An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 1

On (k,n)-closed submodules

Ece Yetkin Celikel

Abstract Let R be a commutative ring with $1 \neq 0$ and M an R-module. We will call a proper submodule N of M as a semi n-absorbing submodule of M if whenever $r \in R$, $m \in M$ with $r^n m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1}m \in N$. We will say N to be a (k,n)-closed submodule of M if whenever $r \in R$, $m \in M$ with $r^k m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1}m \in N$. In this paper we introduce semi n-absorbing and (k,n)-closed submodules of modules over commutative rings, and investigate their basic properties.

Keywords (m,n)-closed ideal · n-absorbing submodule · semi n-absorbing submodule · semi n-absorbing ideal · (k,n)-closed submodule

Mathematics Subject Classification (2010) 06F10 · 06F05 · 13A15

1 Introduction

Let R be a commutative ring with $1 \neq 0$ and I be a proper ideal of R. As stated in [3], I is called an n-absorbing (resp. strongly n-absorbing) ideal of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, ..., x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, ..., I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. Recall that a proper ideal I of R is said to be semi-prime ideal if whenever $r^2 \in I$ for some $r \in R$, then $r \in I$. For generalizations of semi-prime ideals the reader may consult [9]. In [4], D. F. Anderson and A. Badawi said I to be a semi n-absorbing ideal if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. Also A. Badawi said that a proper ideal I of R is a (m,n)-closed ideal if $x^m \in I$ for $x \in R$ implies that $x^n \in I$ [4]. Let M be an R-module. A proper submodule N of M is called n-absorbing (resp. strongly n-absorbing) submodule of M if whenever $a_1 \cdots a_n m \in N$ for $a_1, ..., a_n \in R$ and $m \in M$ (resp. $I_1 \cdots I_n L \subseteq I$ for ideals $I_1, ..., I_n$ of R and a submodule L of M), then either $a_1 \cdots a_n \in (N :_R M)$ (resp. $I_1 \cdots I_n \subseteq (N :_R M)$) or there are $n-1$ of a_i's (I_i's) whose product with m (resp. M) is in N [6]. A proper submodule N of an R-module M is called semi-prime if whenever $r \in R$ and $m \in M$ with $r^2 m \in N$, then $rm \in N$. A proper submodule N of M is called a quasi-prime submodule of M if whenever

$a,b \in R$, $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$. More generally, we define (k,n)-closed submodules of an R-module M as following: let R be a commutative ring with identity and k, n be positive integers. We call a proper submodule N of M as a (k,n)-closed submodule of M if whenever $r \in R$, $m \in M$ with $r^k m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1}m \in N$. In particular, we call N as a semi n-absorbing submodule of M if whenever $r \in R$, $m \in M$ with $r^n m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1}m \in N$. It is clear that a semi n-absorbing submodule is (n,n)-closed.

Throughout we assume that all rings are commutative with $1 \neq 0$, all modules are considered to be unitary and k,n are positive integers. The radical of an ideal I of R is denoted by $\sqrt{I}$. We denote the set of invertible (unit) elements of R by $U(R)$, i.e. $U(R) = \{u \in R : \text{there is a } v \in R \text{ such that } uv = vu = 1_R\}$. Let N be a submodule of an R-module M. We will denote by $(N :_R M)$ *the residual of* N *by* M, that is, the set of all $r \in R$ such that $rM \subseteq N$. An R-module M is called a *multiplication module* if every submodule N of M has the form of IM for some ideal I of R. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$ [7]. For a submodule N of M, if $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $a,b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [1]). It is well-known that if R is a commutative ring and M a non-zero multiplication R-module, then every proper submodule of M is contained in a maximal submodule of M. [7, Theorem 2.5]. As a generalization of Jacobson radical of R, the radical of the module M is defined by the intersection of all maximal submodules of M, that is $Rad(M) = \cap\{N : N \text{ is a maximal submodule of } M\}$. Let N be a proper submodule of a non-zero R-module M. Then the M-radical of N denoted by M-rad(N) is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say M-rad$(N) = M$.

In this study, we give many properties of (k,n)-closed submodules and also obtain relationships among semi n-absorbing submodules, (k,n)-closed submodules and the other concepts. For general background and terminology, the reader may consult [2] and [10].

2 Properties of (k,n)-closed submodules

In this section, we introduce and study basic properties of semi n-absorbing and (k,n)-closed submodules with many examples.

Lemma 2.1 *Let N be a proper submodule of an R-module M. Then the following statements are equivalent:*

1. N is a (k,n)-closed submodule of M.

2. If whenever $r \in R$ and L is a submodule of M with $r^k L \subseteq N$, then $r^{n-1}L \subseteq N$ or $r^n \in (N :_R M)$.

In particular, a proper submodule N of M is a semi n-absorbing submodule of M if and only if whenever $r \in R$, L a submodule of M with $r^n L \subseteq N$ implies either $r^n \in (N :_R M)$ or $r^{n-1}L \subseteq N$.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is a (k,n)-closed submodule of M. Let $r \in R$ and L be a submodule of M with $r^k L \subseteq N$. Assume that $r^{n-1}L \not\subseteq N$. So $r^{n-1}m \notin N$ for some $m \in L$. Since $r^k m \in N$ and $r^{n-1}m \notin N$, we conclude $r^n \in (N :_R M)$, as needed.

(2) $\Longrightarrow$ (1) This part is clear. □

There are some relationships between (k,n)-closed submodules of M and (k,n)-closed ideals of R.

