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Lorentzian stationary surfaces and null curves in 4-dimensional space forms of index 2

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Abstract We discuss a property of the Gaussian curvature and the normal curvature of Lorentzian stationary surfaces in 4-dimensional semi-Riemannian space forms of index 2, which is related to a property of null curves via a characteristic initial value problem.

Keywords Lorentzian stationary surface · null curve · space form

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1 Introduction

Recently, the geometry of Lorentzian surfaces in 4-dimensional semi-Riemannian space forms of index 2 has been studied actively (cf. [1], [2], [4], [5], [8], [12], [13], [16]). In this paper, we will study Lorentzian surfaces with zero mean curvature vector in 4-dimensional space forms of index 2.

Let $N^n_q(c)$ denote the n-dimensional semi-Riemannian space form of constant curvature c and index q. Namely, it is either the n-dimensional semi-Euclidean space R^n_q of index q, the n-dimensional pseudo-sphere $S^n_q(c)$ of constant curvature $c > 0$ and index q, or the n-dimensional pseudo-hyperbolic space $H^n_q(c)$ of constant curvature $c < 0$ and index q. We shall write $N^n_0(c) = N^n(c)$. A surface in $N^n_q(c)$ is called Lorentzian if the induced metric is Lorentzian. We say that a Lorentzian surface is stationary if it has zero mean curvature vector.

For a minimal surface in $N^4(c)$ and a spacelike maximal surface in $N^4_2(c)$, the Gaussian curvature K and the normal curvature K_ν satisfy $(K - c)^2 - K^2_\nu \geq 0$, where the equality holds at isotropic points. Those surfaces have similar geometric properties on the rigidity, isometric deformations and curvatures (cf. [14] and [10]). A related work can be seen also in [7].

For a Lorentzian stationary surface in $N^4_2(c)$, the value $(K - c)^2 - K^2_\nu$ also plays an important role, but its sign is not fixed. According to the sign of $(K-c)^2 - K_\nu^2$, there are different situations on the rigidity and isometric deformability (see [12]). So it should be important to investigate the geometric meaning of the sign of $(K-c)^2 - K_\nu^2$. In a previous paper [13], we consider the case where $c = 0$, and discuss the sign of $K^2 - K_\nu^2$ for Lorentzian stationary surfaces in R_2^4, from the view point of the geometry of null curves in R_2^4. Here we will consider the case where $c \neq 0$.

Let $\langle\ ,\ \rangle$ denote the inner product of the semi-Euclidean space R_q^n. When $c > 0$, $N_2^4(c)$ is the pseudo-sphere $S_2^4(c)$ which is given as

$$S_2^4(c) = \{x \in R_2^5 | \langle x, x\rangle = 1/c\}.$$

When $c < 0$, $N_2^4(c)$ is the pseudo-hyperbolic space $H_2^4(c)$ which is given as

$$H_2^4(c) = \{x \in R_3^5 | \langle x, x\rangle = 1/c\}.$$

Let M be a Lorentzian surface in $N_2^4(c)$ $(c \neq 0)$, and f be the inclusion map. Then f is an R_2^5-valued map with $\langle f, f\rangle = 1/c$ when $c > 0$, and f is an R_3^5-valued map with $\langle f, f\rangle = 1/c$ when $c < 0$. There exists a local coordinate system (u, v) on M such that the induced metric g satisfies

$$g = a(du \otimes dv + dv \otimes du)$$

for some function $a \neq 0$ (cf. p.13 of [15], [6]). Then f_u and f_v are null, and $\langle f_u, f_v\rangle = a \neq 0$. Let H^* be the mean curvature vector of M as a surface in the ambient semi-Euclidean space $R_2^5(\supset S_2^4(c))$ or $R_3^5(\supset H_2^4(c))$. Then we have

$$H^* = \frac{f_{uv}}{\langle f_u, f_v\rangle}.$$

So M is stationary as a surface in $N_2^4(c)$ if and only if

$$f_{uv} = -c\langle f_u, f_v\rangle f. \tag{1.1}$$

Now we state the result as follows:

