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The k-step Fibonacci sequences via Hurwitz matrices

Ömür Deveci · Yeşim Aküzüm ·
Güventürk Uğurlu

Abstract This paper develops properties of recurrence sequences defined from the Hurwitz matrices obtained from the characteristic polynomial of the k-step Fibonacci sequence. Also, we consider the cyclic groups which are generated by the generating matrices and the auxiliary equations of the defined recurrence sequences and then, we examine the orders of these groups. Furthermore, we extend the r-step Fibonacci-Hurwitz sequences to groups. Finally, we obtain the periods of the r-step Fibonacci-Hurwitz sequences in the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ for $m \geq 4$ as applications of the results obtained.

Keywords Hurwitz Matrix · Sequence · Group · Period

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1 Introduction and preliminaries

The k-step Fibonacci sequence $\left\{F_n^k\right\}$ is defined recursively by the equation

$$F_{n+k}^k = F_{n+k-1}^k + F_{n+k-2}^k + \cdots + F_n^k \tag{1.1}$$

for $n \geq 0$, where $F_0^k = F_1^k = \cdots = F_{k-2}^k = 0$ and $F_{k-1}^k = 1$.
For more information on the k-step Fibonacci sequence $\left\{F_n^k\right\}$, see [14,23].
Let an nth degree real polynomial P be given by

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

In [13], A. Hurwitz defined the Hurwitz matrix $H_n = [h_{ij}]_{n\times n}$ associated to P as follows:

$$H_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}.$$

Suppose that the $(n+k)$ th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants.
In [14], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he showed that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [16,21,22,24,25,27–29,31].
In Section 2, we define the generalized order-k Fibonacci Hurwitz sequences and the r-step Fibonacci-Hurwitz sequences by using the Hurwitz matrices which are obtained from the characteristic polynomial of the k-step Fibonacci sequence and then, we obtain the relationship among the elements of the sequences and the generating matrices of the sequences.

In [4–6,19], the authors obtained the cyclic groups via some special matrices. In Section 3, we consider the multiplicative order of the Hurwitz matrix H_F^k and r-step Fibonacci-Hurwitz matrices M_1^k and M_2^k when read modulo m and then, we obtain the cyclic groups. Also in this section, we study the defined recurrence sequences modulo m and we obtain the cyclic groups via the auxiliary equations of the defined recurrence sequences. Then we derive the relationships between the orders of the obtained cyclic groups and the periods of the defined sequences according to modulo m.
The study of linear recurrence sequences in groups began with the earlier work of Wall [30] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1–3,5,7–9,15,17,19,20,26]. In Section 3, we redefine the r-step Fibonacci-Hurwitz sequences by the means of the elements of the groups and then, we examine them in finite groups. Furthermore, we obtain the periods of the r-step Fibonacci-Hurwitz sequences in the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ for $m \geq 4$.

2 Fibonacci sequences via Hurwitz matrices

It is easy to see that the characteristic polynomials of the k-step Fibonacci sequences are as follows:

$$f(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1.$$

Then we can write the following Hurwitz matrices for the polynomial $f(x)$:
If k is positive odd number such that $k \geq 5$,

$$H_F^k = [H_{ij}]_{k\times k} = \begin{cases} -1 \quad 1 \leq i \leq k & \left(\text{if } \ i \text{ is odd}, \quad \frac{i+1}{2} \leq j \leq \frac{k+i}{2}\right) \\ & \text{and}\left(\text{if } \ i \text{ is even}, \quad \frac{i+2}{2} \leq j \leq \frac{k+i-1}{2}\right), \\ 1 \; 1 \leq j \leq \frac{k-1}{2} & (\text{if } i = 2j), \\ 0 & \text{otherwise.} \end{cases}$$

If k is positive even number such that $k \geq 4$,

$$H_F^k = [H_{ij}]_{k\times k} = \begin{cases} -1 \; 1 \leq i \leq k & \left(\text{if } \ i \text{ is odd}, \quad \frac{i+1}{2} \leq j \leq \frac{k+i-1}{2}\right) \\ & \text{and } \left(\text{if } \ i \text{ is even}, \quad \frac{i+2}{2} \leq j \leq \frac{k+i}{2}\right), \\ 1 \; 1 \leq j \leq \frac{k}{2} & (\text{if } i = 2j), \\ 0 & \text{otherwise.} \end{cases}$$

