, A is of *Gödel type.*

$(ii) \Rightarrow (i)$. Follows from [20], Proposition 1.21. $\square$

Proposition 2.9 *For a bounded integral residuated lattice* $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *the following are equivalent:*
(i) $(A, \rightarrow, 1)$ *(* $(A, \rightsquigarrow, 1)$ *) is a Hilbert algebra;*
(ii) $(A, \vee, \wedge, \rightarrow, 0, 1)$ *(* $(A, \vee, \wedge, \rightsquigarrow, 0, 1)$ *) is a relative Stone lattice.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $(A, \rightarrow, 1)$ ($(A, \rightsquigarrow, 1)$) is a Hilbert algebra. Using Proposition 2.8, $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is of *Gödel type.* From Proposition 2.6, by $x^2 = x$, for every $x \in A$, $x \odot y = y \odot x = x \wedge y$, that is $(A, \vee, \wedge, \rightarrow, 0, 1)$ ($(A, \vee, \wedge, \rightsquigarrow, 0, 1)$) is a relative Stone lattice.

$(ii) \Rightarrow (i)$. If $(A, \vee, \wedge, \rightarrow, 0, 1)$ ($(A, \vee, \wedge, \rightsquigarrow, 0, 1)$) is a relative Stone lattice, then $(A, \vee, \wedge, \rightarrow, 0, 1)$ ($(A, \vee, \wedge, \rightsquigarrow, 0, 1)$) is a Heyting algebra, so $(A, \rightarrow, 1)$ ($(A, \rightsquigarrow, 1)$) is a Hilbert algebra.$\square$

Lemma 2.10 *Let A be a bounded integral residuated lattice which satisfies (W). Then*
(i) $x^{-\sim} = x^{\sim -} = x$ *for every* $x \in A$;
(ii) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x = (x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = x \vee y$ *for every* $x, y \in A$.

Proof. (i). For $x \in A$ we have $x^{-\sim} = (x \to 0) \rightsquigarrow 0 \overset{(W)}{=} (0 \to x) \rightsquigarrow x = 1 \rightsquigarrow x = x$ and $x^{\sim -} = (x \rightsquigarrow 0) \to 0 = (0 \rightsquigarrow x) \to x = 1 \to x = x$.

(ii). If $x, y \in A$, then $x, y \leq (x \to y) \rightsquigarrow y$. To prove $(x \to y) \rightsquigarrow y = x \vee y$, consider $t \in A$ such that $x, y \leq t$. Then $t \to y \leq x \to y$, hence $(x \to y) \rightsquigarrow y \leq (t \to y) \rightsquigarrow y = (y \to t) \rightsquigarrow t = 1 \rightsquigarrow t = t$, that is, $x \vee y = (x \to y) \rightsquigarrow y$.

From (W) we deduce $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x = x \vee y$.

If $x, y \in A$, then $x, y \leq (x \rightsquigarrow y) \to y$. To prove $(x \rightsquigarrow y) \to y = x \vee y$, consider $t \in A$ such that $x, y \leq t$. Then $t \rightsquigarrow y \leq x \rightsquigarrow y$, hence $(x \rightsquigarrow y) \to y \leq (t \rightsquigarrow y) \to y = (y \rightsquigarrow t) \to t = 1 \to t = t$, that is, $x \vee y = (x \rightsquigarrow y) \to y$.

Using (W) we deduce $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = x \vee y$. $\square$

3 Boolean center, regular, strong and dense elements in a bounded integral residuated lattice

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We denote the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In bounded integral residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 3.1 *([9], [16]) Let A be a bounded integral residuated lattice and suppose that $a \in A$ has the complement $b \in A$. Then, the following hold:*
(i) *If c is another complement of a in A, then $c = b$;*
(ii) $a^- = a^\sim = b$ *and* $b^- = b^\sim = a$;
(iii) $a^2 = a$.

Let A be a bounded integral residuated lattice and $B(A)$ the set of all complemented elements of the lattice $(A, \vee, \wedge, 0, 1)$. $B(A)$ is the universe of a Boolean subalgebra of A, see [16], called the *Boolean center* of A. If $e \in B(A)$, then $e^{-\sim} = e^{\sim -} = e$. Also, see [16], $e \odot x = x \odot e = e \wedge x$, for every $x \in A$.

Proposition 3.2 *([5], [9], [16]) If A is a bounded integral residuated lattice, then for $e \in A$ the following are equivalent:*
(i) $e \in B(A)$;
(ii) $e \vee e^- = 1$ *and* $e \wedge e^- = 0$;
(iii) $e \vee e^\sim = 1$ *and* $e \wedge e^\sim = 0$.

Remark 3.1 If $e \in B(A)$, then $e^{-} = e^{\sim}$, from Lemma 3.1, and $e^{-} = e^{\sim} \in B(A)$. Indeed, $e^{-} \wedge e^{-\sim} = e^{-} \wedge e = 0$ and $e^{-} \vee e^{-\sim} = e^{-} \vee e = 1$.