Theorem 2.2 *Let M be an R-module, and N be a proper submodule of M. If N is a (k,n)-closed submodule of M, then $(N :_R M)$ is a (k,n)-closed ideal of R. If M is a multiplication R-module, then the presentation ideal of a (k,n)-closed submodule of M is a (k,n)-closed ideal of R.*

Proof. Assume that $r \in R$ with $r^k \in (N :_R M)$ but $r^n \notin (N :_R M)$. Then there is an element $m \in M$ with $r^n m \notin N$ which means that $r^{n-1}m \notin N$. Since $r^k m \in N$, $r^{n-1}m \notin N$ and $r^n \notin (N :_R M)$, this situation contradicts with our hypothesis. Thus $(N :_R M)$ is a (k,n)-closed ideal of R. □

However, the converse of Theorem 2.2 is not true in general. For example, consider $N = 6\mathbb{Z}$ as a submodule of $\mathbb{Z}$-module $\mathbb{Z}$. While $(N :_{\mathbb{Z}} \mathbb{Z}) = 6\mathbb{Z}$ is clearly a $(2,1)$-closed ideal of $\mathbb{Z}$, N is not $(2,1)$-closed submodule of $\mathbb{Z}$. In fact $2^2 \cdot 3^2 \in N$ but $2^1 \notin (N :_{\mathbb{Z}} \mathbb{Z}) = 6\mathbb{Z}$ and $2^0 \cdot 3^2 \notin N$.

Theorem 2.3 *Let N be a proper submodule of the R-module M.*

1. If N is a (k,n)-closed submodule of M, then $(N :_R m)$ is a (k,n)-closed ideal of R for each $m \in M\backslash N$.
2. If $(N :_R m)$ is a (k,n)-closed ideal of R for each $m \in M\backslash N$, then N is a $(k,n+1)$-closed submodule of M.

Proof. (1) Suppose that $r^k \in (N :_R m)$ and $r^n \notin (N :_R m)$ for some $m \in M\backslash N$. Hence $r^k m \in N$ but $r^n m \notin N$ which means $r^{n-1}m \notin N$. Since N is a (k,n)-closed submodule of M, we have $r^n \in (N :_R M) \subseteq (N :_R m)$, a contradiction. Thus $(N :_R m)$ is a (k,n)-closed ideal of R for each $m \in M\backslash N$.

(2) Let $r^k m \in N$ for $r \in R$ and $m \in M$. Assume that $r^{n+1} \notin (N :_R M)$. Since $r^k \in (N :_R m)$ and $(N :_R m)$ is a (k,n)-closed ideal of R for each $m \in M\backslash N$, we conclude that $r^n \in (N :_R m)$. Therefore $r^n m \in N$. This means that N is a $(k,n+1)$-closed submodule of M. □

Lemma 2.4 *Let M be a finitely generated R-module such that $M = Rm_1 + ... + Rm_t$, N be a proper submodule of M and $k > n$. Then*

1. If $(N :_R m_i)$ is a (k,n)-closed ideal of R for all $i = 1,...,t$, then $(N :_R M)$ is a (k,n)-closed ideal of R. In particular, if $M = Rm$ be a cyclic R-module and N is a proper submodule of M, then $(N :_R m)$ is a (k,n)-closed ideal of R if and only if $(N :_R M)$ is a (k,n)-closed ideal of R.
2. Let R be a division ring and $M = Rm$ be a cyclic R-module. Then $(N :_R m)$ is a (k,n)-closed ideal of R if and only if $(N :_R m')$ is a (k,n)-closed ideal of R for all elements $m' \in M$.

Proof. (1) Assume that $(N :_R m_i)$ is a (k,n)-closed ideal of R for all $i = 1,...,t$. Suppose that $r^k \in (N :_R M)$ and $r^n \notin (N :_R M)$ for some $r \in R$. Then $r^n \notin (N :_R m_j)$ for some $j = 1,...,t$. Hence $r^k \notin (N :_R m_j)$, and so $r^k \notin (N :_R M)$ which contradicts with our assumption. Thus $(N :_R M)$ is a (k,n)-closed ideal of R. The "in particular" part is clear.

(2) Suppose that R is a division ring and $M = Rm$ is a cyclic R-module. Then one can easily obtain that $(N :_R m) = (N :_R m')$, so we are done. □

Theorem 2.5 *Let R be a division ring and N be a proper submodule of a cyclic R-module $M = Rm$.*

1. If $(N :_R m)$ is a (k,n)-closed ideal of R, then N is a $(k,n+1)$-closed submodule of M.
2. If $(N :_R m)$ is a semi n-absorbing ideal of R, then N is a semi $(n+1)$-absorbing submodule of M.

Proof. (1) From Theorem 2.3 and Lemma 2.4 (2), we are done.

(2) Since a semi n-absorbing ideal of R is a $(n+1,n)$-closed ideal, N is a $(n+1,n+1)$-closed submodule of M by (1), so it is clear. □

In Theorem 2.5, the condition "division ring" on R is necessary. Otherwise, if $(N :_R m)$ is a (k,n)-closed ideal of R, then N is not need to be $(k,n+1)$-closed submodule of M as in the following example.

Example 2.1 Consider $N = 8\mathbb{Z}$ as a submodule of $\mathbb{Z}$-module $\mathbb{Z}$. Then $(N :_{\mathbb{Z}} 1) = 8\mathbb{Z}$ is $(2,1)$-closed ideal but N is not $(2,2)$-closed submodule of M. In fact $2^2 \cdot 2 \in N$ but neither $2 \cdot 2 \in N$ nor $2^2 \in (N :_{\mathbb{Z}} \mathbb{Z})$.

Proposition 2.6 *Let N be a proper submodule of an R-module M and $k > t$. Then the following statements are equivalent:*

1. N is a (k,n)-closed submodule of M.
2. $(N :_R r^k m) = (N :_R r^{n-1} m)$ or $r^n \in (N :_R M)$ for $r \in R$ and $m \in M$.

Proof. (1)⇒(2) Suppose that N is a (k,n)-closed submodule of M and $r^n \notin (N :_R M)$. Let $s \in (N :_R r^k m)$. Hence $r^k(sm) \in N$. Since N is (k,n)-closed and $r^n \notin (N :_R M)$, we get $r^{n-1}sm \in N$. It follows $s \in (N :_R r^{n-1} m)$, that is $(N :_R r^k m) \subseteq (N :_R r^{n-1} m)$. Since the inverse inclusion is always hold, this completes the proof.

(2)⇒(1) Suppose that $r \in R$, $m \in M$ with $r^k m \in N$. If $r^n \in (N :_R M)$, then we are done. So assume that $(N :_R r^k m) = (N :_R r^{n-1} m)$. Thus $r^{n-1} m \in N$, as needed. □

The relations among the concepts of semi-prime, semi-n-absorbing, quasi-prime, n absorbing submodules and (k,n)-closed submodules are provided in the following theorem.

