An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 1

Complex line q-integrals and q-Green's formula on the plane

Kerim Koca · İlker Gençtürk · Mustafa Aydin

Abstract In this study, we firstly present the line q-integral of a complex function $f(z)$ on a piecewise smooth rectifiable curve γ in complex plane. After that, on a square set, we define the multiple q-integral. At the end of this study, we prove the q-Green's formula.

Keywords q-derivative · line q-integrals · q-multiple integral · q-Green's formula

Mathematics Subject Classification (2010) 05A30 · 30G25

1 Introduction

The q-derivative of a single variable real valued function $F(x)$ is defined as

$$D_q F(x) = \frac{F(x) - F(qx)}{(1-q)x} = \frac{1}{(1-q)x}(1-M_q)F(x) =: f(x), x \neq 0. \tag{1.1}$$

Here $M_q F(x) = F(qx)$. Function $F(x)$ is called the q-antiderivative of the function $f(x)$.

In this study we will consider the parameter q as $0 < q < 1$.

Definition 1.1 *If $\lim_{n\to\infty} F(q^n x)$ exists for all $x \in [0,a]$, the function $F(x)$ is called q-regular at $x = 0$. If $F(x)$ is continuous at $x = 0$, then*

$$\lim_{n\to\infty} F(q^n x) = F(0). \tag{1.2}$$

If $F(0) = 0$, then q-antiderivative of $f(x)$ can be written as,

$$\begin{aligned} F(x) &= \frac{1}{1-M_q}\left[(1-q)xf(x)\right] = (1-q)\sum_{n=0}^{\infty} M_q^n [xf(x)] \\ &= (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x). \end{aligned}$$

Generally for $x \in [0,a]$,

$$\int_0^x f(t)d_qt = F(x) - \lim_{n\to 0} F(q^n x). \tag{1.3}$$

If $x \in [0,a]$, then $qx,...,q^n x \in [0,a]$ where $a > 0$. Let $\lim_{x\to 0} F(x) = F(0) = 0$. The Jackson integral of $f(x)$ is defined as

$$\int_0^x f(t)d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(xq^n) = F(x) \tag{1.4}$$

or

$$\int_0^a f(x)d_qx = (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n). \tag{1.5}$$

More generally,

$$\int_0^x f(t)D_qg(t)d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)D_qg(q^n x) \tag{1.6}$$

or

$$\int_0^x f(t)d_qg(t) = \sum_{n=0}^{\infty} f(q^n x)[g(q^n x) - g(q^{n+1}x)] \tag{1.7}$$

can be written, [6].

Definition 1.2 *If the series in* (1.5)*, in the definition of the Jackson integral, is convergent, then $f(x)$ is called q-integrable.*

The further information about q-calculus can be found in [1], [3], [4] and [6].

Remark 1.1 If the function $F(x)$ is not q-regular at $x = 0$, then the Jackson integral may not be valid.

2 Complex q-integrals

Definition 2.1 *Let $D \subset \mathbb{C}$ and $\lambda \in \mathbb{R}$ be given. For every $z \in D$ if $\lambda z \in D$ holds, D is called λ-geometrical set.*

If $D \subset \mathbb{C}$ is a λ-geometrical set, then for every $z \in D$, the set D includes all the sequences of the form $(z\lambda^n)_\infty$.

Let $D \subset \mathbb{C}$ be a q-geometrical set including the origin, and $f(z)$ be a complex function on D. The complex q-derivative of $f(z)$ is defined as

$$D_q f(z) := \frac{f(z) - f(qz)}{(1-q)z} \; ; \; z \neq 0 \; , \; |q| < 1 \; , \; z \in D - \{0\} \tag{2.1}$$

$$D_q f(0) := \lim_{n\to\infty} \frac{f(0)-f(q^n z)}{q^n z} \ ; \ z \in D - \{0\}, \tag{2.2}$$

see [1].

Definition 2.2 *Let $D \subset \mathbb{C}$ be a q-geometrical set which contains the origin and $f(z)$ be a complex function on D. If for every $z \in D$,*

$$\lim_{n\to\infty} f(q^n z) = f(0) \tag{2.3}$$

holds, $f(z)$ is called q-regular at $z = 0$.

Theorem 2.3 [1] *Let $D \subset \mathbb{C}$ be a q-geometrical set that includes the origin and $f(z)$ be a function in D. Also, let $f(z)$ be q-regular at the origin. Then the function*

$$F(z) := \int_0^z f(t) d_q t \tag{2.4}$$

is q-regular at the origin. Also for every $z \in D$, $D_q F(z) = f(z)$ holds.

Conversely, for all $z \in D$, we have

$$\int_0^z D_q f(t) d_q t = f(z) - f(0). \tag{2.5}$$

Remark 2.1 If the function $f(z)$ is not continuous at $z = 0$, then we have

$$\int_0^z D_q f(t) d_q t = f(z) - \lim_{n\to\infty} f(q^n z) \tag{2.6}$$

or more generally, for $a, b \in D \subseteq \mathbb{C}$,

$$\int_a^b D_q f(t) d_q t = \left[f(b) - \lim_{n\to\infty} f(q^n b)\right] - \left[f(a) - \lim_{n\to\infty} f(q^n a)\right]. \tag{2.7}$$