Proposition 3.3 *([5]) If $e \in B(A)$ and $x, y \in A$ then:*
(c_{24}) $e \to e^{-} = e \rightsquigarrow e^{-} = e^{-}$ *and* $e \to e^{\sim} = e \rightsquigarrow e^{\sim} = e^{\sim}$;
(c_{25}) $e^{-} \to e = e^{-} \rightsquigarrow e = e^{\sim} \to e = e^{\sim} \rightsquigarrow e = e$;
(c_{26}) $(x \to e) \odot x = x \odot (x \rightsquigarrow e) = x \wedge e$ *and* $(e \to x) \odot e = e \odot (e \rightsquigarrow x) = e \wedge x$;
(c_{27}) $e \to (x \to y) = (e \to x) \to (e \to y)$ *and* $e \rightsquigarrow (x \rightsquigarrow y) = (e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y)$.

For any bounded integral residuated lattice A, let us denote by $Id(A) = \{x \in A : x \odot x = x\}$ the set of all *idempotent elements* of A and by $Inv(A) = \{x \in A : x = (x^{\sim})^{-} = (x^{-})^{\sim}\}$ the set of all *involutive elements* of A. $Inv(A)$ is called the *involutive center* of A.

Remark 3.2 ([10]) If A is a pseudo BL-algebra, then $B(A) = Id(A) \cap Inv(A)$.

We give a new characterizations for involutive and idempotent elements in a bounded integral residuated lattice.

Proposition 3.4 *Let A be a bounded integral residuated lattice and $a, x \in A$. Then the following assertions are equivalent:*
(i) $x \in Inv(A)$;
(ii) $a \to x = x^{-} \rightsquigarrow a^{-}$ *and* $a \rightsquigarrow x = x^{\sim} \to a^{\sim}$, *for every* $a \in A$.

Proof. $(i) \Rightarrow (ii)$. Since $x \in Inv(A)$, then $a \to x = a \to x^{-\sim} \overset{c_{12}}{=} x^{-} \rightsquigarrow a^{-}$ and $a \rightsquigarrow x = a \rightsquigarrow x^{\sim -} \overset{c_{12}}{=} x^{\sim} \to a^{\sim}$.

$(ii) \Rightarrow (i)$. For $a = x^{-\sim}$ resp. $x^{\sim -}$ we deduce that $x^{-\sim} \to x = x^{-} \rightsquigarrow x^{-\sim -} \overset{c_{21}}{=} 1$, so, by c_{17}, $x^{-\sim} = x$ and $x^{\sim -} \rightsquigarrow x = x^{\sim} \to x^{\sim -\sim} \overset{c_{21}}{=} 1$, so, by c_{17}, $x^{\sim -} = x$.

We deduce that $x^{-\sim} = x^{\sim -} = x$, so $x \in Inv(A)$. $\square$

Remark 3.3 If $x \in Inv(A)$ then $x^{\sim} \to x = x^{-} \rightsquigarrow x$. Indeed, $x^{\sim} \to x = x^{\sim} \to x^{-\sim} \overset{c_{12}}{=} x^{-} \rightsquigarrow x^{\sim -} = x^{-} \rightsquigarrow x$.

Lemma 3.5 *Let A be a bounded integral residuated lattice and $x \in Id(A)$. Then $x \odot (x \rightsquigarrow y) = x \odot y$ and $(x \to y) \odot x = y \odot x$, for every $y \in A$.*

Proof. We have $x \odot (x \rightsquigarrow y) = x \odot [x \odot (x \rightsquigarrow y)] \leq x \odot (x \wedge y) \overset{c_9}{\leq} (x \odot x) \wedge (x \odot y) = x \wedge (x \odot y) = x \odot y$ and $(x \to y) \odot x = [(x \to y) \odot x] \odot x \leq (x \wedge y) \odot x \overset{c_9}{\leq} (x \odot x) \wedge (y \odot x) = x \wedge (y \odot x) = y \odot x$. Since $x \odot y \leq x \odot (x \rightsquigarrow y)$ and $y \odot x \leq (x \to y) \odot x$, we deduce that $x \odot (x \rightsquigarrow y) = x \odot y$ and $(x \to y) \odot x = y \odot x$, for every $y \in A$.

Proposition 3.6 *If A is an RL- monoid and $x \in Id(A)$, $y \in A$, then*
(i) $x \odot y = x \wedge y = y \odot x$;

(ii) $x \wedge x^{\sim} = 0 = x \wedge x^{-}$;
(iii) $x \rightsquigarrow y = x \to y$;
(iv) $x^{\sim} = x^{-}$;
(v) $x \to x^{-} = x \to x^{\sim} = x \rightsquigarrow x^{-} = x \rightsquigarrow x^{\sim} = x^{-} = x^{\sim}$.

