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Positive periodic solutions for an iterative differential equation related to a discrete derivatives sequence

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Abstract In this paper, we use Krasnoselskii's fixed point theorem to study the existence and uniqueness of periodic solutions of a class of iterative differential equation

$$f'(t) = \frac{1}{K(f_1(t))^{a_1}(f_2(t))^{a_2}\cdots(f_n(t))^{a_n}},$$

where $K \in \mathbb{R} \setminus \{0\}, a_i \in \mathbf{R}, f_i(t)$ denotes ith iterate of $f(t), i = 1,2,\ldots,n$. The above equation is closely related to a discrete derivatives sequence $F'(m)$ (see [Y.-F.S. Pétermann, Jean-Luc Rémy, Ilan Vardi, Discrete derivatives of sequences, Adv. in Appl. Math. 27 (2001) 562-584]).

Keywords Iterative differential equation · Discrete derivatives sequence · Periodic solutions · Fixed point theorem

Mathematics Subject Classification (2010) 39B12 · 39B82

1 Introduction

Consider a sequence $F : \mathbf{N} \to \mathbf{N}$ which is nondecreasing and onto. Then the quantity defined by

$$F'(m) := \frac{1}{|F^{-1}(F(m))|},$$

where as usual $F^{-1}(k)$ denotes the set $\{l|F(l) = k\}$, can be interpreted as the slope of F at m. The function $F' : \mathbf{N} \to \mathbf{R}$ is called the discrete derivative of F. If the sequence F is defined by $|F^{-1}(k)| = F(k)$ in which k appears exactly $F(k)$ times, then its discrete derivative is

$$F'(m) = \frac{1}{F \circ F(m)}.$$

This is known as Golomb's sequence (see [3,8]). Golomb [3] asked for an asymptotic formula for the Golomb's sequence. Marcus in [3] suggested and Fine in [3] (see also [5]) proved that $F(m)$ is approximated by

$$\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{3-\sqrt{5}}{2}} m^{\frac{\sqrt{5}-1}{2}}$$

for sufficiently large m. Marcus' idea is based on the conjecture that an asymptote $f(t)$ of $\{F(m)\}$ satisfies the functional differential equation

$$f'(t) = \frac{1}{f \circ f(t)}. \tag{1.1}$$

Recently, Y.-F.S. Pétermann, in [6] and [7], considered the increasing solutions of Eq. (1.1) and discovered a link between a particular solution of

$$f'(t) = \frac{1}{K(f_1(t))^{a_1}(f_2(t))^{a_2}\cdots(f_n(t))^{a_n}} \tag{1.2}$$

and the asymptotic behaviour of sequence F, where $f_i(t)$ denotes ith iterate of $f(t), i = 1,2,\ldots,n$. In [12], by Schroder transformation, Si and Zhao discussed the analytic solutions of (1.2) under the Brjuno condition.

Since Burton [1] applied Krasnoselskii's fixed theorem to prove the existence of periodic solutions, this theorem has been extensively used in proving stability, periodic of solutions and boundedness of solutions in functional differential (difference) equations. 2005, Raffoul [10] used fixed point theorem to show a nonlinear neutral system

$$\frac{d}{dt}[x(t) - ax(t-\tau)] = r(t)x(t) - f(t, x(t-\tau))$$

has a periodic solution. Some other works can also be found in [2], [4], [9], [11], [14].

In this paper, we consider the existence of periodic solutions of Eq. (1.2).

Throughout this paper, we assume that $K > 0, a_i \in \mathbf{R}$. For convenience, we will make use $C(\mathbf{R},\mathbf{R})$ to denote the set of all real valued continuous functions map $\mathbf{R}$ into $\mathbf{R}$.

For $T > 0$, define

$$\mathscr{P}_T = \Big\{ f \in C(\mathbf{R},\mathbf{R}) : f(t+T) = f(t),\ \forall t \in \mathbf{R} \Big\}.$$

Then $\mathscr{P}_T$ is a Banach space with the norm

$$\|f\| = \max_{t\in[0,T]} |f(t)| = \max_{t\in\mathbf{R}} |f(t)|.$$

For $p, P > 0, L \geq 0$, define the set

$$\mathscr{P}_T(p,P,L) = \Big\{ f \in \mathscr{P}_T : p \leq f(t) \leq P,\ |f(t_2) - f(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbf{R} \Big\},$$

which is a closed convex and bounded subset of $\mathscr{P}_T$, and we wish to find T-periodic functions $f \in \mathscr{P}_T(p,P,L)$ satisfies (1.2).

2 Periodic solutions of (1.2)

In this section, the existence of periodic solutions of equation (1.2) will be proved. Now let us state the Krasnoselskii's fixed point theorem, which will be used to prove our main theorem.