Lemma 2.4 *Let the function $h(x,t)$ be defined on the set $[0,a] \times [0,a]$ and, let $h(x,t)$ be q-integrable for all $t \in [0,a]$. Consider the function*

$$f(x) := \int_0^x h(x,t) d_q t. \tag{2.8}$$

If $h(x, qx) = 0$, then

$$D_q^x \int_0^x h(x,t) d_q t = \int_0^x D_q^x h(x,t) d_q t. \tag{2.9}$$

Proof. By using the definition of the Jackson integral,

$$D_q^x \int_0^x h(x,t)d_qt = h(x,x) + \sum_{n=0}^{\infty} q^n \left[h(q^n x, x) - h(q^n x, qx)\right] \tag{2.10}$$

and

$$\int_0^x D_q^x h(x,t)d_qt = h(x,x) - h(x,qx) + \sum_{n=0}^{\infty} q^n \left[h(q^n x, x) - h(q^n x, qx)\right]. \tag{2.11}$$

By comparing (2.10) and (2.11), the result can be seen. □

On the complex plane $\mathbb{C}$, let

$$\gamma : z(t) = x(t) + iy(t),\ t \in [0,a]\ (a > 0)$$

be a curve which is piecewise smooth and rectifiable. Now, let us consider a function $f(z(t)) = u(z(t)) + iv(z(t))$, $t \in [0,a]$, on the curve γ.

Definition 2.5 *Let $f(z)$ be given on a piecewise smooth, rectifiable curve $\gamma \subset \mathbb{C}$, or indirectly, as a composite function $f(z(t))$ on $I = [0,a]$. It is said to be q-uniformly continuous on the curve γ if for any given positive number ε, we can find a number $\delta = \delta(\varepsilon)$ such that $|f(z(aq^k)) - f(z(aq^{k+1}))| < \varepsilon$ for sufficiently large numbers k, provided $|z(aq^k) - z(aq^{k+1})| < \delta$.*

Lemma 2.6 *Let $\gamma : z(t) = x(t) + iy(t)$ be a continuous curve on $\mathbb{C}$ with $t \in [0,a]$ and $f(z) = u(z) + iv(z) = u(z(t)) + iv(z(t))$ be complex-valued function which is q-uniformly continuous on the given curve γ. If $x(t), y(t)$ have q-derivative and u, v are q-integrable, then the Jackson integral of $f(z)$ on γ is given by*

$$\int_{\gamma} f(z)d_qz = \int_{t=0}^{a} f(z(t))D_qz(t)d_qt = (1-q)a\sum_{n=0}^{\infty} q^n f(z(aq^n))D_qz(aq^n). \tag{2.12}$$

Here the q-antiderivative of $f(z)$ is q-regular at $z = 0$.

Proof. Let $\gamma : z(t) = x(t) + iy(t)$, for $0 \le t \le a$, and $f(z) = f(z(t)) = u(z(t)) + iv(z(t))$. Then

$$\begin{aligned} I = \int_{\gamma} f(z) d_q z &= \int_{t=0}^{a} f(z(t)) D_q z(t) d_q t \\ &= \int_{0}^{a} [u(z(t)) + iv(z(t))][D_q x(t) + i D_q y(t)] d_q t \\ &= \int_{0}^{a} u(z(t)) D_q x(t) d_q t + i \int_{0}^{a} v(z(t)) D_q x(t) d_q t \\ &\quad + i \int_{0}^{a} u(z(t)) D_q y(t) d_q t - \int_{0}^{a} v(z(t)) D_q y(t) d_q t. \end{aligned}$$

By calculating the real integrals and by using (1.5) we get

$$\begin{aligned} I &= \sum_{n=0}^{\infty} [u(z(aq^n)) + iv(z(aq^n))][(x(aq^n) - x(aq^{n+1}) + i(y(aq^n) - y(aq^{n+1}))] \\ &= \sum_{n=0}^{\infty} f(z(aq^n))[z(aq^n) - z(aq^{n+1})] \\ &= (1-q)a \sum_{n=0}^{\infty} q^n f(z(aq^n)) D_q z(aq^n). \end{aligned}$$

The proof is complete. □

If the series in (2.12) is convergent, then the complex function $f(z)$ is called q-integrable on the curve γ.

Lemma 2.7 *If $f(z)$ is q-integrable on the continuous curve γ, then*

$$\lim_{q \to 1^-} \int_{\gamma} f(z) d_q z = \int_{\gamma} f(z) dz. \tag{2.13}$$

Proof. Let $f(z)$ be defined on γ. Since of $1 - q < \delta$, we get

$$|aq^n - aq^{n+1}| = |aq^n||1-q| < |aq^n|\delta = a\delta q^n < a\delta := \varepsilon.$$

Since $z(t)$ is continuous whenever $|aq^n - aq^{n+1}| < \varepsilon$ holds, there is a sufficiently small number ε_1 such that $|z(aq^n) - z(aq^{n+1})| < \varepsilon_1$. By considering the definition of the Riemann integral, we get

$$\int_{\gamma} f(z)dz = \lim_{n\to\infty}\sum_{k=0}^{n} f(z(t_k^*))[z(t_{k+1}) - z(t_k)] \ , 0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < a$$

$$= \sum_{n=0}^{\infty} f(z(t_n^*))[z(t_{n+1}) - z(t_n)] \ , \ t_n < t_n^* \leq t_{n+1}. \tag{2.14}$$

On the other hand,

$$\int_{\gamma} f(z)d_q z = \int_0^a f(z(t))D_q z(t)d_q t = \sum_{n=0}^{\infty} f(z(aq^n))[z(aq^n) - z(aq^{n+1})]. \tag{2.15}$$