For example, the matrices H_F^6 and H_F^7 are as follows:

$$H_F^6=\begin{bmatrix}-1&-1&-1&0&0&0\\1&-1&-1&-1&0&0\\0&-1&-1&-1&0&0\\0&1&-1&-1&-1&0\\0&0&-1&-1&-1&0\\0&0&1&-1&-1&-1\end{bmatrix},\quad H_F^7=\begin{bmatrix}-1&-1&-1&-1&0&0&0\\1&-1&-1&-1&0&0&0\\0&-1&-1&-1&-1&0&0\\0&1&-1&-1&-1&0&0\\0&0&-1&-1&-1&-1&0\\0&0&1&-1&-1&-1&0\\0&0&0&-1&-1&-1&-1\end{bmatrix}.$$

Now we define the generalized order-k Fibonacci Hurwitz sequences by using the matrices H_F^k as follows:
If k is positive odd number,

$$x_n=\begin{cases}-x_{n-\left(\frac{k+t}{2}\right)}-x_{n-\left(\frac{k+t}{2}+1\right)}-\cdots-x_{n-\left(\frac{2k+t-3}{2}\right)}-x_{n-\left(\frac{2k+t-1}{2}\right)}, \\ \qquad n\equiv t \bmod k \ \ (1\le t\le k-2 \text{ and } t \text{ is odd}), \\ -x_{n-\left(\frac{k+t+1}{2}\right)}-x_{n-\left(\frac{k+t+1}{2}+1\right)}-\cdots-x_{n-\left(\frac{2k+t-2}{2}\right)}+x_{n-\left(\frac{2k+t}{2}\right)}, \\ \qquad n\equiv t \bmod k \ \ (2\le t\le k-1 \text{ and } t \text{ is even}), \\ -x_{n-(k)}-x_{n-(k+1)}-\cdots-x_{n-\left(\frac{3k-3}{2}\right)}-x_{n-\left(\frac{3k-1}{2}\right)}, \\ \qquad n\equiv 0 \bmod k;\end{cases}$$

if k is positive even number,

$$x_n=\begin{cases}-x_{n-\left(\frac{k+t+1}{2}\right)}-x_{n-\left(\frac{k+t+1}{2}+1\right)}-\cdots-x_{n-\left(\frac{2k+t-3}{2}\right)}-x_{n-\left(\frac{2k+t-1}{2}\right)}, \\ \qquad n\equiv t \bmod k \ \ (1\le t\le k-1 \text{ and } t \text{ is odd}), \\ -x_{n-\left(\frac{k+t}{2}\right)}-x_{n-\left(\frac{k+t}{2}+1\right)}-\cdots-x_{n-\left(\frac{2k+t-2}{2}\right)}+x_{n-\left(\frac{2k+t}{2}\right)}, \\ \qquad n\equiv t \bmod k \ \ (2\le t\le k-2 \text{ and } t \text{ is even}), \\ -x_{n-(k)}-x_{n-(k+1)}-\cdots-x_{n-\left(\frac{3k-2}{2}\right)}+x_{n-\left(\frac{3k}{2}\right)}, \\ \qquad n\equiv 0 \bmod k,\end{cases}$$

where $x_1=x_2=\cdots=x_{r-1}=0$ and $x_r=1$.

For $n\ge 0$, by an inductive argument, we may write

$$\left(H_F^k\right)^n\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}=\begin{bmatrix}x_{n+k}\\x_{n+k+1}\\x_{n+k+2}\\\vdots\\x_{n+2k-1}\end{bmatrix}. \tag{2.1}$$