Theorem 2.1 (Krasnoselskii [13]) *Let Ω be a closed convex nonempty subset of a Banach space $(\mathbf{B}, \|\cdot\|)$. Suppose that A and B map Ω into $\mathbf{B}$ such that*
(i) A is compact and continuous,
(ii) B is a contraction mapping.
(iii) $x, y \in \Omega$, implies $Ax + By \in \Omega$,
Then there exists $z \in \Omega$ with $z = Az + Bz$.

We begin with the following lemma.

Lemma 2.2 *For any $\varphi, \psi \in \mathscr{P}_T(p, P, L)$,*

$$||\varphi_n - \psi_n|| \leq \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|, \ n = 1, 2, \dots. \tag{2.1}$$

The result can be obtained by the definition of $\mathscr{P}_T(p, P, L)$.

Lemma 2.3 *If $f \in \mathscr{P}_T$, then f is a solution of equation* (1.2) *if and only if*

$$\begin{aligned} f(t) = &\int_t^{t+T} \frac{1}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}} G(t,s)ds \\ &+ \int_t^{t+T} f(s)G(t,s)ds. \end{aligned}$$

where

$$G(t,s) = \frac{e^{K(s-t)}}{e^{KT}-1}. \tag{2.2}$$

Proof. It is easy to find Eq (1.2) can be written in the form of

$$Kf'(t)e^{Kt} + f(t)e^{Kt} = \frac{e^{Kt}}{(f_1(t))^{a_1}(f_2(t))^{a_2}\cdots(f_n(t))^{a_n}} + f(t)e^{Kt}, \tag{2.3}$$

Let $f \in \mathscr{P}_T$ be a solution of (2.3), integrate from t to $t+T$ to obtain

$$\begin{aligned} f(t+T)e^{K(t+T)} - f(t)e^{Kt} = &\int_t^{t+T} \frac{e^{Ks}}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}} ds \\ &+ \int_t^{t+T} f(s)e^{Ks}ds \end{aligned}$$

Using the fact $f(t+T)=f(t)$, the above expression can be put in the form

$$\begin{aligned} f(t) = &\int_t^{t+T} \frac{1}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}} \frac{e^{K(s-t)}}{e^{KT}-1} ds \\ &+ \int_t^{t+T} f(s) \frac{e^{K(s-t)}}{e^{KT}-1} ds. \end{aligned}$$

This completes the proof. □

It is clear that $G(t,s)=G(t+T,s+T)$ for all $(t,s)\in \mathbf{R}^2$, and for $s\in[t,t+T]$, we have

$$m = \frac{1}{|e^{KT}-1|} \le |G(t,s)| \le \frac{e^{KT}}{|e^{KT}-1|} = M. \tag{2.4}$$

Now we will need to construct two mappings that satisfy the hypotheses of Theorem 2.1. To this aim, consider the maps $A,B:\mathscr{P}_T(p,P,L)\to\mathscr{P}_T$ defined as follows:

$$(Af)(t) = \int_t^{t+T} \frac{1}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}} G(t,s)ds,\ t\in\mathbf{R}, \tag{2.5}$$

$$(Bf)(t) = \int_t^{t+T} f(s)G(t,s)ds,\ t\in\mathbf{R}, \tag{2.6}$$

where $G(t,s)$ is defined as in (2.2).

Lemma 2.4 *The operator A is continuous and compact on $\mathscr{P}_T(p,P,L)$.*

Proof. Take $\varphi,\psi\in\mathscr{P}_T(p,P,L), t\in\mathbf{R}$. Using (2.1) we have

$$\begin{aligned} &\left|(\varphi_i(t))^{-a_i} - (\psi_i(t))^{-a_i}\right| \\ &= |a_i||\xi(t)|^{-a_i-1}|\left|\varphi_i(t)-\psi_i(t)\right| \\ &\le |a_i|p^{-a_i-1}\sum_{j=0}^{i-1} L^j\|\varphi-\psi\|, \end{aligned}$$

where $\xi(t)$ lies between $\varphi_i(t)$ and $\psi_i(t)$.

From (2.4),

$$
\begin{aligned}
&|(A\varphi)(t)-(A\psi)(t)|\\
&= M\int_t^{t+T}\Big|(\varphi_1(s))^{-a_1}(\varphi_2(s))^{-a_2}\cdots(\varphi_n(s))^{-a_n}\\
&\quad -(\psi_1(s))^{-a_1}(\psi_2(s))^{-a_2}\cdots(\psi_n(s))^{-a_n}\Big|ds\\
&\le M\int_t^{t+T}\Big|\Big((\varphi_1(s))^{-a_1}-(\psi_1(s))^{-a_1}\Big)(\varphi_2(s))^{-a_2}\cdots(\varphi_n(s))^{-a_n}+\dots\\
&\quad +(\psi_1(s))^{-a_1}\cdots(\psi_{n-1}(s))^{-a_{n-1}}\Big((\varphi_n(s))^{-a_n}-(\psi_n(s))^{-a_n}\Big)\Big|ds\\
&\le TMp^{-(\sum_{i=1}^n a_i+1)}\sum_{i=1}^n\sum_{j=0}^{i-1}L^j|a_i|\|\varphi-\psi\|.
\end{aligned}
$$

This proves the continuity of A.