Here, we notice that $\lim_{q\to 1^-}[z(aq^n) - z(aq^{n+1})] = 0$.
From (2.14), $t_n^* \to aq^n, t_{n+1} \to aq^n$ and $t_n \to aq^{n+1}$ for q enough close to 1,

$$\int_{\gamma} f(z)dz = \lim_{n\to\infty}\sum_{k=0}^{n}\lim_{q\to 1^-}\left\{f(z(aq^k))[z(aq^k) - z(aq^{k+1})]\right\}$$

$$= \lim_{q\to 1^-}\sum_{k=0}^{\infty} f(z(aq^k))[z(aq^k) - z(aq^{k+1})]$$

$$= \lim_{q\to 1^-}\int_{\gamma} f(z)d_q z$$

can be written. Here, the series is a numerical series which is convergent. So, the limit and summation can be interchanged. □

Example 2.1 If we use (2.12) for $\gamma : z(t) = t^2 + it, \ 0 \leq t \leq 2$ and the function $f(z) = \overline{z}$ for $0 < q < 1$, we get

$$\int_{\gamma} \overline{z} d_q z = \int_{t=0}^{2} (t^2 - it)D_q z(t)d_q t = \int_0^2 (t^2 - it)[t(1+q) + i]d_q t$$

$$= 4\left(\frac{4}{1+q^2} - \frac{2i}{1+q+q^2} + \frac{1}{1+q}\right).$$

Note that the result above is the same as the following classical integral result for $q \to 1^-$;

$$\lim_{q\to 1^-}\int_{\gamma} \overline{z} d_q z = \int_{\gamma} \overline{z} dz = 10 - \frac{8i}{3}.$$

Let's assume that the rectifiable piecewise smooth curve

$$\gamma : z(t) = x(t) + iy(t),\ 0 \le t \le a$$

is positively oriented. In that case, the initial point of γ is $z(0) = z_0 = x(0) + iy(0)$ and the final point of γ is $z(a) = z_a = x(a) + iy(a)$.

Let $f(z)$ be a q-integrable function on γ. Let $\widetilde{\gamma}$ be γ with opposite direction of γ, like classically. For the points $aq^n \in [0,a], n = 0,1,2,...$ there correspond $z_n = z(aq^n) = x(aq^n) + iy(aq^n)$, $n = 0,1,2,....$ Observe that z_n are on $\widetilde{\gamma}$, too. In addition, $f(z(aq^n))$ values do not change at the partition $\{z_n = z(aq^n)\}, n = 0,1,2,...$ points on both γ and $\widetilde{\gamma}$. Hence, the line q-integral of $f(z)$ on $\widetilde{\gamma}$ is defined by

$$\int_{\widetilde{\gamma}} f(z)d_qz := (q-1)a\sum_{n=0}^{\infty} q^n f(z(aq^n))D_qz(aq^n) = -\int_{\gamma} f(z)d_qz. \tag{2.16}$$

Let us consider the points $z_n = x(aq^n) + iy(aq^n), n = 0,1,2,...$ on the curve γ, and let $\ell_n := |z(aq^{n+1}) - z(aq^n)|$ for $n = 0,1,2,....$ By using the definition of the Jackson integral, we can write

$$\begin{aligned}\int_0^a |D_qz(t)|d_qt &= (1-q)a\sum_{n=0}^{\infty}\frac{|z(aq^{n+1}) - z(aq^n)|}{|aq^{n+1} - aq^n|}q^n \\ &= \sum_{n=0}^{\infty}|z_{n+1} - z_n| \\ &= \sum_{n=0}^{\infty}\ell_n.\end{aligned} \tag{2.17}$$

For a rectifiable curve γ, the series in (2.17) is convergent. Let us now consider the complex function $f(z) = u(z) + iv(z)$ which is q-uniformly continuous and q-regular at the origin on curve the γ. Here, for $t \in [0,a], 0 < q < 1$, $u = u(z(t)) = u(x(t),y(t))$ and $v = v(z(t)) = v(x(t),y(t))$ are functions and $D_qx(t), D_qy(t)$ derivatives are q-integrable on $I = [0,a]$. Moreover, $u(z(t))$ and $v(z(t))$ are q-uniformly continuous because $f(z(t))$ is q-uniformly continuous. On the other hand, for $z_k = z(aq^k)$, we consider the following finite sums:

$$S_1^{(n)} = \sum_{k=0}^{n} f(z_k)\ell_k, \tag{2.18}$$

$$S_2^{(n)} = \sum_{k=0}^{n} u(z_k)\Delta_q x(aq^k), \tag{2.19}$$

$$S_3^{(n)} = \sum_{k=0}^{n} v(z_k)\Delta_q y(aq^k). \tag{2.20}$$

Here $\Delta_q x(aq^k) := x(aq^k) - x(aq^{k+1})$, $\Delta_q y(aq^k) := y(aq^k) - y(aq^{k+1})$. These representations are used for time scales in [2]. Also, let us consider the following integrals

$$I_1 = \int_0^a f(z(t))|D_q z(t)|d_q t, \tag{2.21}$$

$$I_2 = \int_0^a u(z(t))D_q x(t)d_q t, \tag{2.22}$$

$$I_3 = \int_0^a v(z(t))D_q y(t)d_q t. \tag{2.23}$$

By using (2.22) and (2.23), if we add them, we get

$$\begin{aligned} I_2 + I_3 &= \int_0^a [u(x(t),y(t))D_q x(t) + v(x(t),y(t))D_q y(t)]d_q t \qquad (2.24)\\ :&= \int_\gamma u(x,y)d_q x + v(x,y)d_q y. \end{aligned}$$

Definition 2.8 *(2.21) is called the first type line q-integral of $f(z)$ and (2.24) is called the second type line q-integral of the pair (u,v).*