Define the r-step Fibonacci-Hurwitz sequences as shown:

$$a_n=\begin{cases}-a_{n-\left(\frac{r+1}{2}\right)}-a_{n-\left(\frac{r+1}{2}+1\right)}-\ldots-a_{n-(r-1)}+a_{n-r} & \text{if } r \text{ is odd,}\\ -a_{n-\left(\frac{r}{2}+1\right)}-a_{n-\left(\frac{r}{2}+2\right)}-\ldots-a_{n-(r-1)}+a_{n-r} & \text{if } r \text{ is even}\end{cases} \quad \text{for } n>r \tag{2.2}$$

where $r \geq 5$, $a_1 = a_2 = \cdots = a_{r-1} = 0$ and $a_r = 1$.
The generating functions of the r-step Fibonacci-Hurwitz sequences $\{a_n\}$are as follows:
If r is positive odd number such that $r \geq 5$, then

$$g(x) = \frac{x^{r-1}}{-x^r + x^{r-1} + x^{r-2} + \cdots + x^{\frac{r+1}{2}} + 1}.$$

If r is positive even number such that $r \geq 6$, then

$$g(x) = \frac{x^{r-1}}{-x^r + x^{r-1} + x^{r-2} + \cdots + x^{\frac{r+2}{2}} + 1}.$$

By (2.2), we can write the following companion matrices:

$$M_1^r = [m_{ij}]_{r\times r} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1(j-1)} & m_{1j} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

where r is positive odd number such that $r \geq 5$ and m_{1j}'s are defined as follows:

$$m_{1j} = \begin{cases} m_{1j} = 0 & 1 \leq j \leq \frac{r-1}{2}, \\ m_{1j} = -1 & \frac{r+1}{2} \leq j \leq r-1, \\ m_{1j} = 1 & j = r, \end{cases}$$

and

$$M_2^r = [m_{ij}]_{r\times r} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1(j-1)} & m_{1j} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

where r is positive even number such that $r \geq 6$ and m_{1j}'s are defined as follows:

$$m_{1j} = \begin{cases} m_{1j} = 0 & 1 \leq j \leq \frac{r}{2}, \\ m_{1j} = -1 & \frac{r+2}{2} \leq j \leq r-1, \\ m_{1j} = 1 & j = r. \end{cases}$$

The matrices M_1^k and M_2^k are said to be r-step Fibonacci-Hurwitz matrices.
It is clear that

$$\begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-r+1} \end{bmatrix} = M_s^r \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-r} \end{bmatrix} \text{ for } s = 1, 2.$$

Again by an inductive argument, we may write

$$(M_1^r)^n = \begin{bmatrix} a_{n+r} & a_{n+r+1} & \cdots & a_{n+\left(\frac{3r-1}{2}\right)} & a_{n+\left(\frac{r+1}{2}\right)} - a_{n+\left(\frac{r+1}{2}+1\right)} - \cdots - a_{n+(r-1)} & \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+\left(\frac{3r-1}{2}-1\right)} & a_{n+\left(\frac{r+1}{2}-1\right)} - a_{n+\left(\frac{r+1}{2}\right)} - \cdots - a_{n+(r-2)} & \\ \vdots & \vdots & & \vdots & \vdots & M_1^* \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\left(\frac{r+1}{2}\right)} & a_{n+\left(\frac{-r+3}{2}\right)} - a_{n+\left(\frac{-r+3}{2}+1\right)} - \cdots - a_n & \end{bmatrix} \tag{2.3}$$

where $n \geq \frac{r-1}{2}$ and

$$M_1^* = [m_{ij}]_{(r)\times\left(\frac{r-3}{2}\right)} =$$
$$= \begin{bmatrix} a_{n+\left(\frac{r+1}{2}+1\right)} - a_{n+\left(\frac{r+1}{2}+2\right)} - \cdots - a_{n+(r-1)} & \cdots & a_{n+r-2} - a_{n+r-1} & a_{n+r-1} \\ a_{n+\left(\frac{r+1}{2}\right)} - a_{n+\left(\frac{r+1}{2}+1\right)} - \cdots - a_{n+(r-2)} & \cdots & a_{n+r-3} - a_{n+r-2} & a_{n+r-2} \\ \vdots & & \vdots & \vdots \\ a_{n+\left(\frac{-r+5}{2}\right)} - a_{n+\left(\frac{-r+5}{2}+1\right)} - \cdots - a_n & \cdots & a_{n-1} - a_n & a_n \end{bmatrix},$$