Proof. (i). Since $x \in Id(A)$, from Lemma 3.5, $x \wedge y = x \odot (x \rightsquigarrow y) = x \odot y$ and $x \wedge y = (x \to y) \odot x = y \odot x$. We deduce that $x \wedge y = x \odot y = y \odot x$.

(ii). Follows by (i) and c_{14}.

(iii). We have the following equivalences for any $z \in A$:

$z \leq x \rightsquigarrow y \Leftrightarrow x \odot z \leq y \Leftrightarrow z \odot x \leq y \Leftrightarrow z \leq x \to y$. We deduce $x \rightsquigarrow y = x \to y$.

(iv). Follows taking $y = 0$ in (iii).

(v). $x \to x^{-} = x \to (x \to 0) = (x \odot x) \to 0 = x \to 0 = x^{-}$ and $x \rightsquigarrow x^{\sim} = x \rightsquigarrow (x \rightsquigarrow 0) = (x \odot x) \rightsquigarrow 0 = x \rightsquigarrow 0 = x^{\sim}$.

Using (iii) and (iv) we deduce the conclusion. $\square$

We put in evidence new rules of calculus with boolean elements in a bounded integral residuated lattice A:

Lemma 3.7 *Let $x, y \in A$ and $e, f \in B(A)$. Then:*
(c_{28}) $e \to (e \to x) = e \to x$ *and* $e \rightsquigarrow (e \rightsquigarrow x) = e \rightsquigarrow x$;
(c_{29}) $(e \to x) \vee (x \to e) = (e \rightsquigarrow x) \vee (x \rightsquigarrow e) = 1$;
(c_{30}) $e \vee x = [(e \to x) \rightsquigarrow x] \wedge [(x \to e) \rightsquigarrow e] = [(e \rightsquigarrow x) \to x] \wedge [(x \rightsquigarrow e) \to e]$;
(c_{31}) $e^{\sim} \to x = (e \rightsquigarrow x) \to x = e \vee x = (e \to x) \rightsquigarrow x = e^{-} \rightsquigarrow x$;
(c_{32}) $e \vee (x \to y) = (e \vee x) \to (e \vee y)$ *and* $e \vee (x \rightsquigarrow y) = (e \vee x) \rightsquigarrow (e \vee y)$;
(c_{33}) $(e \vee x) \to (f \vee x) = (e \to f) \vee x$ *and* $(e \vee x) \rightsquigarrow (f \vee x) = (e \rightsquigarrow f) \vee x$;
(c_{34}) $x \to (e \to f) = (x \to e) \to (x \to f)$ *and* $x \rightsquigarrow (e \rightsquigarrow f) = (x \rightsquigarrow e) \rightsquigarrow (x \rightsquigarrow f)$;
(c_{35}) $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$;
(c_{36}) $x \wedge (e \vee f) = (x \wedge e) \vee (x \wedge f)$;
(c_{37}) *If* $e, f \leq x$, *then* $e \odot (x \rightsquigarrow f) = (x \to e) \odot f$.

Proof. (c_{28}). We have $e \to (e \to x) = e^2 \to x = e \to x$ and $e \rightsquigarrow (e \rightsquigarrow x) = e^2 \rightsquigarrow x = e \rightsquigarrow x$.

(c_{29}). Since $e \in B(A)$ then $e \vee e^{-} = e \vee e^{\sim} = 1$. Since $e^{-} \leq e \to x$ and $e \leq x \to e$ we deduce that $1 = e^{-} \vee e \leq (e \to x) \vee (x \to e)$, so $(e \to x) \vee (x \to e) = 1$.

Since $e^{\sim} \leq e \rightsquigarrow x$ and $e \leq x \rightsquigarrow e$ we deduce that $1 = e^{\sim} \vee e \leq (e \rightsquigarrow x) \vee (x \rightsquigarrow e)$, so $(e \rightsquigarrow x) \vee (x \rightsquigarrow e) = 1$.

(c_{30}). Denote $t = [(e \to x) \rightsquigarrow x] \wedge [(x \to e) \rightsquigarrow e]$. We have $e \leq (e \to x) \rightsquigarrow x$ and $x \leq (e \to x) \rightsquigarrow x$; it follows that $e \vee x \leq (e \to x) \rightsquigarrow x$. Analogous, $e \vee x \leq (x \to e) \rightsquigarrow e$. Hence $e \vee x \leq [(e \to x) \rightsquigarrow x] \wedge [(x \to e) \rightsquigarrow e]$, so $e \vee x \leq t$.

We have $t = 1 \odot t \overset{c_{29}}{=} [(e \to x) \vee (x \to e)] \odot t \overset{c_{9}}{=} [(e \to x) \odot t] \vee [(x \to e) \odot t]$; but $(e \to x) \odot t = (e \to x) \odot [((e \to x) \rightsquigarrow x) \wedge ((x \to e) \rightsquigarrow e)] \leq (e \to x) \odot [(e \to x) \rightsquigarrow x] \leq x$; similarly, $(x \to e) \odot t \leq e$. Hence, $t = [(e \to x) \odot t] \vee [(x \to e) \odot t] \leq x \vee e$. It follows that $e \vee x = t$.