Since $f(z)$ is q-uniformly continuous on γ, the number $L_1 = \sup_\gamma |f(z)|$ exists. Also, $\sum_{k=0}^\infty \ell_k = \ell_q$ and the number $\ell_q \in (0,+\infty)$ exists. Then, for every $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$, for sufficiently large n, $|\sum_{k=0}^\infty \ell_k - \ell_q| < \frac{\varepsilon}{L_1}$. In this case, for any $\varepsilon > 0$ and for sufficiently large n, we get

$$\begin{aligned} |S_1^{(n)} - I_1| &= \left| \sum_{k=0}^n f(z(aq^k)) - (1-q)a \sum_{k=0}^\infty q^k f(z(aq^k)) \left| \frac{z(aq^{k+1}) - z(aq^k)}{(q-1)aq^k} \right| \right| \\ &\leq \sum_{k=n+1}^\infty |f(z(aq^k))||z(aq^{k+1}) - z(aq^k)| \\ &\leq L_1 \sum_{k=n+1}^\infty \ell_k \\ &= L_1 |\ell_q - (\ell_0 + \ell_1 + \ldots + \ell_n)| \\ &< L_1 \frac{\varepsilon}{L_1} \\ &= \varepsilon. \end{aligned}$$

Then we get the following result:

$$\lim_{n\to\infty} S_1^{(n)} = \sum_{k=0}^{\infty} f(z(aq^n))\ell_k = \int_0^a f(z(t))|D_q z(t)|d_q t = I_1. \tag{2.25}$$

On the other hand, since $u(z)$ is q-uniformly continuous on γ, there exists a number $L_2 \in (0,+\infty)$ such that

$$\sup_{\gamma} |u(z)| = \sup_{[0,a]} |u(z(t))| = L_2. \tag{2.26}$$

Hence, we can write

$$\begin{aligned}
|I_2 - S_2^{(n)}| &= \left| \int_0^a u(z(t))D_q x(t)d_q t - \sum_{k=0}^{n} u(z_k)\Delta_q x(aq^k) \right| \\
&= \left| (1-q)a \sum_{k=0}^{\infty} q^k u(z(aq^k)) \left[\frac{x(aq^{k+1}) - x(aq^k))}{(q-1)aq^k} \right] \right. \\
&\qquad \left. - \sum_{k=0}^{\infty} u(z_k)[x(aq^k) - x(aq^{k+1})] \right| \\
&= \left| \sum_{k=n+1}^{\infty} u(z(aq^k)) \left[x(aq^k) - x(aq^{k+1}) \right] \right| \\
&\le L_2 \sum_{k=n+1}^{\infty} |x(aq^k) - x(aq^{k+1})| \\
&\le L_2 \sum_{k=n+1}^{\infty} \ell_k.
\end{aligned}$$

Since $\sum_{k=0}^{\infty} \ell_k$ is convergent, $\sum_{k=0}^{\infty} |x(aq^k) - x(aq^{k+1})|$ is also convergent. So, for the remainder term of this series,

$$\sum_{k=n+1}^{\infty} |x(aq^k) - x(aq^{k+1})| < \frac{\varepsilon}{L_2},$$

for some $\varepsilon > 0$. Thus, we have

$$|I_2 - S_2^{(n)}| < \varepsilon.$$

Therefore

$$\lim_{n\to\infty} S_2^{(n)} = \sum_{k=0}^{\infty} u(z(aq^k))\Delta_q x(aq^k) = \int_0^a u(z(t))D_q x(t)d_q t = I_2. \tag{2.27}$$

By using the same argument, it can be shown that

$$I_3 = \lim_{n\to\infty} S_3^{(n)} = \sum_{k=0}^{\infty} v(z(aq^k))\Delta_q y(aq^k) = \int_0^a v(z(t))D_q y(t)d_q t. \tag{2.28}$$

On the other hand,

$$\begin{aligned}\int_\gamma u(x,y)d_q x = \int_\gamma u(z)d_q x = \int_0^a u(z(t))d_q x(t) = \int_0^a u(x(t),y(t))D_q x(t)d_q t\\ = (1-q)a\sum_{n=0}^{\infty} q^n u(x(aq^n),y(aq^n))\frac{x(aq^{n+1})-x(aq^n)}{(q-1)aq^n}\\ = \sum_{n=0}^{\infty} u(z(aq^n))\Delta_q x(aq^n) \end{aligned}\tag{2.29}$$

and by the similarly, we get

$$\begin{aligned}\int_\gamma v(x,y)d_q y = \int_0^a v(x(t),y(t))d_q y(t) = \int_0^a v(x(t),y(t))D_q y(t)d_q t\\ = \sum_{n=0}^{\infty} v(z(aq^n))\Delta_q y(aq^n). \end{aligned}\tag{2.30}$$

From (2.29) and (2.30), we have

$$\int_\gamma u(x,y)d_q x + v(x,y)d_q y = \sum_{n=0}^{\infty}[u(z(aq^n))\Delta_q x(aq^n) + v(z(aq^n))\Delta_q y(aq^n)]. \tag{2.31}$$

3 Multiple q-integrals

Let $h(x,y)$ be a function given in $D = \left\{(x,y)\in\mathbb{R}^2 : 0\le x\le a, 0\le y\le a\right\}$. For $(x,y)\in D$, the multiple q-integral of $h(x,y)$ with $0<q<1$ can be given by,

$$I = \int_0^x \int_0^y h(s,t)d_q t d_q s = (1-q)^2 xy \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{n+k} h(q^n x, q^k y). \tag{3.1}$$

Now, let us investigate under which conditions the series (3.1) will be convergent. For all $(x,y)\in D$, if $M\in(0,+\infty)$ exists such that $|h(x,y)|\le M$, integral (3.1) always exists. However, even if $h(x,y)$ is unbounded, integral (3.1) can exist as well.