Theorem 2.7 *Let M be an R-module and N be a proper submodule of M. Then the following statements hold:*

1. Let N be a semi-prime submodule of M. Then N is a (k,n)-closed submodule of M for all positive integers k and n. Moreover N is a semi n-absorbing submodule of M for all positive integer n.
2. If N is an n-absorbing submodule of M, then N is a semi n-absorbing submodule of M .
3. If N is an n-absorbing submodule of M, then N is a (k,n)-closed submodule of M for every positive integer k.
4. If N is a (k,n)-closed submodule of M, then N is a (k_1,n_1)-closed submodule of M for all $k_1 \leq k$ and $n_1 \geq n$.
5. If N is a semi n-absorbing submodule of M, then N is a semi n_1-absorbing submodule of M for all $n_1 \geq n$.
6. If N is a quasi-prime submodule of M, then N is a (k,n)-closed submodule of M for all positive integers $k \geq n \geq 2$.

Proof. (1), (2), (3) and (4) are clear from the definitions.

(5) Induction method on n. For $n = 1$, it is clear. So suppose that $n \geq 2$ and N is a semi $(n-1)$-absorbing submodule of M. We show that N is semi n-absorbing. Let $r \in R$ and $m \in M$ with $r^n m \in N$. Assume that $r^n \in (N :_R M)$. Hence $r^{n-1}(rm) \in N$ which implies that $r^{n-2}(rm) = r^{n-1}m \in N$ by introduction hypothesis. Thus N is a semi n-absorbing of M for all $n_1 \geq n$.

(6) We show that N is a $(k,2)$-closed submodule of M for all $k \geq 2$ by using mathematical induction on k. Suppose that N is a quasi-prime submodule of M. Then N is a $(k,2)$-closed submodule of M for $k = 2$ directly from their definitions. Now suppose that N is a $(t,2)$-closed submodule of M for all $2 \leq t < k$ and our aim is to show that N is $(k,2)$-closed. Let $r^k m \in N$ for $r \in R$ and $m \in M$. Assume that $r^2 \notin (N :_R M)$. Since $r^{k-1}(rm) \in N$, and N is $(k-1,2)$-closed by induction hypothesis, we conclude that $r(rm) = r^2 m \in N$. Since N is $(2,2)$-closed and $r^2 \notin (N :_R M)$, we get $rm \in N$. Thus N is a $(k,2)$-closed submodule of M for all $k \geq 2$. Consequently, N is a (k,n)-closed submodule of M for all positive integers n with $k \geq n \geq 2$ by (4). □

Example 2.2 The converses of (1)-(6) in Theorem 2.7 are not true in general as these situations are shown in the following examples.

1. Let $N = 30\mathbb{Z}$ as a submodule of the $\mathbb{Z}$-module $\mathbb{Z}$. Since $N = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z}$ is intersection of semi-prime submodules of $\mathbb{Z}$, it is semi 2-absorbing ($(2,2)$-closed) submodule of $\mathbb{Z}$ from Theorem 2.10. Also it is $(3,2)$-closed submodule of $\mathbb{Z}$ from Theorem 2.7 (4). However N is not 2-absorbing submodule of $\mathbb{Z}$. In fact $2 \cdot 3 \cdot 5 \in N$ but $2 \cdot 3 \notin (N :_{\mathbb{Z}} \mathbb{Z})$ and $2 \cdot 5 \notin N$ and $3 \cdot 5 \notin N$. So the converses of (2) and (3) are not true.

2. Consider the submodule $N = \left(\overline{0}\right)$ of $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ where p is a prime and n is positive integer. Then N is a (n,n)-closed submodule of $\mathbb{Z}_{p^n}$, but N is not $(n,n-1)$-closed as $p^n\overline{1} = \overline{0} \in N$ but neither $p^{n-2}\overline{1} \in N$ nor $p^{n-1} \in (N :_{\mathbb{Z}} \mathbb{Z}_{p^n}) = (p^n)$. Note that N in $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is a semi n-absorbing submodule of $\mathbb{Z}_{p^n}$, but it is not quasi-prime as $p^n\overline{1} \in N$ but $p\overline{1} \notin N$. Also it is not semi $(n-1)$-absorbing (it is also not semi-prime clearly) submodule as $p^{n-1}p \in N$ but neither $p^{n-1} \in (N :_{\mathbb{Z}} \mathbb{Z})$ nor $p^{n-2}p = p^{n-1} \in N$. Thus the coverses of (1), (4), (5) and (6) are not true.

Theorem 2.8 *Let N be a proper submodule of M. If N is a semi n-absorbing submodule of M, then N is a (k,n)-closed submodule of M for all positive integer k.*

Proof. If $k \leq n$, the claim is clear. So suppose that $k > n$ and say $t := k-n$. Let $r^k m \in N$ for some $r \in R$ and $m \in M$. Assume that $r^n \notin (N :_R M)$. Hence $r^n(r^t m) \in N$. Since N is semi n-absorbing and $r^n \notin (N :_R M)$, we get $r^{n-1}(r^t m) = r^n(r^{t-1}m) \in N$. This follows $r^{n-1}(r^{t-1}m) = r^n(r^{t-2}) \in N$ as again N is a semi n-absorbing submodule of M. It implies that $r^n(r^{t-3}m) \in N$. So we continue with this argument and obtain that $r^n m \in N$ at the t^{th} step. Finally we conclude $r^{n-1}m \in N$ which means that N is a (k,n)-closed submodule of M. □

Corollary 2.9 *Let N be a proper submodule of M and $k > n$. Then N is a (k,n)-closed submodule of M if and only if N is a semi n-absorbing submodule of M.*

Proof. Suppose that N is (k,n)-closed and $r^n m \in N$ for $r \in R$ and $m \in M$. So $r^k m \in N$, and this implies that either $r^n \in (N :_R M)$ or $r^{n-1}m \in N$. Thus N is a semi n-absorbing submodule of M. The converse part follows from Theorem 2.8. □