Theorem 1.1. *Let M be a Lorentzian stationary surface in $N_2^4(c)$ $(c \neq 0)$, and let $f(u, v)$ be the inclusion map such that f_u and f_v are null. Assume that $\{f_u, f_v, f_{uu}, f_{vv}\}$ are linearly independent. Then the Gaussian curvature K and the normal curvature K_ν satisfy*

$$(K-c)^2 - K_\nu^2 = \frac{\langle f_{uu}, f_{uu}\rangle\langle f_{vv}, f_{vv}\rangle}{\langle f_u, f_v\rangle^4}.$$

So, under the condition of Theorem 1.1, the sign of $(K-c)^2 - K_\nu^2$ is the same as that of $\langle f_{uu}, f_{uu}\rangle\langle f_{vv}, f_{vv}\rangle$. If $\{f_u, f_v, f_{uu}, f_{vv}\}$ are linearly dependent a point p, then $K_\nu = 0$ and $(K-c)^2 - K_\nu^2 = (K-c)^2 \geq 0$ at p.

By [11] we can see the existence of Lorentzian stationary surfaces in $N_2^4(c)$ satisfying $(K-c)^2 - K_\nu^2 = 0$. Next we show the existence of Lorentzian stationary surfaces in $N_2^4(c)$ such that $(K-c)^2 - K_\nu^2$ is positive, or negative.

Theorem 1.2. *Let $P(u)$ and $Q(v)$ be null curves in $N^4_2(c)$ ($c \neq 0$) which satisfy $P(0)=Q(0)$ and $\langle P'(0), Q'(0)\rangle \neq 0$. Suppose that $\{P'(0), Q'(0), P''(0), Q''(0)\}$ are linearly independent, and $\langle P''(0), P''(0)\rangle\langle Q''(0), Q''(0)\rangle > 0$ (resp. < 0). Then there exists a Lorentzian stationary surface in $N^4_2(c)$ whose inclusion map $f(u,v)$ satisfies f_u and f_v are null, $f(u,0) = P(u)$, $f(0,v) = Q(v)$, and $(K-c)^2 - K_\nu^2 > 0$ (resp. < 0) near $(u,v) = (0,0)$.*

In Section 2, we recall the method of moving frames and basic facts for Lorentzian surfaces in $N^4_2(c)$. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, using Theorem 1.2, we give examples. In Section 5, we give a remark on the right hand side of the equation in Theorem 1.1.

2 Preliminaries

We use the following convention on the ranges of indices:

$$1 \leq A, B, \dots \leq 4, \quad 1 \leq i, j, \dots \leq 2, \quad 3 \leq \alpha, \beta, \dots \leq 4.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N^4_2(c)$, and $\{\omega^A\}$ be the dual coframe field, so that the metric g of $N^4_2(c)$ is given by

$$g = \omega^1 \otimes \omega^1 - \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 - \omega^4 \otimes \omega^4.$$

The connection forms ω^A_B are given by

$$de_B = \sum_A \omega^A_B e_A.$$

Then $\omega^A_B = -\varepsilon_A \varepsilon_B \omega^B_A$, where $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = -1$. In other words, $\omega^A_B = -\omega^B_A$ if $|A-B|$ is even, and $\omega^A_B = \omega^B_A$ if $|A-B|$ is odd. The structure equations are given by

$$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \tag{2.1}$$

$$d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \frac{1}{2}\sum_{C,D} R^A{}_{BCD}\,\omega^C \wedge \omega^D, \tag{2.2}$$

$$R^A{}_{BCD} = c\varepsilon_B(\delta^A{}_C \delta_{BD} - \delta^A{}_D \delta_{BC}). \tag{2.3}$$

Let M be a Lorentzian surface in $N^4_2(c)$. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. In the following the argument will be restricted to M. Then $\omega^\alpha = 0$ on M, and by (2.1), we have

$$0 = -\sum_i \omega^\alpha_i \wedge \omega^i.$$

So there exists a symmetric tensor $\{h^\alpha_{ij}\}$ so that

$$\omega^\alpha_i = \sum_j h^\alpha_{ij}\omega^j, \tag{2.4}$$

where h^α_{ij} are the components of the second fundamental form h of M.