The following theorem is related to a function this situation:

Theorem 3.1 *Let $h(x,y)$ be a function given in $D=\{(x,y)\in\mathbb{R}^2:0\le x\le a,0\le y\le a\}$. Assume that for all $(x,y)\in(0,a]\times(0,a]$, $0\le\alpha<1$, $0\le\beta<1$ there is a positive M such that*

$$x^{\alpha}y^{\beta}|h(x,y)|<M. \tag{3.2}$$

Then, the series (3.1) is convergent for all $(x,y)\in(0,a]\times(0,a]$.

Proof. From (3.2), for $x>0$, $y>0$, we have

$$|h(x,y)|<Mx^{-\alpha}y^{-\beta}. \tag{3.3}$$

In (3.3), if we replace x with q^nx and y with q^ky, we get

$$|h(q^nx,q^ky)|<M(q^nx)^{-\alpha}(q^ky)^{-\beta}=Mq^{-n\alpha}x^{-\alpha}q^{-k\beta}y^{-\beta}. \tag{3.4}$$

Thus,

$$q^{n+k}|h(q^nx,q^ky)|<Mq^{(1-\alpha)n}q^{(1-\beta)k}x^{-\alpha}y^{-\beta}. \tag{3.5}$$

For $0<q<1$, we have $0<q^{1-\alpha}<1,0<q^{1-\beta}<1$. The geometric series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}Mx^{-\alpha}y^{-\beta}q^{(1-\alpha)n}q^{(1-\beta)k}=\frac{Mx^{-\alpha}y^{-\beta}}{(1-q^{1-\alpha})(1-q^{1-\beta})} \tag{3.6}$$

is pointwise convergent. Hence, for all $(x,y)\in(0,a]\times(0,a]$, we have

$$(1-q)^2xy\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}q^{n+k}|h(x,y)|<\frac{Mx^{1-\alpha}y^{1-\beta}(1-q)^2}{(1-q^{1-\alpha})(1-q^{1-\beta})}. \tag{3.7}$$

Finally, the series in (3.1) is absolute convergent, therefore it is convergent. □

Remark 3.1 The condition (3.2) is sufficient but not necessary for the convergence of the series in (3.1).

Geometrically, the double integral

$$\int_0^a\int_0^a h(x,y)d_qxd_qy=(1-q)^2a^2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}q^{n+k}h(aq^n,aq^k) \tag{3.8}$$

means the sum of the function values at right top corner of rectangles $D_{n,k}=\{(x,y)\in\mathbb{R}^2:aq^{n+1}\le x\le aq^n,aq^{k+1}\le y\le aq^k,0<q<1\},(n,k:0,1,\dots)$ multiplied with the area of $D_{n,k}$. On the other hand, it is easy to see that

$$\bigcup_{n,k=0}^{\infty}D_{n,k}=(0,a]\times(0,a]. \tag{3.9}$$

Remark 3.2 For all $(x,y)\in D$ if we take $h(x,y)=1$, then

$$\int_0^a\int_0^a d_qxd_qy=a^2=Area(D). \tag{3.10}$$

Also, by considering the geometric meaning of the multiple q-integral we can see that

$$\lim_{q\to 1^-}\int_0^a\int_0^a h(x,y)d_qxd_qy=\int_0^a\int_0^a h(x,y)dxdy. \tag{3.11}$$

Definition 3.2 *Let $h(x,y)$ be a function on a q-geometrical set $D=\left\{(x,y)\in\mathbb{R}^2 : 0\le x\le a, 0\le y\le a\right\}$ with $0<q<1$. For all $(x,y)\in D$ and $x\neq 0, y\neq 0$, if*

$$\lim_{n\to\infty} h(q^n x,y)=h(0,y) \tag{3.12}$$

and

$$\lim_{k\to\infty} h(x,q^k y)=h(x,0) \tag{3.13}$$

hold, then $h(x,y)$ is called q-regular, respectively, on line segments $D_1=\left\{(0,y)\in\mathbb{R}^2 : 0<y\le a\right\}$ and $D_2=\left\{(x,0)\in\mathbb{R}^2 : 0<x\le a\right\}$.
Also, if

$$\lim_{n,k\to\infty} h(q^n x,q^k y):=\lim_{n\to\infty}[\lim_{k\to\infty} h(q^n x,q^k y)]=\lim_{k\to\infty}[\lim_{n\to\infty} h(q^n x,q^k y)]=h(0,0), \tag{3.14}$$

then $h(x,y)$ is called q-regular at the point $(0,0)$.

Remark 3.3 If $h(x,y)$ is not q-regular on $D_1=\left\{(0,y)\in\mathbb{R}^2 : 0<y\le a\right\}$, $D_2=\left\{(x,0)\in\mathbb{R}^2 : 0<x\le a\right\}$ and at $(0,0)$, the q-integral which is defined by (3.1) may not be true. If $h(x,y)$ is q-regular on D_1, D_2 and also at $(0,0)$ and if it equals zero at these points, then (3.1) is always true.