Analogously, $e \vee x = [(e \rightsquigarrow x) \to x] \wedge [(x \rightsquigarrow e) \to e]$.

(c_{31}). From $e^- \leq e \to x$ and $x \leq e \to x \Rightarrow e^- \vee x \leq e \to x$. Also from $e \leq (e \to x) \rightsquigarrow x \leq (e \to x) \rightsquigarrow (e^- \vee x)$ and $e^- \leq (e \to x) \rightsquigarrow e^- \leq (e \to x) \rightsquigarrow (e^- \vee x)$, since $e \vee e^- = 1 \Rightarrow (e \to x) \rightsquigarrow (e^- \vee x) = 1 \Rightarrow e \to x \leq e^- \vee x \Rightarrow e \to x = e^- \vee x$. If we replace e with $e^\sim$, then $\Rightarrow e^\sim \to x = e \vee x$.

Since $e, x \leq (e \rightsquigarrow x) \to x \Rightarrow e \vee x \leq (e \rightsquigarrow x) \to x$.

Since $e^\sim \leq e \rightsquigarrow x \Rightarrow (e \rightsquigarrow x) \to x \leq e^\sim \to x = e \vee x \Rightarrow (e \rightsquigarrow x) \to x = e \vee x$.

Analogously, $e \vee x = (e \to x) \rightsquigarrow x = e^- \rightsquigarrow x$.

(c_{32}). By c_{31} we have $e \vee (x \to y) = e^\sim \to (x \to y) \overset{c_{27}}{=} (e^\sim \to x) \to (e^\sim \to y) = (e \vee x) \to (e \vee y)$ and $e \vee (x \rightsquigarrow y) = e^- \rightsquigarrow (x \rightsquigarrow y) \overset{c_{27}}{=} (e^- \rightsquigarrow x) \rightsquigarrow (e^- \rightsquigarrow y) = (e \vee x) \rightsquigarrow (e \vee y)$.

(c_{33}). We have $(e \vee x) \to (f \vee x) = (e^\sim \to x) \to (f^\sim \to x) = f^\sim \to [(e^\sim \to x) \to x] = f^\sim \to [(e^- \rightsquigarrow x) \to x] = f^\sim \to (e^- \vee x) = f^\sim \to (e^{-\sim} \to x) = f^\sim \to (e \to x) = (e \odot f^\sim) \to x = (e \wedge f^\sim) \to x$ and $(e \to f) \vee x = (e \to f)^\sim \to x = (e^{-\sim} \to f)^\sim \to x = (e^- \vee f)^\sim \to x = (e^{-\sim} \wedge f^\sim) \to x = (e \wedge f^\sim) \to x$, so $(e \vee x) \to (f \vee x) = (e \to f) \vee x$.

Analogously, $(e \vee x) \rightsquigarrow (f \vee x) = (e \rightsquigarrow f) \vee x$.

(c_{34}). $(x \to e) \to (x \to f) = [(x \to e) \odot x] \to f = (x \wedge e) \to f = (x \odot e) \to f = x \to (e \to f)$ and $(x \rightsquigarrow e) \rightsquigarrow (x \rightsquigarrow f) = [x \odot (x \rightsquigarrow e)] \rightsquigarrow f = (x \wedge e) \rightsquigarrow f = (e \odot x) \rightsquigarrow f = x \rightsquigarrow (e \rightsquigarrow f)$.

(c_{35}). We have $e \wedge (x \vee y) = e \odot (x \vee y) = (e \odot x) \vee (e \odot y) = (e \wedge x) \vee (e \wedge y)$.

(c_{36}). We have $x \wedge (e \vee f) = x \odot (e \vee f) = (x \odot e) \vee (x \odot f) = (x \wedge e) \vee (x \wedge f)$.

(c_{37}). We have $e \odot (x \rightsquigarrow f) = (e \wedge x) \odot (x \rightsquigarrow f) = [(x \to e) \odot x] \odot (x \rightsquigarrow f) = (x \to e) \odot [x \odot (x \rightsquigarrow f)] = (x \to e) \odot (x \wedge f) = (x \to e) \odot f$.$\square$

Proposition 3.8 *Let A be a bounded integral residuated lattice. For $e \in A$ we consider the following assertions:*

(i) $e \in B(A)$;

(ii) $(e \rightsquigarrow e^-) \vee (e^- \rightsquigarrow e) = (e \to e^\sim) \vee (e^\sim \to e) = 1$.

Then (i) $\Rightarrow$ (ii) but (ii) $\not\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) If $e \in B(A)$ from c_{29}, for $x = e^\sim$ respectively e^-, we deduce that $(e \rightsquigarrow e^-) \vee (e^- \rightsquigarrow e) = (e \to e^\sim) \vee (e^\sim \to e) = 1$.

