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Cesàro integrability and Tauberian theorems in quantum calculus

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Abstract In this paper, we show some properties regarding Cesàro integrability method in quantum calculus. Moreover, we extend and generalize q-analogues of some classical Tauberian theorems given for Cesàro integrability by Fitouhi and Brahim [J. Nonlinear Math. Phys., 14 (3) (2007), 324–340].

Keywords Tauberian theorems · Cesàro integrability · q-calculus · Jackson integral

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1 Introduction and Preliminaries

Quantum calculus or q-calculus is the modern name for the investigation of calculus which is focused on the idea of deriving q-analogues of results belonging to standard calculus without using limits. The q-calculus served as a bridge between mathematics and physics. It has gained noticeable importance and popularity during the past three decades or so, due mainly to its applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric series and other sciences such as statistical mechanics, quantum theory, and the theory of relativity.

The summability theory has several uses throughout mathematical analysis and applied mathematics and also has links to number theory. It results from the the summation of series and integrals. The methods of summability for the continuous case have become a popular area of research in recent years. Now, we begin with introducing some significant notions of the classical summability theory of standard calculus for the continuous case.

Let $f(x)$ be a real valued continuous function on $[0,\infty)$ and $s(x)=\int_0^x f(t)dt$. A function $s(x)$ is said to be Cesàro integrable to a fi-

nite number A if

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x s(t)dt = A. \tag{1.1}$$

If the limit

$$\lim_{x\to\infty} s(x) = A \tag{1.2}$$

exists, then $s(x)$ is clearly Cesàro integrable to A. That is, Cesàro integrability method is regular. The converse is not necessarily true. However, (1.1) may imply (1.2) by adding some suitable condition on $s(x)$. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem. The first theorem of this type was given by Tauber [8].

The following theorem was given by Hardy [5] for Cesàro integrability method is the continuous analogue of Tauber's first theorem.

Theorem 1.1 *If $s(x)$ is Cesàro integrable to A and $\lim_{x\to\infty} x\frac{d}{dx}s(x) = 0$, then $s(x)$ converges to A as $x \to \infty$.*

Schmidt [7] defined the concept of slowly oscillation of a sequence and obtained a Tauberian theorem for Cesàro integrability for which the slow oscillation is a Tauberian condition. Also, Hardy [5, page 144] defined the concept of slow oscillation for functions of real variables motivated by the concept of the slow oscillation for sequences of real numbers: A function $s(x)$ is said to be slowly oscillating if for for every $\varepsilon > 0$ there exists $x_0 > 0$ such that $|s(x) - s(y)| < \epsilon$ whenever $x > y > x_0$ and $x/y \to 1$.

It can be easily seen that if $s(x)$ converges to A as $x \to \infty$, then it is slowly oscillating.

Çanak and Totur [2] proved a Schmidt type theorem for functions of one-variable as follows:.

Theorem 1.2 *If $s(x)$ is Cesàro integrable to A and slowly oscillating, then $s(x)$ converges to A as $x \to \infty$.*

Note that the slow oscillation of a function $s(x)$ implies the slow oscillation of its Cesàro means $\sigma(s(x)) := \frac{1}{x}\int_0^x s(t)dt$. (see [1]).

We now give some basic concepts of q-calculus and some properties of q-Cesàro integrability method. Throughout this paper, we fix $q \in (0, 1)$ and we use the notations and the definitions in the book of Kac and Cheung [6].

We denote by

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}. \tag{1.3}$$

The q-derivative $D_q f$ of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \text{ if } x \neq 0, \tag{1.4}$$

$D_q f(0) = f'(0)$ provided $f'(0)$ exists. If f is differentiable, then $D_q f(x)$ tends to $f'(x)$ as q tends to 1.

Notice that the q-derivative satisfies the following q-analogue of Leibniz rule

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x). \tag{1.5}$$

The Jackson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1.6}$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{1.7}$$

provided the sums converge absolutely.

The Jackson integral on a general interval $[a,b]$ is defined by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{1.8}$$

The Jackson integral and q-derivative are related by the fundamental theorem of $q-$calculus which can be stated as follows:
If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, we have

$$\int_a^b f(x)d_qx = F(b)-F(a), \quad 0 \le a < b \le \infty. \tag{1.9}$$

Also, for any function f, we have

$$D_q\left(\int_0^x f(t)d_qt\right) = f(x). \tag{1.10}$$

Further, reversing the order of integration in q-calculus is given by

$$\int_0^x\int_0^t f(s)d_qsd_qt = \int_0^x\int_{qs}^x f(s)d_qtd_qs. \tag{1.11}$$

Note that for $n\in\mathbb{Z}$, it is easy to see that

$$\int_0^x f(q^nt)d_qt = \frac{1}{q^n}\int_0^{xq^n} f(t)d_qt. \tag{1.12}$$

A function f is said to be q-integrable on $[0,\infty)$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely. We denote the set of all functions that are q-integrable on $[0,\infty)$ by $L_1(\mathbb{R}_{q,+})$.