The Gaussian curvature K and the normal curvature K_ν of M are given by

$$d\omega^1_2 = -K\omega^1 \wedge \omega^2, \qquad d\omega^3_4 = -K_\nu\omega^1 \wedge \omega^2. \tag{2.5}$$

Then by (2.2), (2.3), (2.4) and (2.5), we have

$$K = c - h^3_{11}h^3_{22} + (h^3_{12})^2 + h^4_{11}h^4_{22} - (h^4_{12})^2, \tag{2.6}$$

and

$$K_\nu = h^3_{11}h^4_{12} - h^3_{12}h^4_{11} - h^3_{12}h^4_{22} + h^3_{22}h^4_{12}. \tag{2.7}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{2}\sum_\alpha (h^\alpha_{11} - h^\alpha_{22})e_\alpha.$$

We say that M is stationary if $H = 0$ on M. When M is stationary, by (2.6) and (2.7), we have

$$K = c - (h^3_{11})^2 + (h^3_{12})^2 + (h^4_{11})^2 - (h^4_{12})^2, \tag{2.8}$$

and

$$K_\nu = 2(h^3_{11}h^4_{12} - h^3_{12}h^4_{11}). \tag{2.9}$$

3 Proof of Theorems

Proof of Theorem 1.1. Let M be a Lorentzian stationary surface in $N^4_2(c)$ $(c \neq 0)$, and let $f(u, v)$ be the inclusion map such that f_u and f_v are null. Then the equation (1.1) is satisfied. We assume that $\{f_u, f_v, f_{uu}, f_{vv}\}$ are linearly independent.

Set $a = \langle f_u, f_v\rangle \neq 0$. Changing the parameter, we assume that $a > 0$. We note that the Gaussian curvature K is intrinsically given by

$$K = -\frac{1}{a}(\log a)_{uv} = -\frac{1}{a^2}\left(a_{uv} - \frac{a_u a_v}{a}\right). \tag{3.1}$$

Since f_u and f_v are null, and noting (1.1), we can see that

$$a_u = \langle f_{uu}, f_v\rangle, \qquad a_v = \langle f_u, f_{vv}\rangle, \tag{3.2}$$

$$\begin{aligned} a_{uv} = \langle f_{uuv}, f_v\rangle + \langle f_{uu}, f_{vv}\rangle &= \langle (-caf)_u, f_v\rangle + \langle f_{uu}, f_{vv}\rangle \\ &= -ca^2 + \langle f_{uu}, f_{vv}\rangle. \end{aligned} \tag{3.3}$$

Set

$$e_1 = \frac{1}{\sqrt{2a}}(f_u + f_v), \qquad e_2 = \frac{1}{\sqrt{2a}}(f_u - f_v). \tag{3.4}$$

Then $\{e_1, e_2\}$ is an orthonormal frame field on M of signature $(+,-)$. Set

$$V = f_{uu} - \frac{a_u}{a} f_u,$$

which is nonzero, because $\{f_u, f_v, f_{uu}, f_{vv}\}$ are linearly independent. Using (3.2), we see that

$$\langle V, f\rangle = \langle V, f_u\rangle = \langle V, f_v\rangle = 0, \qquad \langle V, V\rangle = \langle f_{uu}, f_{uu}\rangle.$$

So V is a nonzero normal vector to M.