Remark 3.4 Let $F(x,y)$ satisfy that $D_q^x D_q^y F(x,y)=h(x,y)$. In this case the sufficient condition for

$$\int_0^x\int_0^y h(s,t)d_qtd_qs=F(x,y) \tag{3.15}$$

is that $F(x,y)$ be q-regular on $D_1=\left\{(0,y)\in\mathbb{R}^2 : 0<y\le a\right\}$, $D_2=\left\{(x,0)\in\mathbb{R}^2 : 0<x\le a\right\}$ and at $(0,0)$, and $F(x,y)$ be zero at these points. Under these conditions, we have

$$\int_0^x\int_0^y h(s,t)d_qtd_qs=F(x,y)=(1-q)^2xy\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{n+k}h(q^n x,q^k y). \tag{3.16}$$

Example 3.1 Let $F(x,y) = \dfrac{q-1}{(1+q)\ln q} x^2 \ln y$ be. For $x > 0,\ y > 0$, it is clear that

$$\begin{aligned} D_q^x D_q^y F(x,y) &= \frac{q-1}{(1+q)logq}\left[D_q^x x^2\right]\left[D_q^y logy\right] \\ &= \frac{q-1}{(1+q)logq}\frac{1-q^2}{(1-q)}x\frac{logq}{(q-1)}\frac{1}{y} \\ &= \frac{x}{y} = h(x,y). \end{aligned}$$

So, by using (3.16), we get

$$\int_0^x \int_0^y \frac{s}{t} d_q t d_q s = \frac{q-1}{(1+q)\ln q} x^2 \ln y.$$

On the other hand, by considering the definition of the Jackson Integral, we get for $0 < q < 1$

$$\begin{aligned} \int_0^x \int_0^y \frac{s}{t} d_q t d_q s &= \left(\int_0^x \frac{d_q t}{t}\right)\left(\int_0^y s d_q s\right) \\ &= (1-q)x \sum_{n=0}^{\infty} q^n \frac{1}{q^n x}(1-q)y \sum_{k=0}^{\infty} q^k q^k y \\ &= (1-q)^2 y^2 \frac{1}{1-q^2} \sum_{n=0}^{\infty} 1 = +\infty. \end{aligned}$$

However, this is not true, because $F(x,y)$ is not q- regular on D_1.
If $h(x,y)$ is q-integrable with respect to both x and y on $D^* = (0,a] \times (0,a]$, we have

$$\int_0^x \int_0^y h(s,t) d_q t d_q s = \int_0^y \int_0^x h(s,t) d_q s d_q t. \tag{3.17}$$

Now, let us define the function

$$F(x,y) = \int_0^x \int_0^y h(s,t) d_q t d_q s. \tag{3.18}$$

Now, we will find the conditions under which the following equality is true:

$$D_q^x D_q^y \int_0^x \int_0^y h(s,t) d_q t d_q s = \int_0^x \int_0^y D_q^s D_q^t h(s,t) d_q t d_q s. \tag{3.19}$$

For $x \neq 0$ and $y \neq 0$, we get

$$
\begin{aligned}
D_q^x D_q^y F(x,y) &= D_q^x D_q^y \int_0^x \int_0^y h(s,t)d_qt d_qs \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{n+k}\Big[h(q^n x, q^k y) - qh(q^n x, q^{k+1}y) \\
&\qquad -qh(q^{n+1}x, q^k y) + q^2 h(q^{n+1}x, q^{k+1}y)\Big] \\
&= h(x,y),
\end{aligned} \tag{3.20}
$$

by simple calculations.
On the other hand, if we compute the right side of (3.19), we get

$$
\begin{aligned}
\int_0^x \int_0^y D_q^t h(s,t)d_qt d_qs &= \int_0^x \int_0^y \frac{h(s,qt)-h(s,t)}{(q-1)t} d_qt d_qs \\
&= \int_0^x (1-q)y \sum_{k=0}^{\infty} q^k \left[\frac{h(s,q^{k+1}y)-h(s,q^k y)}{(q-1)q^k y}\right] d_qs \\
&= \int_0^x \sum_{k=0}^{\infty} [h(s,q^k y) - h(s,q^{k+1}y)]d_qs.
\end{aligned} \tag{3.21}
$$

Nextly, we see that

$$
\sum_{k=0}^{N} [h(s,q^k y) - h(s,q^{k+1}y)] = h(s,y) - h(s,q^{N+1}y) \tag{3.22}
$$

$$
\lim_{N\to\infty} \sum_{k=0}^{N} h(s,q^k y) - h(s,q^{k+1}y) = h(s,y) - h(s,0). \tag{3.23}
$$

Therefore, we have

$$
\begin{aligned}
\int_0^x \int_0^y D_q^t h(s,t)d_qt d_qs &= \int_0^x \sum_{k=0}^{\infty} [h(s,q^k y) - h(s,q^{k+1}y)]d_qs \\
&= \int_0^x [h(s,y) - h(s,0)]d_qs \\
&= (1-q)x \sum_{n=0}^{\infty} q^n [h(q^n x, y) - h(q^n x, 0)].
\end{aligned} \tag{3.24}
$$

Finally, we get

$$\int_0^x\int_0^y D_q^s D_q^t h(s,t)d_qt d_qs = \int_0^x D_q^s[h(s,y)-h(s,0)]d_qs$$
$$= h(x,y)-h(x,0)-h(0,y)+h(0,0).$$

So, we can give the following lemma:

Lemma 3.3 *Let $h(x,y)$ be q-regular on $D_1 = \{(0,y)\in\mathbb{R}^2 : 0<y\le a\}$, $D_2 = \{(x,0)\in\mathbb{R}^2 : 0<x\le a\}$ and at the point $(0,0)$.*
If $h(0,y)=h(x,0)=h(0,0)=0$, then

$$D_q^x D_q^y \int_0^x\int_0^y h(s,t)d_qt d_qs = \int_0^x\int_0^y D_q^s D_q^t h(s,t)d_qt d_qs. \tag{3.25}$$

Theorem 3.4 *Let the real valued functions $u(x,y), v(x,y)$ be given on set $D = [0,a]\times[0,a]$ and let the functions $u, v, D_q^x v, D_q^y u$ be q-integrable on D with respect to both x and y. Then the q-Green's formula*

$$\int_{\partial D} u(x,y)d_qx + v(x,y)d_qy = \iint_D [D_q^x v(x,y) - D_q^y u(x,y)]d_qx d_qy \tag{3.26}$$

is valid. Here ∂D is oriented positively.