$$(M_2^r)^n = \begin{bmatrix} a_{n+r} & a_{n+r+1} & \cdots & a_{n+\frac{3r}{2}} & a_{n+\frac{r+2}{2}} - a_{n+\left(\frac{r+2}{2}+1\right)} - \cdots - a_{n+(r-1)} & \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+\frac{3r}{2}-1} & a_{n+\left(\frac{r+2}{2}-1\right)} - a_{n+\left(\frac{r+2}{2}\right)} - \cdots - a_{n+(r-2)} & \\ \vdots & \vdots & & \vdots & \vdots & M_2^* \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\left(\frac{r+2}{2}\right)} & a_{n+\left(\frac{-r+4}{2}\right)} - a_{n+\left(\frac{-r+4}{2}+1\right)} - \cdots - a_n & \end{bmatrix} \tag{2.4}$$

where $n \geq \frac{r-2}{2}$ and

$$M_2^* = [m_{ij}]_{(r)\times\left(\frac{r-4}{2}\right)} =$$
$$\begin{bmatrix} a_{n+\left(\frac{r+2}{2}+1\right)} - a_{n+\left(\frac{r+2}{2}+2\right)} - \cdots - a_{n+(r-1)} & \cdots & a_{n+r-2} - a_{n+r-1} & a_{n+r-1} \\ a_{n+\left(\frac{r+2}{2}\right)} - a_{n+\left(\frac{r+2}{2}+1\right)} - \cdots - a_{n+(r-2)} & \cdots & a_{n+r-3} - a_{n+r-2} & a_{n+r-2} \\ \vdots & & \vdots & \vdots \\ a_{n+\left(\frac{-r+6}{2}\right)} - a_{n+\left(\frac{-r+6}{2}+1\right)} - \cdots - a_n & \cdots & a_{n-1} - a_n & a_n \end{bmatrix}.$$

Note that $\det M_1^r = 1$ and $\det M_2^r = -1$. It is well-known that the Simpson formula for a recurrence sequence can be obtained from the determinant of their generating matrices.

3 The Cyclic Groups via the matrices H_F^k, M_1^r and M_2^r

For given a matrix $B = [b_{ij}]$ with b_{ij}'s being integers, $B \,(\mathrm{mod}\, m)$ means that each element of B are reduced modulo m, that is, $B \ \ (mod\, m) = (b_{ij} \ \ (mod\, m))$. Let us consider the set $\langle B\rangle_m = \left\{ B^i \ (\mathrm{mod}\, m) \middle| \, i \geq 0 \right\}$. If $\gcd(m, \det B) = 1$, then the set $\langle B\rangle_m$ is a cyclic group. Let the notation $|\langle B\rangle_m|$ denotes the order of the set $\langle B\rangle_m$. Since $\det M_1^r = 1$ and $\det M_2^r = -1$, it is clear that the sets $\langle M_1^r\rangle_m$and $\langle M_2^r\rangle_m$ are cyclic groups for every positive integer m. Similarly, the set $\left\langle H_F^k\right\rangle_m$ is a cyclic group if $\gcd\left(m,\ H_F^k\right) = 1$.
Now we consider the cyclic groups which are generated by the matrices H_F^k, M_1^r and M_2^r.

Theorem 3.1 *Let p be a prime and let $\left\langle H_F^k\right\rangle_{p^\alpha}$, $\langle M_1^r\rangle_{p^\alpha}$ and $\langle M_2^r\rangle_{p^\alpha}$ be cyclic groups. Suppose that $\langle G\rangle_{p^\alpha}$ is any of the cyclic groups of $\left\langle H_F^k\right\rangle_{p^\alpha}$, $\langle M_1^r\rangle_{p^\alpha}$ and $\langle M_2^r\rangle_{p^\alpha}$ such that $\alpha \in N$. If u is the largest positive integer such that $\left|\langle G\rangle_p\right| = \left|\langle G\rangle_{p^u}\right|$, then $\left|\langle G\rangle_{p^v}\right| = p^{v-u} \cdot \left|\langle G\rangle_p\right|$ for every $v \geq u$. In particular, if $\left|\langle G\rangle_p\right| \neq \left|\langle G\rangle_{p^2}\right|$, then $\left|\langle G\rangle_p\right| = p^{v-1} \cdot \left|\langle G\rangle_p\right|$ for every $v \geq 2$.*