Theorem 2.10 *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a family of semi-prime submodules of M. Then $\cap_{\lambda\in\Lambda} N_\lambda$ is a (k,n)-closed submodule of M for all positive integers k and n.*

Proof. Suppose that $r^k m \in \cap_{\lambda\in\Lambda} N_\lambda$ for $r \in R$ and $m \in M$. Then $r^k m \in N_\lambda$ for all $\lambda \in \Lambda$. Since each N_i is semi-prime, we conclude that $rm \in N_\lambda$ for all $\lambda \in \Lambda$. Thus $rm \in \cap_{\lambda\in\Lambda} N_\lambda$ which means that $r^{n-1}m \in \cap_{\lambda\in\Lambda} N_\lambda$ for all n. From Theorem 2.7 (4), $\cap_{\lambda\in\Lambda} N_\lambda$ is (k,n)-closed for all integers k and n. □

Corollary 2.11 *Let N be a proper submodule of an R-module M. Then $M - rad(N)$ and $Rad(M)$ are (k,n)-closed submodules of M for all integers k and n.*

Proof. The result is follows from Theorem 2.10. □

Lemma 2.12 *[5] Let R be a commutative ring, M a finitely generated multiplication R-module and $N_1,...,N_t$ are pairwise comaximal R-submodules of M. Then the following statements hold:*

1. $N_1 N_2 = N_1 \cap N_2$.
2. $N_1 \cap \cdots \cap N_{t-1}$ and N_t are comaximal.
3. $N_1 \cdots N_t = N_1 \cap \cdots \cap N_t$.

Theorem 2.13 *Let M is finitely generated multiplication R-module and $N_1,...,N_t$ be semi-prime submodules of M. If $N_1,...,N_t$ are pairwise comaximal, then $N_1...N_t$ is a (k,n)-closed submodule of M for all positive integers k and n. In particular, if N is semi-prime, then N^n is a (k,n)-closed submodule of M.*

Proof. It follows from Theorem 2.10 and Lemma 2.12. □

D.F. Anderson and A. Badawi proved in Theorem 2.3 [4] that the intersection of two semi n-absorbing ideals is also a semi n-absorbing ideal of R. However this situation is not true for submodules of any module. The intersection of two semi n-absorbing submodule may not to be semi n-absorbing as the following:

Example 2.3 Consider $\mathbb{Z}$ as $\mathbb{Z}$-module and two submodules $N = p^n\mathbb{Z}$ and $K = q^n\mathbb{Z}$ of $\mathbb{Z}$ where p and q are prime integers. Clearly both of them are semi n-absorbing submodules of $\mathbb{Z}$. However $N \cap K = p^n q^n \mathbb{Z}$ is not semi n-absorbing since $p^n(q^n) \in N \cap K$ but $p^{n-1}(q^n) \notin N \cap K$ and $p^n \notin (N \cap K :_{\mathbb{Z}} \mathbb{Z})$.

Theorem 2.14 *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a chain of (k,n)-closed submodules of an R-module M. Then $\bigcap\limits_{\lambda\in\Lambda} N_\lambda$ is a (k,n)-closed submodule of M.*

Proof. Let $r^k m \in N$ for $r \in R$ and $m \in M$. If $r^n \in (N_\lambda :_R M)$ for all $\lambda \in \Lambda$, then $r^n \in \cap(N_\lambda :_R M) = (\cap N_\lambda :_R M)$, we are done. Supose that $r^n \notin (N_{\lambda_0} :_R M)$ for some $\lambda_0 \in \Lambda$. Then $r^n \notin (N_\lambda :_R M)$ for all $N_\lambda \subseteq N_{\lambda_0}$. Hence $r^{n-1}m \in N_\lambda$ for all $N_\lambda \subseteq N_{\lambda_0}$ as each N_λ is (k,n)-closed. Therefore $r^{n-1}m \in \bigcap\limits_{\lambda\in\Lambda} N_\lambda$ which means that $\bigcap\limits_{\lambda\in\Lambda} N_\lambda$ is a (k,n)-closed submodule of M. □

Theorem 2.15 *Let N_1 and N_2 be proper submodules of an R-module M.*

1. If N_1 is a semi n_1-absorbing and N_2 is a semi n_2-absorbing submodule of M, then $N_1 \cap N_2$ is semi $(n+1)$-closed submodule of M where $n = \max\{n_1, n_2\}$.
2. If $N_1,...,N_t$ be semi n-absorbing submodules of M. Then $N_1 \cap \cdots \cap N_t$ is a semi $(n_t + t)$-absorbing submodule of M.
3. If $N_1,...,N_t$ be semi n_t-absorbing submodules of M. Then $N_1 \cap \cdots \cap N_t$ is a semi $(n_t + 2)$-absorbing submodule of M where $n = \max\{n_1,...,n_t\}$.

Proof. (1) Let $r \in R$ and $m \in M$ such that $r^{n+1}m \in N_1 \cap N_2$. First observe from Corollary 2.9 that N_1 and N_2 are (n,n_1)-closed and (n,n_2)-closed submodules of M, respectively. Hence we have $r^{n_1} \in (N_1 :_R M)$ or $r^{n_1-1}m \in N_1$ and $r^{n_2} \in (N_2 :_R M)$ or $r^{n_2-1}m \in N_2$. If $r^{n_1} \in (N_1 :_R M)$ and $r^{n_2} \in (N_2 :_R M)$, then $r^n \in (N_1 :_R M) \cap (N_2 :_R M) = (N_1 \cap N_2 :_R M)$. If $r^{n_1} \in (N_1 :_R M)$ and $r^{n_2-1}m \in N_2$, then $r^n m \in N_1 \cap N_2$. If symmetrically $r^{n_1-1}m \in N_1$, and $r^{n_2} \in (N_2 :_R M)$, then again we have $r^n m \in N_1 \cap N_2$. For the last, if $r^{n_1-1}m \in N_1$ and $r^{n_2-1}m \in N_2$, then $r^{n-1}m \in N_1 \cap N_2$. Thus we conclude either $r^{n+1} \in (N_1 \cap N_2 :_R M)$ or $r^n m \in (N_1 \cap N_2 :_R M)$, as needed.