(i) We consider the case where $\langle f_{uu}, f_{uu}\rangle > 0$. We can write $\langle f_{uu}, f_{uu}\rangle = r_1^2$ for some positive function r_1. Then the vector

$$e_3 = \frac{1}{r_1} V = \frac{1}{r_1}\left(f_{uu} - \frac{a_u}{a} f_u\right)$$

is a spacelike unit normal vector to M. Set

$$W = f_{vv} - \frac{a_v}{a} f_v - \frac{1}{r_1}\left(ca^2 + a_{uv} - \frac{a_u a_v}{a}\right) e_3,$$

which is nonzero, because $\{f_u, f_v, f_{uu}, f_{vv}\}$ are linearly independent. Using (3.2) and (3.3), we can see that

$$\langle W, f\rangle = \langle W, f_u\rangle = \langle W, f_v\rangle = \langle W, e_3\rangle = 0.$$

So W is a nonzero normal vector which is orthogonal to e_3. Since the normal plane is 2-dimensional Minkowskian, W is a timelike vector.

By (3.1) we can rewrite W as follows:

$$W = f_{vv} - \frac{a_v}{a} f_v + \frac{a^2}{r_1}(K - c) e_3.$$

Then, by (3.1), (3.2) and (3.3),

$$\langle W, W\rangle = \langle f_{vv}, f_{vv}\rangle - \frac{a^4}{r_1^2}(K - c)^2 =: -r_2^2 < 0, \tag{3.5}$$

for some positive function r_2, and the vector

$$e_4 = \frac{1}{r_2} W = \frac{1}{r_2}\left(f_{vv} - \frac{a_v}{a} f_v + \frac{a^2}{r_1}(K - c) e_3\right)$$

is a timelike unit normal vector which is orthogonal to e_3. So $\{e_3, e_4\}$ is a normal orthonormal frame field to M of signature $(+,-)$.

Using (3.1), (3.2), (3.3) and (3.5), we can compute that

$$\langle f_{uu}, e_3\rangle = r_1, \quad \langle f_{vv}, e_3\rangle = -\frac{a^2}{r_1}(K-c), \tag{3.6}$$

$$\langle f_{uu}, e_4\rangle = 0, \quad \langle f_{vv}, e_4\rangle = -r_2. \tag{3.7}$$

We denote by D the flat connection of R^5_2 when $c > 0$, or R^5_3 when $c < 0$. Using (3.4), (1.1), (3.6) and (3.7), we have the components of the second fundamental form h as follows:

$$\begin{aligned} h^3_{11} &= \langle D_{e_1}e_1, e_3\rangle = \frac{1}{2a}\langle f_{uu} + 2f_{uv} + f_{vv}, e_3\rangle \\ &= \frac{1}{2a}\left(r_1 - \frac{a^2}{r_1}(K-c)\right) = h^3_{22}, \\ h^3_{12} &= \langle D_{e_1}e_2, e_3\rangle = \frac{1}{2a}\langle f_{uu} - f_{vv}, e_3\rangle \\ &= \frac{1}{2a}\left(r_1 + \frac{a^2}{r_1}(K-c)\right), \end{aligned}$$

$$h^4_{11} = -\langle D_{e_1}e_1, e_4\rangle = \frac{r_2}{2a} = h^4_{22}, \quad h^4_{12} = -\langle D_{e_1}e_2, e_4\rangle = -\frac{r_2}{2a}.$$

By (2.9), the normal curvature K_ν is given by

$$K_\nu = -\frac{r_1 r_2}{a^2}. \tag{3.8}$$

By (3.5) and (3.8), we get

$$(K-c)^2 - K_\nu^2 = \frac{r_1^2}{a^4}\langle f_{vv}, f_{vv}\rangle = \frac{\langle f_{uu}, f_{uu}\rangle\langle f_{vv}, f_{vv}\rangle}{\langle f_u, f_v\rangle^4}.$$