(ii) $\not\Rightarrow$ (i). Consider the bounded integral residuated lattice $A = \{0, a, b, c, 1\}$ from the Example 2.1; it is easy to verify that $B(A) = \{0, 1\}$. We have $(c \to c^\sim) \vee (c^\sim \to c) = (c \to 0) \vee (0 \to c) = 1$ and $(c \rightsquigarrow c^-) \vee (c^- \rightsquigarrow c) = (c \rightsquigarrow 0) \vee (0 \rightsquigarrow c) = 1$, but $c \notin B(A)$.

We can consider another example, Example 2.2. It is easy to verify that $B(A) = \{0, 1\}$. We have $(b \to b^\sim) \vee (b^\sim \to b) = (b \to b) \vee (b \to b) = 1$ and $(b \rightsquigarrow b^-) \vee (b^- \rightsquigarrow b) = (b \rightsquigarrow b) \vee (b \rightsquigarrow b) = 1$, but $b \notin B(A)$. $\square$

Corollary 3.9 *If A is an RL-monoid, then the following assertions are equivalent:*

(i) $e \in B(A)$;

(ii) $e \in Id(A) \cap Inv(A)$ *and* $(e \rightsquigarrow e^-) \vee (e^- \rightsquigarrow e) = 1$;

(iii) $e \in Id(A) \cap Inv(A)$ *and* $(e \to e^{\sim}) \vee (e^{\sim} \to e) = 1$.

Proof. $(i) \Rightarrow (ii), (iii)$. See [16] and Proposition 3.8.

$(ii) \Rightarrow (i)$. Since $e \in Inv(A)$, from Remark 3.3, we deduce that $e^{\sim} \to e = e^{-} \rightsquigarrow e$. Since $e \in Id(A)$, from Proposition 3.6, we deduce that $e^{\sim} = e^{-}$ and $e \to e^{-} = e \to e^{\sim} = e \rightsquigarrow e^{\sim} = e \rightsquigarrow e^{-} = e^{-} = e^{\sim}$.

Hence, $e \wedge e^{-} = (e \to e^{-}) \odot e = e^{-} \odot e = 0$. Since $e \wedge e^{-} = 0$ then $e^{-} \odot (e^{-} \rightsquigarrow e) = 0$, so we obtain that $e^{-} \rightsquigarrow e \leq e^{-\sim} = e$. But, $e \leq e^{-} \rightsquigarrow e$, so $e^{-} \rightsquigarrow e = e$.

We deduce that $e^{-} \rightsquigarrow e = e^{\sim} \to e = e$.

We have $e \vee e^{-} = (e^{-} \rightsquigarrow e) \vee (e \rightsquigarrow e^{-}) = 1$. From Proposition 3.2, we deduce that $e \in B(A)$.

$(iii) \Rightarrow (i)$. The same proof with $(ii) \Rightarrow (i)$, because $e^{\sim} = e^{-}$. $\square$

Definition 3.10 *Let A be a bounded integral residuated lattice. We say that an element $x \in A$ is* regular *if for every $y \in A$ we have $(x \to y) \rightsquigarrow x = (x \rightsquigarrow y) \to x = x$. We denote by $R(A)$ the set of all regular elements of A. We say that an element $x \in A$ is* dense *if for every $r \in R(A)$ we have $x \to r = x \rightsquigarrow r = r$. We denote by $D(A)$ the set of all dense elements of A.*

We give new characterizations for regular and dense elements.

Theorem 3.11 *Let A be a bounded integral residuated lattice. For $x \in A$ the following assertions are equivalent:*

(i) $x \in R(A)$;

(ii) $x^{-} \rightsquigarrow x = x^{\sim} \to x = x$;

(iii) $x = x^{-\sim} = x^{\sim -}$ *and* $x^{-} \odot (x^{-} \rightsquigarrow x) = (x^{\sim} \to x) \odot x^{\sim} = 0$.

Proof. $(i) \Rightarrow (ii)$. If $x \in R(A)$, then $(x \to y) \rightsquigarrow x = (x \rightsquigarrow y) \to x = x$, for every $y \in A$, so for $y = 0$ we obtain $x^{-} \rightsquigarrow x = x^{\sim} \to x = x$.

$(ii) \Rightarrow (i)$. Suppose that $x^{-} \rightsquigarrow x = x$ and let $y \in A$. Then $0 \leq y \Rightarrow x \to 0 \leq x \to y \Rightarrow x^{-} \leq x \to y \Rightarrow (x \to y) \rightsquigarrow x \leq x^{-} \rightsquigarrow x = x$.

Since $x \leq (x \to y) \rightsquigarrow x$ we deduce that $(x \to y) \rightsquigarrow x = x$.

Since $x^{\sim} \to x = x$, then $x^{\sim} \leq x \rightsquigarrow y$ implies $(x \rightsquigarrow y) \to x \leq x^{\sim} \to x = x$, so, $(x \rightsquigarrow y) \to x = x$. We deduce that $x \in R(A)$.