Throughout this paper, we assume that $f(t)$ is a function defined on $(0,\infty)$ satisfying

$$\int_0^x |f(t)|d_qt < \infty \tag{1.13}$$

for all $x\in\mathbb{R}_{q,+}$.

Cieśliński [3] defined the q-exponential function ε_q^z by $\varepsilon_q^z = \sum_{n=0}^{\infty} \frac{z^n}{\{n\}!}$, for $|z| < \frac{2}{1-q}$, where $\{n\} = \frac{1+q+\cdots+q^{n-1}}{\frac{1}{2}(1+q^{n-1})}$ and $\{n\}! = \{1\}\{2\}\dots\{n\}$. Cieśliński also introduced the q-sine and q-cosine functions defined by $\mathcal{S}in_q x = \frac{\varepsilon_q^{ix} - \varepsilon_q^{-ix}}{2i}$ and $\mathcal{C}os_q x = \frac{\varepsilon_q^{ix} + \varepsilon_q^{-ix}}{2}$ such that $-1 \leq \mathcal{C}os_q x \leq 1$ and $-1 \leq \mathcal{S}in_q x \leq 1$ for all $x \in \mathbb{R}$ and he showed that $D_q \mathcal{S}in_q x = \frac{\mathcal{C}os_q x + \mathcal{C}os_q(qx)}{2}$ and $D_q \mathcal{C}os_q x = -\frac{\mathcal{S}in_q x + \mathcal{S}in_q(qx)}{2}$.

Now, we set

$$s(x) = \int_0^x f(t) d_q t. \tag{1.14}$$

The q-Cesàro means of $s(x)$ are defined by

$$\sigma(s(x)) = \frac{1}{x} \int_0^x s(t) d_q t. \tag{1.15}$$

The function $s(x)$ is q-Cesàro integrable to A if

$$\lim_{x \to \infty} \sigma(s(x)) = A. \tag{1.16}$$

If the limit $\lim_{x\to\infty} s(x) = A$ exists, then the limit (1.16) also exists (see Theorem 3.2 from below).

But the existence of the limit (1.16) does not imply the existence of the limit $\lim_{x\to\infty} s(x) = A$, in general. For example, $\int_0^x \frac{\mathcal{C}os_q t + \mathcal{C}os_q(qt)}{2} d_q t$ is q-Cesàro integrable to zero, but the integral $\int_0^\infty \frac{\mathcal{C}os_q t + \mathcal{C}os_q(qt)}{2} d_q t$ does not exist.

However, (1.16) may imply the existence of the limit $\lim_{x\to\infty} s(x) = A$ by adding some Tauberian condition on $s(x)$ just as in the standard calculus.

Our main aim in this paper is to recover convergence of $s(x)$ as $x \to \infty$ from its q-Cesàro integrability by imposing some condition on $s(x)$.

2 Lemmas

We need the following Lemmas in proving our main results.

Lemma 2.1 *The difference between $s(x)$ and its q-Cesàro mean $\sigma(s(x))$, which is so-called the q-Kronecker identity, is given by*

$$s(x) - \sigma(s(x)) = q v(x), \tag{2.1}$$

where $v(x) := \dfrac{1}{x} \displaystyle\int_0^x t f(t) d_q t.$

Proof. By definition, we have

$$\begin{aligned}\sigma(s(x)) &= \frac{1}{x}\int_0^x s(t)d_qt \\ &= \frac{1}{x}\int_0^x \int_0^t f(s)d_qsd_qt.\end{aligned}$$

Reversing the order of integration gives

$$\begin{aligned}\sigma(s(x)) &= \frac{1}{x}\int_0^x f(s)\int_{qs}^x d_qtd_qs \\ &= \frac{1}{x}\int_0^x (x-qs)\, f(s)d_qs \\ &= \int_0^x \left(1-\frac{qs}{x}\right) f(s)d_qs \\ &= \int_0^x f(s)d_qs - q\frac{1}{x}\int_0^x sf(s)d_qs,\end{aligned}$$

which completes the proof. □

Lemma 2.2 *For a function $s(x)$ defined on $(0,\infty)$,*

$$\sigma(xD_q(s(x))) = xD_q(\sigma(s(x))), \tag{2.2}$$

and

$$xD_q(\sigma(s(x))) = v(x). \tag{2.3}$$

Proof. The proof of first identity can be easily obtained by taking q-Cesàro means of both sides of the identity $xD_q(s(x)) = xf(x)$.