(ii) Next we consider the case where $\langle f_{uu}, f_{uu}\rangle < 0$. We can write $\langle f_{uu}, f_{uu}\rangle = -r_1^2$ for some positive function r_1. Then the vector

$$e_4 = \frac{1}{r_1}V = \frac{1}{r_1}\left(f_{uu} - \frac{a_u}{a}f_u\right)$$

is a timelike unit normal vector to M. Set

$$\begin{aligned} W &= f_{vv} - \frac{a_v}{a}f_v + \frac{1}{r_1}\left(ca^2 + a_{uv} - \frac{a_u a_v}{a}\right)e_4 \\ &= f_{vv} - \frac{a_v}{a}f_v - \frac{a^2}{r_1}(K-c)e_4. \end{aligned}$$

Then we can see that W is a nonzero normal vector which is orthogonal to e_4. So W is spacelike, and

$$\langle W, W\rangle = \langle f_{vv}, f_{vv}\rangle + \frac{a^4}{r_1^2}(K-c)^2 =: r_2^2 > 0, \tag{3.9}$$

for some positive function r_2. Then the vector

$$e_3 = \frac{1}{r_2}W = \frac{1}{r_2}\left(f_{vv} - \frac{a_v}{a}f_v - \frac{a^2}{r_1}(K-c)e_4\right)$$

is a spacelike unit normal vector which is orthogonal to e_4. So $\{e_3, e_4\}$ is a normal orthonormal frame field to M of signature $(+,-)$.

Using (3.1), (3.2), (3.3) and (3.9), we can get

$$\langle f_{uu}, e_4\rangle = -r_1, \quad \langle f_{vv}, e_4\rangle = -\frac{a^2}{r_1}(K-c),$$

$$\langle f_{uu}, e_3\rangle = 0, \quad \langle f_{vv}, e_3\rangle = r_2.$$

Then, as in the case (i),

$$h^3_{11} = h^3_{22} = \frac{r_2}{2a}, \quad h^3_{12} = -\frac{r_2}{2a},$$

$$h^4_{11} = h^4_{22} = \frac{1}{2a}\left(r_1 + \frac{a^2}{r_1}(K-c)\right), \quad h^4_{12} = \frac{1}{2a}\left(r_1 - \frac{a^2}{r_1}(K-c)\right),$$

and

$$K_\nu = \frac{r_1 r_2}{a^2}. \tag{3.10}$$

By (3.9) and (3.10), we have

$$(K-c)^2 - K_\nu^2 = -\frac{r_1^2}{a^4}\langle f_{vv}, f_{vv}\rangle = \frac{\langle f_{uu}, f_{uu}\rangle\langle f_{vv}, f_{vv}\rangle}{\langle f_u, f_v\rangle^4}.$$

Remark. Changing the roles of u and v, we can show the theorem also in the case where $\langle f_{vv}, f_{vv}\rangle > 0$ or $\langle f_{vv}, f_{vv}\rangle < 0$.

(iii) We consider the case where both f_{uu} and f_{vv} are null at a point $p \in M$. Set

$$N_1 = f_{uu} - \frac{a_u}{a}f_u, \quad N_2 = f_{vv} - \frac{a_v}{a}f_v.$$

Then N_1 and N_2 are null and linearly independent normal vector at p. So $\langle N_1, N_2\rangle \neq 0$ at p, because the normal plane is 2-dimensional Minkowskian. By (3.1), (3.2) and (3.3), we have

$$\langle N_1, N_2\rangle = \langle f_{uu}, f_{vv}\rangle - \frac{a_u a_v}{a} = -a^2(K-c).$$

So $K \neq c$ at p. When $K < c$ at p, set

$$e_3 = \frac{1}{a\sqrt{2(c-K)}}(N_1 + N_2), \quad e_4 = \frac{1}{a\sqrt{2(c-K)}}(N_1 - N_2).$$

When $K > c$ at p, set

$$e_3 = \frac{1}{a\sqrt{2(K-c)}}(N_1 - N_2), \quad e_4 = \frac{1}{a\sqrt{2(K-c)}}(N_1 + N_2).$$

In each case, $\{e_3, e_4\}$ is a normal orthonormal basis at p of signature $(+, -)$.