$(ii) \Rightarrow (iii)$. Let $x \in A$ such that $x^{-} \rightsquigarrow x = x^{\sim} \to x = x$. Then $x^{-} \odot (x^{-} \rightsquigarrow x) = x^{-} \odot x = 0$ and $(x^{\sim} \to x) \odot x^{\sim} = x \odot x^{\sim} = 0$.

To prove that $x = x^{-\sim} = x^{\sim -}$ we use c_{17} and the relation $x^{\sim -} \leq x^{\sim} \to x = x$ and $x^{-\sim} \leq x^{-} \rightsquigarrow x = x$.

$(iii) \Rightarrow (ii)$. Since $x^{-} \odot (x^{-} \rightsquigarrow x) = 0$ we deduce that $x^{-} \rightsquigarrow x \leq x^{-\sim} = x$. But, $x \leq x^{-} \rightsquigarrow x$, so $x^{-} \rightsquigarrow x = x$. Since $(x^{\sim} \to x) \odot x^{\sim} = 0$ we deduce that $x^{\sim} \to x \leq x^{\sim -} = x$, hence $x^{\sim} \to x = x$. $\square$

Proposition 3.12 *Let A be a bounded integral residuated lattice. For $e \in A$ we consider the following assertions:*

(i) $e \in B(A)$;

(ii) $e \in Id(A) \cap Inv(A)$;

(iii) $e \in Id(A) \cap R(A)$.
Then $(i) \Rightarrow (ii), (iii)$ *but* $(ii) \nRightarrow (i), (iii) \nRightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$. Follows from [16].
$(i) \Rightarrow (iii)$. Follows from [16] and c_{25}.
Consider the bounded integral residuated lattice $A = \{0, a, b, c, 1\}$ from the Example 2.1; it is easy to verify that $B(A) = \{0, 1\}$.
$(ii) \nRightarrow (i)$. We have $a^2 = a, a^- = a^\sim = b, b^- = b^\sim = a$, hence $a = a^{\sim -} = a^{-\sim}$, but $a \notin B(A)$.
$(iii) \nRightarrow (i)$. We have $a^2 = a$ and $a^- \rightsquigarrow a = a^\sim \rightarrow a = a$, but $a \notin B(A)$. $\square$

Remark 3.4 From Proposition 3.12 and Theorem 3.11 we deduce that $B(A) \subsetneqq Id(A) \cap Inv(A) \cap R(A)$.

Lemma 3.13 *If* $x, y \in R(A)$, *then* $x^{-\sim}, x^{\sim -} \in R(A)$ *and* $x \wedge y \in R(A)$.

Proof. Let $x \in R(A)$; by Theorem 3.11, $x^{-\sim} = x^{\sim -} = x$, so $x^{-\sim}, x^{\sim -} \in R(A)$.
From $x \wedge y \leq x \Rightarrow x^- \leq (x \wedge y)^- \Rightarrow (x \wedge y)^- \rightsquigarrow x \leq x^- \rightsquigarrow x = x \Rightarrow (x \wedge y)^- \rightsquigarrow x = x$. Analogously we deduce that $(x \wedge y)^- \rightsquigarrow y = y$. Then $(x \wedge y)^- \rightsquigarrow (x \wedge y) = [(x \wedge y)^- \rightsquigarrow x] \wedge [(x \wedge y)^- \rightsquigarrow y] = x \wedge y$.
Analogously, $(x \wedge y)^\sim \rightarrow (x \wedge y) = (x \wedge y)$, so $x \wedge y \in R(A)$. $\square$

Remark 3.5 If A is of *Gödel type*, then for every $x \in A$ we deduce that $x^* \overset{not}{=} x^- = x^\sim \in R(A)$. Indeed, x^* verifies the conditions (iii) from the Theorem 3.11 because $x^* = (x^*)^{**}$ and $(x^*)^* \odot [(x^*)^* \rightarrow x^*] = x^{**} \odot (x^{**} \rightarrow x^*) = x^{**} \odot x^* = 0$.

Definition 3.14 *Let A be a bounded integral residuated lattice. Using the model of [14], we say that an element $x \in A$ is* strong *if $x^- = x^\sim$. We denote by $S(A)$ the set of all strong elements of A. Clearly, if A is a commutative bounded integral residuated lattice, then $S(A) = A$.*

Remark 3.6 $S(A) \nsubseteq Inv(A)$ and $Inv(A) \nsubseteq S(A)$. Obviously, if consider the bounded integral residuated lattice $A = \{0, a, b, c, 1\}$ from Example 2.1, then $S(A) = A$ and $Inv(A) = \{0, a, b, 1\}$, so $S(A) \nsubseteq Inv(A)$. In [14], page 435, there are examples of pseudo MV-algebras with elements which are not strong, so $Inv(A) \nsubseteq S(A)$.

Open problem. Find another characterization for strong elements in a bounded integral residuated lattice.