We shall prove the second identity. Using the definition of q-derivative of the product, we obtain

$$\begin{aligned}D_q(\sigma(s(x))) &= D_q\left(\frac{1}{x}\int_0^x s(t)d_qt\right) \\ &= \frac{1}{qx}s(x) - \frac{1}{qx^2}\int_0^x s(t)d_qt \\ &= \frac{1}{qx}\left(s(x)-\sigma(s(x))\right)\end{aligned}$$

Hence, using q-Kronecker identity, we get $D_q(\sigma(s(x))) = \dfrac{v(x)}{x}$. □

3 q-Analogues of Classical Tauberian Theorems for the Cesàro integrability

We now define the property $(\mathcal{P})$ which will be used as a Tauberian condition for q-Cesàro integrability.

Definition 3.1 *A function f is said to satisfy the property $(\mathcal{P})$ if for all $\epsilon > 0$, there exists $K > 0$ such that*

$$|f(x) - f(qx)| < \epsilon \tag{3.1}$$

for all $x > K$.

Notice that the property $(\mathcal{P})$ is similar to the concept of slowly oscillation of a function in standard calculus. However, the convergence need not imply the property $(\mathcal{P})$ in quantum calculus. We start by showing that convergence of $\sigma(s(x))$ to A may not imply convergence of $\sigma(s(qx))$ to the same value. Indeed, by (1.6), we have

$$\begin{aligned}
\int_0^{qx} s(t)d_qt &= (1-q)qx\sum_{n=0}^{\infty} s(xq^{n+1})q^n \\
&= (1-q)x\sum_{n=0}^{\infty} s(xq^{n+1})q^{n+1} \\
&= (1-q)x\sum_{n=1}^{\infty} s(xq^{n})q^{n} \\
&= (1-q)x\left(\sum_{n=0}^{\infty} s(xq^{n})q^{n} - s(x)\right) \\
&= \int_0^{x} s(x)d_qx - (1-q)xs(x),
\end{aligned}$$

and

$$\frac{q}{1-q}(\sigma(s(x)) - \sigma(s(qx))) = s(x) - \sigma(s(x)).$$

From the last identity it follows that convergence of $\sigma(s(x))$ to A implies convergence of $\sigma(s(qx))$ to the same value whenever $s(x) \to A$ as $x \to \infty$.

Our first result is the regularity theorem for q-Cesàro integrability.

Theorem 3.2 *If $\lim\limits_{x\to\infty} s(x) = A$, then $s(x)$ is q-Cesàro integrable to A.*

Proof. Let $\lim_{x\to\infty} s(x) = A$. Then for all $\epsilon > 0$ there exists $K_0 > 0$ such that $|s(x) - A| \leq \dfrac{\epsilon}{2}$ whenever $x > K_0$ and $|s(x) - A| < M$ whenever $x \leq$

K_0. So, we have

$$\begin{aligned}
|\sigma(s(x)) - A| &= \left| \frac{1}{x}\int_0^x s(t)d_qt - \frac{1}{x}\int_0^x Ad_qt \right| \\
&= \left| \frac{1}{x}\int_0^x (s(t) - A)d_qt \right| \\
&\leq \frac{1}{x}\int_0^x |s(t) - A|\, d_qt \\
&= \frac{1}{x}\int_0^{K_0} |s(t) - A|\, d_qt + \frac{1}{x}\int_{K_0}^x |s(t) - A|\, d_qt \\
&\leq \frac{K_0M}{x} + \frac{\epsilon}{2}\frac{(x - K_0)}{x} \\
&\leq \frac{K_0M}{x} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{K_0M}{x} = 0$, there exists $K_1 > 0$ such that $\left|\frac{K_0M}{x}\right| \leq \frac{\epsilon}{2}$ whenever $x > K_1$. So there exists $K = \max\{K_0, K_1\}$ such that

$$|\sigma(s(x)) - A| < \epsilon \tag{3.2}$$

whenever $x > K$. This completes the proof. □

The following is a Tauberian theorem for q-Cesàro integrability which generalizes Proposition 2 in [4].