Using (3.1), (3.2) and (3.3), we have at p,

$$\langle f_{uu}, N_1 \rangle = 0, \quad \langle f_{uu}, N_2 \rangle = -a^2(K-c),$$
$$\langle f_{vv}, N_1 \rangle = -a^2(K-c), \quad \langle f_{vv}, N_2 \rangle = 0.$$

So, when $K < c$ at p, we get

$$h^3_{11} = h^3_{22} = \sqrt{\frac{c-K}{2}}, \quad h^3_{12} = 0,$$

$$h^4_{11} = h^4_{22} = 0, \quad h^4_{12} = \sqrt{\frac{c-K}{2}},$$

and

$$K_\nu = c - K, \quad (K-c)^2 - K_\nu^2 = 0.$$

When $K > c$ at p, we get

$$h^3_{11} = h^3_{22} = 0, \quad h^3_{12} = \sqrt{\frac{K-c}{2}},$$

$$h^4_{11} = h^4_{22} = \sqrt{\frac{K-c}{2}}, \quad h^4_{12} = 0,$$

and

$$K_\nu = -(K-c), \quad (K-c)^2 - K_\nu^2 = 0.$$

Now, from (i), (ii) and (iii), we have proved Theorem 1.1.

Proof of Theorem 1.2. Let $P(u)$ and $Q(v)$ be null curves in $N^4_2(c)$ $(c \neq 0)$ which satisfy $P(0) = Q(0)$ and $\langle P'(0), Q'(0) \rangle \neq 0$. By Theorem 2 of [3], there exists a Lorentzian conformal stationary immersion $F(s,t)$ from a domain containing $(0,0)$ of R^2 into $N^4_2(c)$, which satisfies

$$\langle F_s, F_s \rangle = -\langle F_t, F_t \rangle \neq 0, \quad \langle F_s, F_t \rangle = 0,$$
$$F(s,s) = P(s), \quad F(s,-s) = Q(s).$$

It is a kind of characteristic initial value problem. Set $f(u,v) = F(u+v, u-v)$, which gives a Lorentzian stationary surface M in $N^4_2(c)$ such that f_u and f_v are null, and $f(u,0) = P(u), f(0,v) = Q(v)$.

Assume that $\{P'(0), Q'(0), P''(0), Q''(0)\}$ are linearly independent, and $\langle P''(0), P''(0) \rangle \langle Q''(0), Q''(0) \rangle > 0$ (resp. < 0). Then, using Theorem 1.1 for the surface M, we have $(K-c)^2 - K_\nu^2 > 0$ (resp. < 0) near $(u,v) = (0,0)$. Thus we have proved Theorem 1.2.

4 Examples

Let R^5_2 be the 5-dimensional semi-Euclidean space of index 2 with standard coordinate system $(x_1, x_2, x_3, x_4, x_5)$ and signature $(+,+,+,-,-)$. Then

$$S^4_2(1) = \{(x_1, x_2, x_3, x_4, x_5) \in R^5_2 | x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = 1\}.$$

Example 4.1. Let $P(u)$ be a null curve in $S^4_2(1)$ given by

$$P(u) = \left(\cos\left(\frac{u^2}{2}\right),\ \sin\left(\frac{u^2}{2}\right),\ u\cosh u,\ u,\ u\sinh u\right).$$

Then

$$P(0) = (1,0,0,0,0), \quad P'(0) = (0,0,1,1,0), \quad P''(0) = (0,1,0,0,2),$$

and $\langle P''(0), P''(0)\rangle = -3$. Next, let $Q(v)$ be a null curve in $S^4_2(1)$ given by

$$Q(v) = (2\cos v - \cos(\sqrt{2}v), \sqrt{2}\sin v, 0, \sin(\sqrt{2}v), -\sqrt{2}\cos v + \sqrt{2}\cos(\sqrt{2}v)).$$

Then

$$Q(0) = (1,0,0,0,0), \quad Q'(0) = (0,\sqrt{2},0,\sqrt{2},0), \quad Q''(0) = (0,0,0,0,-\sqrt{2}),$$

and $\langle Q''(0), Q''(0)\rangle = -2$.