(2) One can easily obtain the proof by using induction method on t.

(3) We use induction method on t. If $t = 3$, then the claim is clear from (1) and (2). So assume that $t > 3$ and the claim is satisfied for $t-1$. Then $N_1 \cap \cdots \cap N_{t-1}$

is semi $(n_{t-1}+2)$-absorbing. If $n_{t-1}+2<n_t$, then $N_1\cap\cdots\cap N_t$ is semi $(n_{t-1}+1)$-absorbing submodule of M by part (1). Thus $N_1\cap\cdots\cap N_t$ is semi $(n_{t-1}+2)$-absorbing submodule of M by Theorem 2.7 (5). If $n_{t-1}+2=n_t$, then $N_1\cap\cdots\cap N_t$ is semi $(n_{t-1}+2)$-absorbing submodule of M by part (2). If $n_{t-1}+2>n_t$, then $N_1\cap\cdots\cap N_t$ is $(n_{t-1}+3)$-absorbing by part (1). Here observe that $n_{t-1}+3=n_t+2$ as $n_{t-1}+2>n_t$ and $n_{t-1}<n_t$. Therefore $N_1\cap\cdots\cap N_t$ is semi (n_t+2)-absorbing submodule of M. □

Theorem 2.16 *Let R be a division ring, M a cyclic R-module, and $N_1,...,N_t$ be (k_j,n_j)-closed submodules of M. Then $N_1\cap...\cap N_t$ is a $(k,n+1)$-closed submodule of M for all integers $k\leq\min\{k_1,...,k_t\}$ and $n\geq\min\{k,\max\{n_1,...,n_t\}\}$.*

Proof. Suppose that $N_1,...,N_t$ are (k_j,n_j)-closed submodules of M. Hence $(N_1:_R M),...,(N_t:_R M)$ are (k_j,n_j)-closed ideals of R by Theorem 2.2. Then $\bigcap_{j=1}^{t}(N_j:_R M)=(\bigcap_{j=1}^{t}N_j:_R M)$ is a (k,n)-closed ideal of R for $k\leq\min\{k_1,...,k_t\}$ and $n\geq\min\{k,\max\{n_1,...,n_t\}\}$ by Theorem 2.3 in [4]. Thus we conclude that $\bigcap_{j=1}^{t}N_j$ is a $(k,n+1)$-closed submodule of M by Theorem 2.5. □

A non-zero submodule N of an R-module M is called a secondary submodule of M if for each $r\in R$ the homothety $N\xrightarrow{r}N$ is surjective or nilpotent (resp. surjective or zero). In this case $P=\sqrt{(0:_R N)}$ is a prime ideal, and we call N a P-secondary submodule of M. For more details concerning secondary submodule of a module refer to [8].

Theorem 2.17 *Let N be a secondary submodule of an R-module M. If K is a semi n-absorb-ing submodule of M, then $N\cap K$ is a secondary submodule of M.*

Proof. Suppose that N is a P-secondary submodule of M and $r\in R$. If $r\in P=\sqrt{(0:_R N)}$, then clearly $r\in\sqrt{(0:_R N\cap K)}$. So assume that $r\notin P$. Since $r^n\notin P$ for all $n\geq 0$, this implies that $r^nN=N$. It is needed to show that $r(N\cap K)=(N\cap K)$. Let $m\in N\cap K$. Since $N=r^nN$, there is an element m_1 of N such that $m=r^nm_1\in N\cap K\subseteq K$. Since K is semi n-absorbing, we conclude either $r^n\in(K:_R M)$ or $r^{n-1}m\in K$. If $r^n\in(K:_R M)$, then $N=r^nN\subseteq K$, and so $r(N\cap K)=rN=N\cap K$. If $r^{n-1}m\in K$, then $m=r^nm_1\in r(N\cap K)$, we are done. □

Corollary 2.18 *Let N and K be proper submodules of an R-module M with $K\subseteq N$. If N is a secondary semi n-absorbing submodule of M, then K is a semi n-absorbing submodule of M.*

Proof. This is a direct consequence of Theorem 2.17. □

Let N and K be submodules of M with $K\subseteq N$. If N is a semi n-absorbing submodule of M, then K is not need to be a semi n-absorbing submodule of M as the following example verifying this case. So Example 2.4 shows that the condition "secondary" in Corollary 2.18 is necessary.

Example 2.4 Consider a submodule $N = 4\mathbb{Z}$ of $\mathbb{Z}$-module $\mathbb{Z}$ and $K = 12\mathbb{Z}$. Then K is clearly a semi 2-absorbing submodule and $K \subseteq N$, but N is not semi 2-absorbing submodule of M as $2^2 \cdot 3 \in K$ but $2^2 \notin (K :_R M)$ and $2 \cdot 3 \notin K$.

Let R be an integral domain. Recall that if for every element r of its field of fractions F, at least one of r or r^{-1} belongs to R, then R is called valuation domain.

Proposition 2.19 *Let R be a valuation domain with quotient field K. Let M be an R-module and N a proper submodule of M Then N is a semi n-absorbing submodule of M if and only if whenever $r \in K$, $m \in M$ with $r^{n+1} \in N$ implies that $r^n m \in N$.*

Proof. Suppose that N is a semi n-absorbing submodule of M. Assume that $r^{n+1}m \in N$ but $r^{n+1} \notin (N :_R M)$ for some $r \in K$ and $m \in M$. If $r \in R$, then we are done. So assume that $r \notin R$. Since R is a valuation domain, $r^{-1} \in R$. Hence we get $r^{-1}(r^{n+1}m) = r^n m \in N$. The converse part is clear. □

Definition 2.20 *Let N be a proper submodule of M.*

1. N is said to be strongly semi n-absorbing submodule if whenever I is an ideal and L is a submodule of M with $I^n L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1}L \subseteq N$.
2. N is said to be strongly (k,n)-closed submodule if whenever I is an ideal and L is a submodule of M with $I^k L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1}L \subseteq N$.