Lemma 3.15 *Let A be a bounded integral residuated lattice and $x \in S(A)$. Then:*
(i) $x^n \in S(A)$, *for every* $n \geq 1$;
(ii) $x \rightarrow x^- = x \rightarrow x^\sim = x \rightsquigarrow x^- = x \rightsquigarrow x^\sim$;
(iii) $x \leq x^=, x \leq x^{\approx}$.

Proof. (i). We prove the result by induction: $(x^{n+1})^- = x^n \to x^- = x^n \to x^\sim \overset{c_8}{=} x \rightsquigarrow (x^n \to 0) = x \rightsquigarrow (x^n)^- = x \rightsquigarrow (x^n)^\sim = (x^{n+1})^\sim$.

(ii). $x \to x^- = (x^2)^- = (x^2)^\sim = x \rightsquigarrow x^\sim = x \rightsquigarrow x^- \overset{c_8}{=} x \to x^\sim$.

(iii). Clearly, using c_{14}.

We give a new characterization for boolean elements:

Theorem 3.16 *Let A be a bounded integral residuated lattice. For $x \in A$ the following assertions are equivalent:*

(i) $x \in B(A)$;

(ii) $x \in R(A) \cap Id(A) \cap S(A)$ *and* $(x \rightsquigarrow x^-) \vee (x^- \rightsquigarrow x) = (x \to x^\sim) \vee (x^\sim \to x) = 1$.

Proof. $(i) \Rightarrow (ii)$. If $x \in B(A)$, then by Proposition 3.12 and Remark 3.1, $x \in R(A) \cap Id(A) \cap S(A)$. From Proposition 3.8, we have that $(x \rightsquigarrow x^-) \vee (x^- \rightsquigarrow x) = (x \to x^\sim) \vee (x^\sim \to x) = 1$.

$(ii) \Rightarrow (i)$. Since $x \odot x = x$ and $x^- = x^\sim$ then $x \to x^- = x \to x^\sim = x \rightsquigarrow x^- = x \rightsquigarrow x^\sim = x^- = x^\sim$. Since $x \in R(A)$, then by Theorem 3.11, (ii), $x^\sim \to x = x^- \rightsquigarrow x = x$. Then $x \vee x^- = (x^- \rightsquigarrow x) \vee (x \rightsquigarrow x^-) = 1$ and $x \vee x^\sim = (x^\sim \to x) \vee (x \to x^\sim) = 1$. Also, we have $0 = (x \vee x^\sim)^- = x^- \wedge x^{\sim -}$. Because $x \leq x^{\sim -}$ we deduce that $x^- \wedge x = 0$. Using Proposition 3.2, we conclude that $x \in B(A)$. $\square$

Corollary 3.17 *If A is a pseudo $MTL-$ algebra, then $B(A) = R(A) \cap Id(A) \cap S(A)$.*

From Theorems 3.11 and 3.16 we deduce that:

Corollary 3.18 *If A is a bounded integral residuated lattice, then $B(A) \subsetneqq R(A) \subsetneqq Inv(A)$.*

Proof. By Theorem 3.16, $B(A) \subseteq R(A)$.

To prove that $B(A) \neq R(A)$ we consider the bounded integral residuated lattice $A = \{0, a, b, c, 1\}$ from Example 2.1; it is easy to verify that $B(A) = \{0, 1\}$. We have $a^- = a^\sim = b, b^- = b^\sim = a$, hence $a^{-\sim} = a^{\sim -} = a$ and $a^- \odot (a^- \rightsquigarrow a) = (a^\sim \to a) \odot a^\sim = b \odot (b \to a) = b \odot a = 0$, hence $a \in R(A)$ but $a \notin B(A)$. We concluding that $B(A) \subsetneqq R(A)$.

By Theorem 3.11, $R(A) \subseteq Inv(A)$. To prove that $R(A) \neq Inv(A)$, we consider the bounded integral residuated lattice $A = \{0, a, b, c, 1\}$ from Example 2.2. We have $b^{-\sim} = b^{\sim -} = b$, so $b \in Inv(A)$, but $b^- \rightsquigarrow b = b^\sim \to b = 1 \neq b$, so $b \notin R(A)$. We concluding that $R(A) \subsetneqq Inv(A)$. $\square$

We characterize the bounded integral residuated lattices which are Boolean algebras:

Theorem 3.19 *For a bounded integral residuated lattice A the following assertions are equivalent :*

(i) *A is a Boolean algebra relative to the natural ordering*;

(ii) A is of Gödel type and $x^{-\sim} = x^{\sim -} = x$, *for every* $x \in A$.

Proof. $(i) \Rightarrow (ii)$. If A is a Boolean algebra relative to the natural ordering, then A becomes a residuated lattice and $x \odot x = x \wedge x = x$, $x^{-\sim} = x^{\sim -} = x$, for every $x \in A$.