Theorem 3.3 *If $s(x)$ is q-Cesàro integrable to A and $\sigma(s(x))$ satisfies the property $(\mathcal{P})$, then*

$$\lim_{x\to\infty} s(x) = A.$$

Proof. We can suppose that $A = 0$. From the definition of $\sigma(x)$, we have

$$\begin{aligned}
\sigma(s(x)) - \sigma(s(qx)) &= \frac{1}{x}\int_0^x s(t)d_qt - \frac{1}{qx}\int_0^{qx} s(t)d_qt \\
&= \frac{1}{x}\int_0^{qx} s(t)d_qt + \frac{1}{x}\int_{qx}^x s(t)d_qt - \frac{1}{qx}\int_0^{qx} s(t)d_qt \\
&= (q-1)\frac{1}{qx}\int_0^{qx} s(t)d_qt + (1-q)\frac{1}{x - qx}\int_{qx}^x s(t)d_qt \\
&= (1-q)[s(x) - \sigma(s(qx))]. \qquad (3.3)
\end{aligned}$$

Given $\epsilon > 0$. Then there exists $K > 0$ such that

$$-\epsilon < \sigma(s(qx)) - s(x) < \epsilon \tag{3.4}$$

for all $x > K$, by the property $(\mathcal{P})$ of $\sigma(x)$. From (3.4), we obtain

$$s(x) - \epsilon \leq \sigma(s(qx)) \leq s(x) + \epsilon. \tag{3.5}$$

By the property $(\mathcal{P})$ and q-Cesàro integrability of $s(x)$ to zero, we obtain

$$\lim_{x\to\infty} \sigma(s(qx)) = 0. \tag{3.6}$$

It follows from (3.5) that

$$-\epsilon \leq \liminf_{x\to\infty} s(x) \leq \limsup_{x\to\infty} s(x) \leq \epsilon, \tag{3.7}$$

which completes the proof. □

Corollary 3.4 *[4] If $s(x)$ is q-Cesàro integrable to A and satisfies the property $(\mathcal{P})$, then*

$$\lim_{x\to\infty} s(x) = A.$$

Proof. It suffices to show that if a function $s(x)$ defined on $(0,\infty)$ satisfies the property $(\mathcal{P})$, then $\sigma(s(x))$ satisfies the property $(\mathcal{P})$. Given $\epsilon > 0$, there exists $K_0 > 0$ such that

$$|s(x) - s(qx)| < \frac{\epsilon}{2} \tag{3.8}$$

for all $x > K_0$ and $|s(x) - s(qx)| < M$ whenever $x \leq K_0$. Using the property $(\mathcal{P})$ of $s(x)$, we have

$$\begin{aligned}
\sigma(s(x)) - \sigma(s(qx)) &= \frac{1}{x}\int_0^x s(t)d_qt - \frac{1}{qx}\int_0^{qx} s(t)d_qt \\
&= \frac{1}{x}\int_0^x s(t)d_qt - \frac{1}{x}\int_0^x s(qt)d_qt \\
&= \frac{1}{x}\int_0^x [s(t) - s(qt)]d_qt,
\end{aligned}$$

and

$$\begin{aligned}
|\sigma(s(x)) - \sigma(s(qx))| &\leq \frac{1}{x}\int_0^x |s(t) - s(qt)|d_qt \\
&\leq \frac{1}{x}\int_0^x |s(t) - s(qt)|d_qt \\
&\leq \frac{1}{x}\int_0^{K_0} |s(t) - s(qt)|d_qt + \frac{1}{x}\int_{K_0}^x |s(t) - s(qt)|d_qt \\
&\leq \frac{K_0M}{x} + \frac{\epsilon}{2}\frac{(x - K_0)}{x} \\
&\leq \frac{K_0M}{x} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{K_0M}{x} = 0$, there exists $K_1 > 0$ such that $\left|\frac{K_0M}{x}\right| \leq \frac{\epsilon}{2}$ whenever $x > K_1$. So there exists $K = \max\{K_0, K_1\}$ such that

$$|\sigma(s(x)) - \sigma(s(qx))| < \epsilon$$

whenever $x > K$. This completes the proof. □

The following example shows that there is some function $s(x)$ defined on $(0,\infty)$ such that $\sigma(s(x))$ satisfies the property $(\mathcal{P})$, but $s(x)$ doesn't.

Example 3.1 Define $s(x)$ by $s(x)=\frac{\mathcal{C}os_q x+\mathcal{C}os_q(qx)}{2}$ for all x. Then $\sigma(s(x))=\frac{\mathcal{S}in_q x}{x}$. It is clear from the definition that $\sigma(s(x))$ satisfies the property $(\mathcal{P})$, but $s(x)$ doesn't.