Proof. Let $\partial D = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ be. Here, γ_1 is following form that
$\gamma_1 : x=t, 0\le t\le a, y=0, d_qx(t)=d_qt, d_qy(t)=0.$
Then, if the left side of (3.26) is calculated on γ_1,

$$\int_{\gamma_1} u(x,y)d_qx + v(x,y)d_qy = \int_0^a u(t,0)d_qt. \tag{3.27}$$

By the same way,
$\gamma_2 : x=a, y=t$, $0\le t\le a$, $d_qx(t)=0$, $d_qy(t)=d_qt$, and

$$\int_{\gamma_2} u(x,y)d_qx + v(x,y)d_qy = \int_0^a v(a,t)d_qt. \tag{3.28}$$

On the other hand, the curve γ_3 is the curve $\widetilde{\gamma}_3 : x=t$, $y=a$, $0\le t\le a$ oriented oppositely. Hence from (2.16) on $\widetilde{\gamma}_3$, $d_qy=0$ and $d_qx=d_qt$, we have

$$\int_{\gamma_3} u(x,y)d_qx + v(x,y)d_qy = -\int_{\widetilde{\gamma}_3} u(x,y)d_qx + v(x,y)d_qy = -\int_0^a u(t,a)d_qt. \tag{3.29}$$

By the same way, the curve γ_4 is the curve $\widetilde{\gamma}_4 : x = 0,\ y = t,\ 0 \leq t \leq a$ oriented oppositely. Thus,

$$\int_{\gamma_4} u(x,y)d_qx + v(x,y)d_qy = -\int_{\widetilde{\gamma}_4} u(x,y)d_qx + v(x,y)d_qy = -\int_0^a v(0,t)d_qt. \quad (3.30)$$

Hence we have,

$$\int_{\partial D} u(x,y)d_qx + v(x,y)d_qy = \int_0^a [u(t,0) + v(a,t) - u(t,a) - v(0,t)]d_qt. \quad (3.31)$$

On the other hand, from (3.24) we can write

$$\iint_D D_q^x v(x,y)d_qxd_qy = \int_0^a \int_0^a D_q^x v(x,y)d_qxd_qy = \int_0^a [v(a,t) - v(0,t)]d_qt \quad (3.32)$$

and

$$\iint_D D_q^y u(x,y)d_qxd_qy = \int_0^a \int_0^a D_q^y u(x,y)d_qxd_qy = \int_0^a [u(t,a) - u(t,0)]d_qt. \quad (3.33)$$

By using (3.32) and (3.33), we have

$$\iint_D [D_q^x v(x,y) - D_q^y u(x,y)]d_qxd_qy = \int_0^a [v(a,t) - v(0,t) - u(t,a) + u(t,0)]d_qt. \quad (3.34)$$

If we compare (3.31) and (3.34), it is seen that the q-Green's formula (3.26) holds.
□

Remark 3.5 Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be a complex valued function in D. The partial q-derivatives of $f(z)$ with respect to x and y are

$$\begin{aligned} D_q^x f(z) &= \frac{f(x+iy) - f(qx+iy)}{(1-q)x} = \frac{u(x,y) + iv(x,y) - u(qx,y) - iv(qx,y)}{(1-q)x} \\ &= D_q^x u + iD_q^x v \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} D_q^y f(z) &= \frac{f(x+iy) - f(x+iqy)}{(1-q)iy} \\ &= -iD_q^y u + D_q^y v. \end{aligned} \quad (3.36)$$

Then, the partial q-derivatives of $f(z)$ with respect to z and $\bar{z}$ are

$$D_q^z f(z) = \frac{1}{2}[D_q^x f(z) - iD_q^y f(z)] , \quad (3.37)$$

$$D_q^{\bar{z}} f(z) = \frac{1}{2}[D_q^x f(z) + D_q^y f(z)]. \tag{3.38}$$

Hence, the complex partial q-derivative operators are given as;

$$D_q^z = \frac{1}{2}(D_q^x - iD_q^y) \tag{3.39}$$

$$D_q^{\bar{z}} = \frac{1}{2}(D_q^x + iD_q^y). \tag{3.40}$$

Definition 3.5 [5] *Let $f(z)$ be a function in D. If $D_q^{\bar{z}} f(z) = 0$ holds, then the function $f(z)$ is called q-analytic in D.*

Example 3.2 $f(z) = \frac{x^2-y^2}{1+q} + ixy$ is a q-analytic function.

If $f(z) = u(x,y) + iv(x,y)$ is a q-analytic function in D, from definition, the q-Cauchy Riemann system for u, v is

$$D_q^x u(x,y) = D_q^y v(x,y) \ , \ D_q^y u(x,y) = -D_q^x v(x,y). \tag{3.41}$$

From (3.41) and $\Delta_q := \Delta_q^x + \Delta_q^y \equiv D_q^x D_q^x + D_q^y D_q^y$, it can be seen that the q-Laplace equations

$$\Delta_q u = 0 \ , \ \Delta_q v = 0 \tag{3.42}$$

are valid.