$(ii) \Rightarrow (i)$. Let A be of *Gödel type* such that $x^{-\sim} = x^{\sim -} = x$, for every $x \in A$. Then $x \odot y = x \wedge y$ for every $x, y \in A$. We deduce that A is commutative and $\rightarrow$ coincides with $\rightsquigarrow$. We denote $x^{-} = x^{\sim} = x^{*}$.

First we shall prove that $x \vee y = x^{*} \rightarrow y$ for every $x, y \in A$.

Indeed, $x, y \leq x^{*} \rightarrow y$. Let $t \in A$ such that $x, y \leq t$. From $x \leq t$ we deduce that $t^{*} \leq x^{*}$, hence $x^{*} \rightarrow y \leq t^{*} \rightarrow y \leq t^{*} \rightarrow t = t^{*} \rightarrow t^{**} = (t^{*} \odot t^{*}) \rightarrow 0 = t^{*} \rightarrow 0 = t^{**} = t$.

Following Proposition 3.2, to prove that A is a Boolean algebra it will suffice to prove that $x \vee x^{*} = 1$, for every $x \in A$.

Obviously, $x \vee x^{*} = x^{*} \rightarrow x^{*} = 1$. $\square$

Theorem 3.20 *Let A be a bounded integral residuated lattice. For* $x \in A$ *the following assertions are equivalent:*
(i) $x \in D(A)$;
(ii) $x^{-} = x^{\sim} = 0$;
(iii) $(a \odot x)^{-} = a^{-}$ *and* $(x \odot a)^{\sim} = a^{\sim}$, *for every* $a \in A$.

Proof. $(i) \Rightarrow (ii)$. Since $(0 \rightarrow y) \rightsquigarrow 0 = 1 \rightsquigarrow 0 = 0$ and $(0 \rightsquigarrow y) \rightarrow 0 = 1 \rightarrow 0 = 0$ for every $y \in A$, we deduce that $0 \in R(A)$. Let $x \in D(A)$; since $0 \in R(A)$, we obtain $x \rightarrow 0 = x \rightsquigarrow 0 = 0$, hence $x^{-} = x^{\sim} = 0$.

$(ii) \Rightarrow (i)$. Let now $x \in A$ such that $x^{-} = x^{\sim} = 0$ and $r \in R(A)$ (hence $r^{-\sim} = r^{\sim -} = r$, by Theorem 3.11). Then $x \rightarrow r = x \rightarrow r^{-\sim} \overset{c_{12}}{=} r^{-} \rightsquigarrow x^{-} = r^{-} \rightsquigarrow 0 = r^{-\sim} = r$ and $x \rightsquigarrow r = x \rightsquigarrow r^{\sim -} \overset{c_{12}}{=} r^{\sim} \rightarrow x^{\sim} = r^{\sim} \rightarrow 0 = r^{\sim -} = r$, hence $x \rightarrow r = x \rightsquigarrow r = r$, so, $x \in D(A)$.

$(ii) \Rightarrow (iii)$. From c_{22}, $(a \odot x)^{-} = a \rightarrow x^{-} = a \rightarrow 0 = a^{-}$ and $(x \odot a)^{\sim} = a \rightsquigarrow x^{\sim} = a \rightsquigarrow 0 = a^{\sim}$.

$(iii) \Rightarrow (ii)$. For $a = 1$, we obtain $x^{-} = (1 \odot x)^{-} = 1^{-} = 0$ and $x^{\sim} = (x \odot 1)^{\sim} = 1^{\sim} = 0$, so $x^{-} = x^{\sim} = 0$. $\square$

Example 3.1 If we consider the bounded integral residuated lattice A from Example 2.2, it is easy to verify that $D(A) = \{1, c\}$.

Remark 3.7
1. $D(A) \subseteq S(A)$;
2. $D(A) \cap B(A) = D(A) \cap R(A) = D(A) \cap Inv(A) = \{1\}$;
3. If $x \in D(A)$, then $(x^{n})^{-} = (x^{n})^{\sim} = 0$, so, $x^{n} \in D(A)$, for every $n \geq 1$.
4. $D(A)$ is a filter (see [5], [17]) of A. Indeed, $1 \in D(A)$, because $1^{-} = 1^{\sim} = 0$. If $x \in D(A)$ and $y \in A$ such that $x \leq y$, then $y^{-} \leq x^{-}$ and $y^{\sim} \leq x^{\sim}$, so $y^{-} = y^{\sim} = 0$. We deduce that $y \in D(A)$. If $x, y \in D(A)$. We have $(x \odot y)^{-} \overset{c_{22}}{=} x \rightarrow y^{-} = x \rightarrow 0 = x^{-} = 0$ and $(x \odot y)^{\sim} \overset{c_{22}}{=} y \rightsquigarrow x^{\sim} = y \rightsquigarrow 0 = y^{\sim} = 0$, so $(x \odot y)^{-} = (x \odot y)^{\sim} = 0$. We deduce that $x \odot y \in D(A)$.

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