Theorem 3.5 *If $s(x)$ is q-Cesàro integrable to A and $\sigma(v(x))$ satisfies the property $(\mathcal{P})$, then*

$$\lim_{x\to\infty} s(x)=A.$$

Proof. We can suppose that $A=0$. From the definition of the q-derivative, we have $v(x)=\frac{1}{x-qx}\int_{qx}^{x} v(t)d_qt$. Since $\sigma(v(x))=\frac{1}{x}\int_0^x v(t)d_qt$, and $\sigma(v(qx))=\frac{1}{qx}\int_0^{qx} v(t)d_qt$, we deduce that

$$\begin{aligned} v(x) &= \frac{1}{x-qx}\int_{qx}^{x} v(t)d_qt \\ &= \frac{1}{1-q}\frac{1}{x}\int_0^x v(t)d_qt-\frac{q}{1-q}\frac{1}{qx}\int_0^{qx} v(t)d_qt \\ &= \frac{1}{1-q}\sigma(v(x))-\frac{q}{1-q}\sigma(v(qx)). \end{aligned} \tag{3.9}$$

Subtracting $\sigma(v(x))$ from the identity (3.9), we obtain

$$v(x)-\sigma(v(x))=\frac{q}{1-q}\left[\sigma(v(x))-\sigma(v(qx))\right] \tag{3.10}$$

From (3.10), we have

$$|v(x)-\sigma(v(x))|=\frac{q}{1-q}|\sigma(v(x))-\sigma(v(qx))|. \tag{3.11}$$

Since $\sigma(v(x))$ satisfies the property $(\mathcal{P})$ by the hypothesis, for given $\epsilon>0$ there exists $K>0$ such that

$$|\sigma(v(x))-\sigma(v(qx))|<\epsilon. \tag{3.12}$$

It follows from Lemma 2.1 and the q-Cesàro integrability of $s(x)$ to zero that

$$\lim_{x\to\infty}\sigma(v(x))=0. \tag{3.13}$$

Taking (3.11), (3.12) and (3.13) into consideration, we have

$$\lim_{x\to\infty} v(x)=0. \tag{3.14}$$

By Lemma 2.2, we get $|xD_q\sigma(s(x))|<\epsilon$ for all $x>M$.

On the other hand, it follows from the identity

$$\begin{aligned} s(x) &= \frac{1}{x-qx}\int_{qx}^{x} s(t)d_qt \\ &= \frac{1}{1-q}\frac{1}{x}\int_0^x s(t)d_qt - \frac{q}{1-q}\frac{1}{qx}\int_0^{qx} s(t)d_qt \\ &= \frac{1}{1-q}\sigma(s(x)) - \frac{q}{1-q}\sigma(s(qx)). \end{aligned}$$

that we get

$$s(x) - \sigma(s(x)) = \frac{q}{1-q}\left[\sigma(s(x)) - \sigma(s(qx))\right].$$

By Lemma 2.1 and Lemma 2.2, we have the identity $\sigma(s(x)) - \sigma(s(qx)) = (1-q)xD_q(\sigma(s(x)))$. Therefore, we obtain that $\sigma(s(x))$ satisfies the property $(\mathcal{P})$. By Theorem 3.3, we have convergence of $s(x)$ to zero as $x \to \infty$. □

Corollary 3.6 *If $s(x)$ is q-Cesàro integrable to A and $v(x)$ satisfies the property $(\mathcal{P})$, then*

$$\lim_{x\to\infty} s(x) = A.$$

Proof. By writing $v(x)$ instead of $s(x)$ in the proof of the Corollary 3.4, we obtain that if $v(x)$ satisfies the property $(\mathcal{P})$, then $\sigma(v(x))$ satisfies the property $(\mathcal{P})$. Therefore, it follows from Theorem 3.5 that $s(x) \to A$ as $x \to \infty$. □

Corollary 3.7 *If $s(x)$is q-Cesàro integrable to A and satisfies the condition $\lim_{x\to\infty} xD_q(\sigma(s(x))) = 0$, then*

$$\lim_{x\to\infty} s(x) = A. \tag{3.15}$$

Proof. We have

$$\sigma(s(x)) - \sigma(s(qx)) = \int_{qx}^{x} (D_q(\sigma(s(t))))d_qt = (1-q)xD_q(\sigma(s(x)))$$

as in the proof of Theorem 3.5. By the hypothesis, we easily see that $\sigma(s(x))$ satisfies the property $(\mathcal{P})$. By Theorem 3.3, we have convergence of $s(x)$ as $x \to \infty$. □

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