Note that every strongly (k,n)-closed submodule is a (k,n)-closed submodule of M. Clearly a $(k,1)$-closed submodule is also a strongly $(k,1)$-closed submodule of M. Also observe that a strongly semi n-absorbing submodule is a n-absorbing submodule of M.

Lemma 2.21 *Let N be a proper submodule of M. Then the following statements are equivalent:*

1. N is a strongly (k,n)-closed submodule of M.
2. If I is an ideal of R and $m \in M$ with $I^k m \subseteq N$, then $I^n \subseteq (N :_R M)$ or $I^{n-1}m \subseteq N$.

Proof. (1) $\Longrightarrow$ (2) It is obvious.

(2) $\Longrightarrow$ (1) Suppose that $I^k L \subseteq N$ for an ideal I of R and a submodule L of M. Assume that $I^{n-1}L \nsubseteq N$. Then there is an element m of L such that $I^{n-1}m \nsubseteq N$ for some $m \in L$. Since $I^k m \subseteq N$, we have $I^n \subseteq (N :_R M)$ by (2). Thus N is a strongly (k,n)-closed submodule of M. □

Theorem 2.22 *Let R be a principal ideal domain and N be a proper submodule of an R-module M. Then the following are equivalent:*

1. N is a (k,n)-closed submodule of M.
2. N is a strongly (k,n)-closed submodule of M.

Proof. (1) $\Longrightarrow$ (2) Since I is principal, $I = (a)$ for some $a \in R$. So we are done by Lemma 2.21.

(2) $\Longrightarrow$ (1) It is clear. □

Proposition 2.23 *Let N be a proper submodule of an R-module M. If N is a (k,n)-closed submodule of M, then $(N :_M I) = \{m \in M : Im \subseteq N\}$ is a (k,n)-closed submodule of M for all ideal I of R. Moreover if N is a strongly (k,n)-closed submodule of M, then $(N :_M I^k) = (N :_M I^{n-1})$.*

Proof. Suppose that $r^k m \in (N :_M I)$ for $r \in R$ and $m \in M$. Hence $r^k Im \subseteq N$, which implies that either $r^n \in (N :_R M)$ or $r^{n-1}Im \subseteq N$ by Lemma 2.1. This means $r^n \in ((N :_R M) :_R I) = ((N :_M I) :_R M)$ or $r^{n-1}m \in (N :_M I)$. Thus $(N :_M I)$ is a (k,n)-closed submodule of M for all ideal I of R. Now suppose that N is a strongly (k,n)-closed submodule of M. Since $(N :_M I^{n-1}) \subseteq (N :_M I^k)$ is always true, it is sufficient to show the inverse inclusion. Let $m \in (N :_M I^k)$. Then $I^k m \in N$, and we have $I^n \subseteq (N :_R M)$ or $I^{n-1}m \in N$ from Lemma 2.21. If $I^{n-1}m \in N$, then $m \in (N :_M I^{n-1})$, so we are done. So suppose that $I^n \subseteq (N :_R M)$. Thus $I^k \subseteq (N :_R M)$, as needed. □

Theorem 2.24 *Let N be a proper submodule of M. Then the following statements are equivalent:*

1. *N is a strongly (k,n)-closed submodule of M.*
2. *For any ideal I of R and $N \subseteq L$ a submodule of M with $I^k L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1}L \subseteq N$.*

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Let K be a submodule of M and I an ideal of R such that $I^k K \subseteq N$. Hence $I^k(K+N) = I^k K + I^k N \subseteq N$. Put $L = K+N$. Since N is strongly (k,n)-closed, we conclude that either $I^n \subseteq (N :_R M)$ or $I^{n-1}L \subseteq N$ by hypothesis (2). Thus $I^n \subseteq (N :_R M)$ or $I^{n-1}K \subseteq N$. □

Theorem 2.25 *Let N be a $(k,2)$-closed submodule of M, and L a submodule of M. Then:*

1. *If $L^k M \subseteq N$, then $2L^2 \subseteq (N :_R M)$.*
2. *If $2 \in U(R)$, then N is a strongly $(k,2)$-closed submodule of M.*

Proof. (1) Suppose that $L^k M \subseteq N$. Then $l_1^k m$, $l_2^k m$, $(l_1+l_2)^k m \in N$ for all $m \in M$, for all l_1, l_2. Since N is $(k,2)$-closed, we conclude that (either $l_1^2 \in (N :_R M)$ or $l_1 m \in N$) and (either $l_2^2 \in (N :_R M)$ or $l_2 m \in N$) and (either $(l_1+l_2)^2 \in (N :_R M)$ or $(l_1+l_2)m \in N$) which means $l_1^2 m$, $l_2^2 m$, $(l_1+l_2)^2 \in N$. Then $2l_1 l_2 m = ((l_1+l_2)^2 - l_1^2 - l_2^2)m \in N$. Thus $2L^2 M \subseteq N$, and so $2L^2 \subseteq (N :_R M)$.

(2) Let $2 \in U(R)$. Since $2L^2 M \subseteq N$ from (1), we conclude that $L^2 \subseteq (N :_R M)$. □

Now we extend well-known results about prime submodules, n-absorbing submodules and (m,n)-closed ideals to (k,n)-closed submodules.

Theorem 2.26 *Let N be a proper submodule of M, and S be a multiplicatively closed subset of R such that $(N :_R M) \cap S = \emptyset$. If N is a (k,n)-closed submodule*

of M, then $S^{-1}N$ is a (k,n)-closed submodule of $S^{-1}M$. In particular, if N is a semi n-absorbing submodule of M, then $S^{-1}N$ is a semi n-absorbing submodule of $S^{-1}M$.