Now we will show that A is a compact map. It is easy to see that $\mathscr{P}_T(p,P,L)$ is uniformly bounded and equicontinuous on $\mathbf{R}$, then using Arzela-Ascoli theorem we know $\mathscr{P}_T(p,P,L)$ is a compact set. Since A is continuous, it maps compact sets into compact sets, therefore A is compact. This completes the proof. □

Lemma 2.5 *If $MT<1$, then the operator B is a contraction mapping on $\mathscr{P}_T(p,P,L)$.*

Proof. Take $\varphi,\psi\in\mathscr{P}_T(p,P,L)$. Then

$$\|B\varphi-B\psi\|\le M\max_{t\in[0,T]}\int_t^{t+T}\Big|\varphi(s)-\psi(s)\Big|ds\le MT\|\varphi-\psi\|,$$

hence B defines a contraction mapping. □

Theorem 2.6 *Suppose $MT<1$ and the following inequalities hold*

$$MTp^{-\sum_{i=1}^n a_i}\le(1-MT)P,\ \ (1-mT)p\le mTP^{-\sum_{i=1}^n a_i},\tag{2.7}$$

$$2(p^{-\sum_{i=1}^n a_i}+P)\le L,\tag{2.8}$$

then Eq. (2.1) *has a periodic solution in $\mathscr{P}_T(p,P,L)$.*

Proof. For any $\varphi,\psi\in\mathscr{P}_T(p,P,L)$, by (2.7) and (2.8),

$$
\begin{aligned}
&(A\varphi)(t)+(B\psi)(t)\\
&\le M\int_t^{t+T}\frac{1}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds+M\int_t^{t+T}\psi(s)ds\\
&\le MT(P+p^{-\sum_{i=1}^n a_i})\\
&\le P.
\end{aligned}
$$

$$
\begin{aligned}
&(A\varphi)(t)+(B\psi)(t)\\
&\geq m\int_t^{t+T}\frac{1}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds+m\int_t^{t+T}\psi(s)ds\\
&\geq mT(p+P^{-\sum_{i=1}^n a_i})\\
&\geq p.
\end{aligned}
$$

Without loss of generality, we assume $t_2\geq t_1$. By (2.8),

$$
\begin{aligned}
&\Big|(A\varphi)(t_2)-(A\varphi)(t_1)\Big|\\
&\leq \frac{1}{|e^{KT}-1|}\Bigg|e^{-Kt_2}\int_{t_2}^{t_2+T}\frac{e^{Ks}}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds\\
&\quad -e^{-Kt_1}\int_{t_1}^{t_1+T}\frac{e^{Ks}}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds\Bigg|\\
&\leq \frac{1}{|e^{KT}-1|}\Big|e^{-Kt_2}-e^{-Kt_1}\Big|\Bigg|\int_{t_1}^{t_1+T}\frac{e^{Ks}}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds\Bigg|\\
&\quad +\frac{e^{-Kt_2}}{|e^{KT}-1|}\Bigg|\int_{t_2}^{t_1}\frac{e^{Ks}}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds\\
&\quad +\int_{t_1+T}^{t_2+T}\frac{e^{Ks}}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}ds\Bigg|\\
&\leq 2p^{-\sum_{i=1}^n a_i}|t_2-t_1| \qquad (2.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|(B\psi)(t_2)-(B\psi)(t_1)\Big|\\
&\leq \frac{1}{|e^{KT}-1|}\Big|e^{-Kt_2}\int_{t_2}^{t_2+T}\psi(s)e^{Ks}ds-e^{-Kt_1}\int_{t_1}^{t_1+T}\psi(s)e^{Ks}ds\Big|\\
&\leq \frac{1}{|e^{KT}-1|}\Big|e^{-Kt_2}-e^{-Kt_1}\Big|\Big|\int_{t_1}^{t_1+T}\psi(s)e^{Ks}ds\Big|\\
&\quad +\frac{e^{-Kt_2}}{|e^{KT}-1|}\Big|\int_{t_2}^{t_1}\psi(s)e^{Ks}ds+\int_{t_1+T}^{t_2+T}\psi(s)e^{Ks}ds\Big|\\
&\leq 2P|t_2-t_1| \qquad (2.10)
\end{aligned}
$$