Theorem 3.6 *Let D and ∂D be as in Theorem 3. If $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ has the partial q-derivatives with respect to z and $\bar{z}$, and these q-derivatives are q-integrable, then*

$$\int_{\partial D} f(z) d_q z = 2i \iint_D D_q^{\bar{z}} f(z) d_q x d_q y, \tag{3.43}$$

$$\int_{\partial D} f(z) d_q \bar{z} = -2i \iint_D D_q^z f(z) d_q x d_q y. \tag{3.44}$$

Proof. The proof of theorem is seen by directly calculations. □

Definition 3.7 *(3.43) and (3.44) are called the q-Gauss formulas in the complex plane.*

Remark 3.6 If the complex valued function $f(z)$ is q-analytic in $D = [0,a] \times [0,a]$, then for $\gamma = \partial D$, because of $D_{\bar{z}} f(z) = 0$, and from (3.43) we have

$$\int_\gamma f(z) d_q z = 0. \tag{3.45}$$

Proof. The proof is clear since $D_{\bar{z}} f(z) = 0$, from (3.43). □

If $f(z)$ and $g(z)$, which are complex valued functions, satisfy the condition of the Theorem 4, by using (3.43) and (3.44), we have

$$\int\limits_{\gamma} f(z)d_qz + g(z)d_q\bar{z} = 2i \iint\limits_{D} \left[D_q^{\bar{z}} f(z) - D_q^{z} g(z)\right] d_qxd_qy. \tag{3.46}$$

Remark 3.7 In [7], the partial q-derivatives of $f(z)$ with respect to z and $\bar{z}$, are given as

$$D_z f = \frac{1}{2}\left[D_q^x f(z) - iM_{\frac{1}{q}} D_q^y f(z)\right], \tag{3.47}$$

$$D_{\bar{z}} f = \frac{1}{2}\left[D_q^x f(z) + iM_{\frac{1}{q}} D_q^y f(z)\right]. \tag{3.48}$$

Here is $M_q^y f(x,y) = f(x,qy)$. Moreover, the q-analyticity of $f(z)$ is defined as $D_{\bar{z}} f = 0$. However, the Theorem 3 cannot be used, because, the operator $M_{\frac{1}{q}}$ changes the definition set of $f(z)$.

Remark 3.8 If the real valued function $v(x,y)$ is q-harmonic, i.e. it satisfies $D_q^x D_q^x v(x,y) + D_q^y D_q^y v(x,y) = 0$, then $u(x,y)$, which is q-conjugate of $v(x,y)$, can be calculated for $y \neq 0$ by using

$$\begin{aligned} u(x,y) = {} & \frac{x}{y} \sum_{n=0}^{\infty} q^n \left[v(q^n x, y) - v(q^n x, qy)\right] \\ & - \lim_{n\to\infty} \frac{(1-q)y}{q^n x} \sum_{k=0}^{\infty} q^k \left[v(q^n x, q^k y) - v(0, q^k y)\right] + c \end{aligned} \tag{3.49}$$

where c is an arbitrary real constant.

Example 3.3 $v(x,y) = ax + by + c, (a,b,c$ are real constants) is q-harmonic. By using (3.49), the q-conjugate of $v(x,y)$ can be found as $u(x,y) = bx - ay + k$, where k is a real constant. Hence, $f(z) = bx - ay + k + i(ax + by + c)$ is q-analytic, in addition being an analytic function.

Example 3.4 $v(x,y) = x(1-y)$ is q-harmonic. By using (3.49), the q-conjugate of $v(x,y)$ can be found as $u(x,y) = \dfrac{y^2 - x^2}{1+q} - y + c$.

Hence, $f(z) = \dfrac{y^2 - x^2}{1+q} - y + c + ix(1-y)$ is q-analytic.

Remark 3.9 In this paper, we present examples of quadratic q-analytic functions because we would like to compare the convergence value of the series in the Jackson Integral with the classical value. One can study convergence of the series in the Jackson Integral for a broader class of q-analytic functions, however, it might be difficult to find the convergence value.

Acknowledgements The authors thank to the referee for his suggestions which helped improve the paper.

References

1. ANNABY, M.H.; MANSOUR, Z.S. – *q-Fractional calculus and equations*, Lecture Notes in Mathematics 2056, Springer, Heidelberg (2012), 318 pp.
2. BOHNER, M.; GUSEINOV G.SH. – *Line integrals and Green's formula on time scales*, J. Math. Anal. Appl. **326** (2007), no. 2, 1124-1141.
3. ERNST, T. – *A comprehensive treatment of q-calculus*, Birkhäuser/Springer Basel AG (2012), 491 pp.
4. ERNST, T. – *q-complex numbers, a natural consequence of umbral calculus*, U. U. D. M. Report, **44** (2007).
5. HARMAN, C.J. – *Discrete geometric function theory. I*, Applicable Analysis, **7** (1978), 315-336.
6. KAC, V.; CHEUNG, P. – *Quantum calculus*, Universitext, Springer-Verlag, New York (2002), 112 pp.
7. PASHAEV, O.K.; NALCI, S. – *q-analytic functions, fractals and generalized analytic functions*, J. Phys. A, **47** (2014), no. 4, 25 pp.

Received: 13.VII.2015 / Revised: 29.I.2016 / Accepted: 28.IV.2016

AUTHORS

KERIM KOCA,
Department of Mathematics,
Faculty of Science and Arts,
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey,
E-mail: kerimkoca@gmail.com

İLKER GENÇTÜRK (Corresponding author),
Department of Mathematics,
Faculty of Science and Arts,
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey,
E-mail: ilkergencturk@gmail.com

MUSTAFA AYDIN,
Department of Mathematics,
Faculty of Science and Arts,
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey,
E-mail: mustafa.aydin868@gmail.com