Proof. Let $\left(\frac{r}{s_1}\right)^k \left(\frac{m}{s_2}\right) \in S^{-1}N$. Hence $ur^k m \in N$ for some $u \in S$. Hence $(ur)^k m \in N$. Since N is (k,n)-closed, $(ur)^{n-1}m \in N$ or $(ur)^n \in (N :_R M)$ which follows either $\left(\frac{r}{s_1}\right)^{n-1} \left(\frac{m}{s_2}\right) = \frac{u^{n-1}r^{n-1}}{u^{n-1}s_1^n}\frac{m}{s_2} \in S^{-1}N$ or $\left(\frac{r}{s_1}\right)^n = \frac{u^n r^n}{u^n s_1^n} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. "In particular" part is clear as a semi n-absorbing submodule is a (n,n)-closed submodule of M. □

Corollary 2.27 *Let S be a multiplicatively closed subset of R such that $S \cap (N :_R M) = \emptyset$ with $2 \in S$. If N is a strongly $(k,2)$-closed submodule of M, then $S^{-1}N$ is a strongly $(k,2)$-closed submodule of $S^{-1}M$.*

Proof. Let $S^{-1}K$ be a submodule of $S^{-1}M$ such that $(S^{-1}K)^k(S^{-1}M) \subseteq S^{-1}N$. Since $2 \in S$, $2 \notin U(S^{-1}R)$, we are done by Theorem 2.25 (2). □

Corollary 2.28 *Let N be a proper submodule of M, and P a prime submodule of M containing N. Then N is a (k,n)-closed submoule of M if and only if N_P is a (k,n)-closed submodule of M_P.*

Proof. If N is a (k,n)-closed submodule of M, then N_P is a (k,n)-closed submodule of M_P by Theorem 2.26. Conversely suppose that $r \in R$, $m \in M$ with $r^k m \in N$. Let $\Omega = \{u \in R : ur^n m \in N\}$. Then $\left(\frac{r}{1}\right)^k \frac{m}{1} \in N_P$ implies that $\left(\frac{r}{1}\right)^{n-1} \frac{m}{1} \in N_P$ or $\left(\frac{r}{1}\right)^n \in (N_P :_{R_P} M_P)$ as N_P is (k,n)-closed. Therefore $ur^n m \in N_P$ for some $u \in R \backslash P$. Hence $\Omega \nsubseteq P$. Also $\Omega \nsubseteq P'$ where P' is any prime submodule of M with $I \nsubseteq P'$. Therefore $\Omega = R$, which means that $r^n m \in N$. Thus N is a (k,n)-closed submodule of M. □

Theorem 2.29 *Let M, M' be R-modules, and $f : M \to M'$ an R-module homomorphism.*

1. If N' is a (k,n)-closed (resp. semi n-absorbing) submodule of M', then $f^{-1}(N')$ is a (k,n)-closed (resp. semi n-absorbing) submodule of M.
2. If f is onto and N is a (k,n)-closed (resp. semi n-absorbing) submodule of M containing $Ker\, f$, then $f(N)$ is a (k,n)-closed (resp. semi n-absorbing) submodule of M'

Proof. The reader can easily obtain the proof, so it is omitted. □

Corollary 2.30 *Let M, M' be R-modules and N, K be proper submodules of M. Then the following statements hold:*

1. If $M \subseteq M'$ and N is a (k,n)-closed (resp. semi n-absorbing) submodule of M', then $N \cap M$ is a (k,n)-closed (resp. semi n-absorbing) submodule of M.

2. If $K \subseteq N$, then N/K is a (k,n)-closed (resp. semi n-absorbing) submodule of M/K if and only if K is a (k,n)-closed (resp. semi n-absorbing) submodule of M.

Theorem 2.31 *Let M_1, M_2 be R-modules with $M = M_1 \oplus M_2$, and let N_1, N_2 be proper submodules of M_1, M_2, respectively.*

1. N_1 is a (k_1,n_1)-closed submodule of M_1 if and only if $N_1 \oplus M_2$ is a (k,n)-closed submodule of $M_1 \oplus M_2$ for all positive integers $k_1 \leq k$ and $n \geq n_1$.
2. N_2 is a (k_2,n_2)-closed submodule of M_2 if and only if $M_1 \oplus N_2$ is a (k,n)-closed submodule of $M_1 \oplus M_2$ for all positive integers $k_2 \leq k$ and $n \geq n_2$.

Proof. (1) Suppose that N_1 is a (k_1,n_1)-closed submodule of M_1. Assume that $r^{k_1}(m_1,m_2) \in N_1 \oplus M_2$ but $r^{n_1-1}(m_1,m_2) \notin N_1 \oplus M_2$. Then $r^{n_1-1}m_1 \notin N_1$, which implies $r^{n_1} \in (N_1 :_R M_1)$. Thus $r^{n_1} \in (N_1 \oplus M_2 :_R M)$. Consequently, $N_1 \oplus M_2$ is a (k,n)-closed submodule of $M_1 \oplus M_2$ for all positive integers $k_1 \leq k$ and $n \geq n_1$ by Theorem 2.7 (4). The converse part can be obtained easily by using the similar argument.

(2) It can be easily verified similar to (1). □

Theorem 2.32 *Let M_1, M_2 be R-modules, N_1 be a (k_1,n_1)-closed submodule of M_1, and N_2 be a (k_2,n_2)-closed submodule of M_2. Then $N_1 \oplus N_2$ is a (k,n)-closed submodule of $M_1 \oplus M_2$ for all positive integers $k \leq \min\{k_1,k_2\}$ and $n \geq \max\{n_1, n_2\}+1$.*

Proof. Suppose that $r \in R$ and $(m_1,m_2) \in M$ such that $r^k(m_1,m_2) \in N_1 \oplus N_2$. Hence $r^k m_1 \in N_1$. Since $r^{k_1}m_1 \in N_1$ and N_1 is a (k_1,n_1)-closed submodule of M_1, we have $r^{n_1}m_1 \in N_1$. Similarly, since $r^{k_2}m_2 \in N_2$ and N_2 is a (k_2,n_2)-closed submodule of M_2, we get $r^{n_2}m_2 \in N_2$. Thus $r^n m_1 \in N_1$ and $r^n m_2 \in N_2$ for all $n \geq \max\{n_1,n_2\}$. Therefore $r^n(m_1,m_2) \in N_1 \oplus N_2$, as needed. □

D.F. Anderson and A. Badawi determined in [4] when the powers of principal prime ideal or the ideals of the form $p_1^{t_1}...p_i^{t_i}$ where $p_1,...,p_t$ are non associate prime elements of R and $t_1,...,t_n$ are positive integers are (m,n)-closed ideal of an integral domain R. Analogous to them, we may conclude many results for submodules of multiplication modules over commutative rings. Some of them are presented as the following.