We can see that $P(0) = Q(0)$, $\langle P'(0), Q'(0)\rangle = -\sqrt{2} \neq 0$, $\{P'(0), Q'(0), P''(0), Q''(0)\}$ are linearly independent, and

$$\langle P''(0), P''(0)\rangle\langle Q''(0), Q''(0)\rangle = 6 > 0.$$

So by Theorem 1.2, there exists a Lorentzian stationary surface in $S^4_2(1)$ whose inclusion map $f(u,v)$ satisfies f_u and f_v are null, $f(u,0) = P(u)$, $f(0,v) = Q(v)$, and $(K-1)^2 - K_\nu^2 > 0$ near $(u,v) = (0,0)$.

Example 4.2. Let $P(u)$ be a null curve in $S^4_2(1)$ given by

$$P(u) = \left(\cosh\left(\frac{u^2}{2}\right),\ u\cos u,\ u\sin u,\ u,\ \sinh\left(\frac{u^2}{2}\right)\right),$$

which satisfies

$$P(0) = (1,0,0,0,0), \quad P'(0) = (0,1,0,1,0), \quad P''(0) = (0,0,2,0,1),$$

and $\langle P''(0), P''(0)\rangle = 3$. Let $Q(v)$ be a null curve in $S^4_2(1)$ given by

$$Q(v) = \left(\cos\left(\frac{v^2}{2}\right),\ \sin\left(\frac{v^2}{2}\right),\ v,\ v\cos v,\ v\sin v\right),$$

which satisfies

$$Q(0) = (1,0,0,0,0), \quad Q'(0) = (0,0,1,1,0), \quad Q''(0) = (0,1,0,0,2),$$

and $\langle Q''(0), Q''(0)\rangle = -3$.

We can see that $P(0) = Q(0)$, $\langle P'(0), Q'(0)\rangle = -1 \neq 0$, $\{P'(0), Q'(0), P''(0), Q''(0)\}$ are linearly independent, and

$$\langle P''(0), P''(0)\rangle\langle Q''(0), Q''(0)\rangle = -9 < 0.$$

By Theorem 1.2, there exists a Lorentzian stationary surface in $S_2^4(1)$ whose inclusion map $f(u,v)$ satisfies f_u and f_v are null, $f(u,0) = P(u)$, $f(0,v) = Q(v)$, and $(K-1)^2 - K_\nu^2 < 0$ near $(u,v) = (0,0)$.

Let $\sigma : R_2^5 \to R_3^5$ be the canonical anti-isometry from R_2^5 to R_3^5, which naturally induces the anti-isometry from $S_2^4(1)$ to $H_2^4(-1)$ (cf. p.110 of [9]). For the null curves $P(u)$ and $Q(v)$ in $S_2^4(1)$ in Example 4.1 or 4.2, considering null curves $\sigma(P(u))$ and $\sigma(Q(v))$ in $H_2^4(-1)$, we can get Lorentzian stationary surfaces in $H_2^4(-1)$ such that $(K+1)^2 - K_\nu^2$ is positive, or negative.

5 A remark

Let $f(u,v)$ be a Lorentzian surface in a semi-Euclidean space R_q^n with coordinate system (u,v) such that f_u and f_v are null. Then the value

$$\frac{\langle f_{uu}, f_{uu}\rangle\langle f_{vv}, f_{vv}\rangle}{\langle f_u, f_v\rangle^4}$$

is an invariant, which is independent of the choice of such coordinate systems. So, some more general treatment will be possible. Furthermore, it should be interesting to find and study higher order invariants for Lorentzian surfaces in R_q^n.

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