From (2.9), (2.10) and (2.8) we have

$$
\begin{aligned}
&\Big|\Big((A\varphi)(t_2)+(B\psi)(t_2)\Big)-\Big((A\varphi)(t_1)+(B\psi)(t_1)\Big)\Big|\\
&\leq 2(p^{-\sum_{i=1}^n a_i}+P)|t_2-t_1|\\
&\leq L|t_2-t_1|.
\end{aligned}
$$

This shows that $(A\varphi)(t)+(B\psi)(t)\in\mathscr{P}_T(p,P,L)$. By Lemma 2.4 and Lemma 2.5, we see that all the conditions of Krasnoselskii's theorem are satisfied on the set $\mathscr{P}_T(p,P,L)$. Thus there exists a fixed point f in $\mathscr{P}_T(p,P,L)$, hence

$$\begin{aligned}f(t) &= (Af)(t)+(Bf)(t)\\ &= \int_t^{t+T}\frac{1}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}}G(t,s)ds\\ &\quad+\int_t^{t+T}f(s)G(t,s)ds.\end{aligned}$$

From Lemma 2.3, f is a T-periodic solution of equation (1.2). This completes the proof. □

3 Uniqueness and example

In this section, we show that under a suitable condition equation (2.1) has a unique solution. An example is also provided to illustrate that the assumptions of Theorem 2.6 do not self-contradict.

Theorem 3.1 *In addition to the assumption of Theorem* 2.6*, suppose that*

$$TM\Big(1+p^{-(\sum_{i=1}^n a_i+1)}\sum_{i=1}^{n}\sum_{j=0}^{i-1}L^j|a_i|\Big)<1, \tag{3.1}$$

then (1.2) *has a unique solution in* $\mathscr{P}_T(p,P,L)$.

Proof. Define an operator H from $\mathscr{P}_T(p,P,L)$ into $\mathscr{P}_T$ by

$$\begin{aligned}(Hf)(t) &= (Af)(t)+(Bf)(t)\\ &= \int_t^{t+T}\frac{1}{(f_1(s))^{a_1}(f_2(s))^{a_2}\cdots(f_n(s))^{a_n}}G(t,s)ds\\ &\quad+\int_t^{t+T}f(s)G(t,s)ds,\end{aligned} \tag{3.2}$$

where $G(t,s)$ is defined as in (2.2). Denote by $\varphi,\psi \in \mathscr{P}_T(p,P,L)$ two different T-periodic solutions of (1.1). From Lemma 2.4 we have

$$\begin{aligned}
&|\varphi(t)-\psi(t)| \\
=& \left|(H\varphi)(t)-(H\psi)(t)\right| \\
\le& \int_t^{t+T}\left|\frac{1}{(\varphi_1(s))^{a_1}(\varphi_2(s))^{a_2}\cdots(\varphi_n(s))^{a_n}}\right. \\
&\left.-\frac{1}{(\psi_1(s))^{a_1}(\psi_2(s))^{a_2}\cdots(\psi_n(s))^{a_n}}\right||G(t,s)|ds \\
&+\int_t^{t+T}|\varphi(s)-\psi(s)||G(t,s)|ds \\
\le& TM\Big(1+p^{-(\Sigma_{i=1}^n a_i+1)}\sum_{i=1}^n\sum_{j=0}^{i-1}L^j|a_i|\Big)\|\varphi-\psi\| \\
=& \Gamma\|\varphi-\psi\|,
\end{aligned}$$

where $\Gamma = TM\Big(1+p^{-(\Sigma_{i=1}^n a_i+1)}\sum_{i=1}^n\sum_{j=0}^{i-1}L^j|a_i|\Big)$. Thus

$$\|\varphi-\psi\|\le\Gamma\|\varphi-\psi\|.$$

By (3.1), we know $\Gamma<1$ and the fixed point φ must be unique. $\square$

Example 3.1 Now, we will show that the conditions in Theorem 2.6 do not self-contradict. Consider the following equation:

$$f'(t)=\frac{(f(f(t)))^2}{2f(t)}, \tag{3.3}$$

where $a_1=1,\ a_2=-2,\ K=2$, take $T=\frac{1}{2}, p=1, P=5, L=15$. A simple calculation yields

$$1.58<M=\frac{e}{e-1}<1.59,\ \ 0.79<MT<0.8<1,$$

$$0.58<m=\frac{1}{e-1}<0.59,\ \ 0.29<mT<0.3.$$

and

$$MTp<0.8<1\le(1-MT)P,$$

$$(1-mT)p<0.71<1.45<mTP,$$

$$2(p+P)<12<15=L,$$

then (2.7) and (2.8) are satisfied. By Theorem 2.6, equation (3.3) has a $\frac{1}{2}$-periodic solution f such that $1\le f(t)\le 5$, and $|f(t_2)-f(t_1)|\le 15|t_2-t_1|,\ \forall t_1,t_2\in\mathbf{R}$.

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