Theorem 2.33 *Let R be an integral domain and M a multiplication R-module. Let $(N :_R M) = p^t R$ where p is prime element of R and $k > 0$. If N is a (k,n)-closed submodule of M, then the following statements are hold:*

1. $t = ka + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(k \bmod n) + r \leq n$, and if $a \neq 0$, then $k = n + c$ for an integer c with $1 \leq c \leq n-1$.
2. If $k = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $t \in \{1,...,n\}$. If $k = n + c$ for an integer c with $0 \leq c \leq n-1$, then $t \in \cup_{h=1}^{n}\{ki + h : i \in \mathbb{Z}$ and $0 \leq ic \leq n-h\}$.

Proof. Suppose that N is a (k,n)-closed submodule of M. Then $(N :_R M)$ is a (k,n)-closed ideal of M by Theorem 2.2. So we are done from Theorem 3.1 in [4]. □

Corollary 2.34 *Let M be a multiplication R-module where R is an integral domain, and $(N :_R M) = p^t R$ where p is prime element of R, $t > 0$. If N is a semi n-absorbing submodule of M, then $t = na + r$, where a and r are integers such that $a \geq 0$, $1 \leq r < n$; that is $t \in \cup_{h=1}^{n} \{ni + h : i \in \mathbb{Z}$ and $0 \leq i < n - h\}$.*

Proof. Since a semi n-absorbing submodule is a (n,n)-closed submodule of M, the resut is clear by Theorem 2.33. □

Corollary 2.35 *Let R be an integral domain and $(N :_R M) = p^t R$ where p is a prime element of R and t is a positive integer. Then N is a semi 2-absorbing submodule of M, then $t \in \{1,2\}$.*

Consider a $\mathbb{Z}$-module $M = \mathbb{Z}$ and a submodule $N = 2^3\mathbb{Z}$ of M. It is shown in Example 2.1 that N is not a semi 2-absorbing submodule of M for $t = 3$.

Theorem 2.36 *Let R be a principal ideal domain, N a proper submodule of a multiplication R-module M and k, n be integers with $1 \leq n \leq k$. If N is a (strongly) (k,n)-closed submodule of M, then $N = P_1^{t_1} \cdots P_i^{t_i}$ where $P_1, ..., P_i$ are nonassociate prime submodules of M, $t_1, ..., t_i$ are positive integers, and one of the following two conditions holds:*

1. If $k = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $t_j \in \{1, ..., n\}$ for every $1 \leq j \leq i$.
2. If $k = n + c$ for an integer c with $0 \leq c \leq n-1$, then $t_j \in \cup_{h=1}^{n} \{kv + h : v \in \mathbb{Z}$ and $0 \leq vc \leq n - h\}$ for every $1 \leq j \leq i$.

Proof. Suppose that N is (k,n)-closed submodule of M. Then $(N :_R M)$ is a (k,n)-closed ideal of R by Theorem 2.2. Hence $(N :_R M) = p_1^{t_1} \cdots p_i^{t_i} R$ for some nonassociate prime elements of R, $t_1, ..., t_i$ are positive integers, and the conditions (1) or (2) is satisfied for k and n by [4]. Thus $N = p_1^{t_1} \cdots p_i^{t_i} M$. Put $P_i^{t_i} = p_i^{t_i} M$ for all $i = 1, ..., t$, so $N = P_1^{t_1} \cdots P_i^{t_i}$, we are done. □

Theorem 2.37 *Let N be a proper submodule of a multiplication R-module M where R is an integral domain and k, n be integers with $1 \leq n \leq k$. Suppose that $N = P^t$, where P is a prime submodule of M and t is a positive integer. If N is a (k,n)-closed submodule of M, then one of the following statements holds:*

1. $1 \leq t \leq n$.
2. There is a positive integer a such that $t = ka + r = na + d$ for an integer r and d with $1 \leq r, d \leq n-1$.
3. There is a positive integer a such that $t = ka + r = n(a+1)$ for an integer r with $1 \leq r \leq n-1$.

Proof. From Theorem 2.2 and Theorem 3.8 in [4], the result is clear. □

References

1. AMERI, R. – *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci., **27** (2003), 1715-1724.
2. ANDERSON, D.D. – *Abstract commutative ideal theory without chain condition*, Algebra Universalis, **6** (1976), no. 2, 131-145.
3. ANDERSON, D.F.; BADAWI, A. – *On n-absorbing ideals of commutative rings*, Comm. Algebra, **39** (2011), no. 5, 1646-1672.
4. ANDERSON, D.F.; BADAWI, A. – *On (m,n)-closed ideals of commutative rings*, Journal of Algebra and Its Applications, **16** (2017), 21 pages.
5. ATANI, S.E.; DARANI, A.Y. – *On quasi-primary submodules*, Chiang Mai J. Sci., **33** (2006), no. 3, 249-254.
6. DARANI, A.Y.; SOHEILNIA, F. – *On n-absorbing submodules*, Math. Commun., **17** (2012), no. 2, 547-557.
7. EL-BAST, Z.A.; SMITH, P.F. – *Multiplication modules*, Comm. Algebra, **16** (1988), no. 4, 755-779.
8. MACDONALD, I.G. – *Secondary representation of modules over a commutative ring*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 23-43, Academic Press, London, (1973).
9. MOSTAFANASAB, H.; DARANI, A.Y. – *On n-absorbing ideals and two generalizations of semiprime ideals*, Thai J. Math., **15** (2017), no. 2, 387-408.
10. SMITH, P.F. – *Some remarks on multiplication modules*, Arch. Math. (Basel), **50** (1988), no. 3, 223-235.

Received: 16.V.2016 / Accepted: 22.XI.2016

AUTHOR

ECE YETKIN CELIKEL,
Department of Mathematics,
Faculty of Art and Science,
Gaziantep University,
Gaziantep, Turkey,
E-mail: yetkinece@gmail.com