Proof. Let us consider the cyclic group $\langle M_1^r\rangle_{p^\alpha}$. Suppose that b is a positive integer and $\left|\langle M_1^r\rangle_{p^\alpha}\right|$ is denoted by $h(p^\alpha)$. If $(M_1^r)^{h\left(p^{b+1}\right)} \equiv I \ \left(\mathrm{mod}\ p^{b+1}\right)$, then $(M_1^r)^{h\left(p^{b+1}\right)} \equiv I \ \left(\mathrm{mod}\ p^b\right)$ where I is a $r \times r$ identity matrix . Thus we obtain that $h\left(p^b\right)$ divides $h\left(\beta^{b+1}\right)$. Also, writing $(M_1^r)^{h\left(p^b\right)} = I +$

$\left(m_{ij}^{(b)} \cdot p^b\right)$, by the binomial theorem, we obtain

$$\left(M_1^r\right)^{h\left(p^b\right) \cdot p} = \left(I + \left(m_{ij}^{(b)} \cdot p^b\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(m_{ij}^{(b)} \cdot p^b\right)^i \equiv I \left(\bmod p^{b+1}\right),$$

which yields that $h\left(p^{b+1}\right)$ divides $h\left(p^b\right) \cdot p$. Thus, $h\left(p^{b+1}\right) = h\left(p^b\right)$ or $h\left(p^{b+1}\right) = h\left(p^b\right) \cdot p$. It is clear that $h\left(p^{b+1}\right) = h\left(p^b\right) \cdot p$ holds if and only if there exists an integer $m_{ij}^{(b)}$ which is not divisible by p. Since u is the largest positive integer such that $h(p) = h(p^u)$, we have $h(p^u) \neq h\left(p^{u+1}\right)$. Then, there exists an integer $m_{ij}^{(u+1)}$ which is not divisible by p. So we get that $h\left(p^{u+1}\right) \neq h\left(p^{u+2}\right)$. To complete the proof we may use an inductive method on u . There are similar proofs for the cyclic groups $\left\langle H_F^k \right\rangle_{p^\alpha}$ and $\left\langle M_2^r \right\rangle_{p^\alpha}$. □

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the *period* of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, \ldots$ is periodic after the initial element a and has period 3. *A sequence is simply periodic with period k* if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

Reducing the generalized order-k Fibonacci-Hurwitz sequence and the r-step Fi-bonacci-Hurwitz sequence of the by a modulus m, we can get the repeating sequences, respectively denoted by

$$\{x_n(m)\} = \{x_1(m),\ x_2(m),\ x_3(m), \ldots, x_j(m), \ldots\}$$

and

$$\{a_n(m)\} = \{a_1(m),\ a_2(m),\ a_3(m), \ldots, a_j(m), \ldots\}$$

where $x_j(m) = x_j\ (\bmod m)$ and $a_j(m) = a_j\ (\bmod m)$. They have the same recurrence relations as in the definitions of the generalized order-k Fibonacci Hurwitz sequence and the r-step Fibonacci-Hurwitz sequences, respectively.

Theorem 3.2 *The sequence $\{a_n(m)\}$ is simply periodic for every positive integer m. Similarly, the sequence $\{x_n(m)\}$ is a periodic sequence for every positive integer m. In particular, if $\gcd\left(\det H_F^k, m\right) = 1$, then $\{x_n(m)\}$ is simply periodic.*

Proof. Let us consider the r-step Fibonacci-Hurwitz sequence $\{a_n(m)\}$. Suppose that $U = \{(u_1, u_2, \ldots, u_r) | \, 0 \leq u_i \leq m-1\}$. Then we have $|U| = m^r$. Since there are m^r distinct r-tuples of elements of $\mathbb{Z}_m$, at least one of the r-tuples appears twice in the sequence $\{a_n(m)\}$. Thus, the subsequence following this r-tuple repeats; that is the sequence is periodic. Let

$a_{i+1}(m) \equiv a_{j+1}(m)$, $a_{i+2}(m) \equiv a_{j+2}(m)$, …, $a_{i+r}(m) \equiv a_{j+r}(m)$ be such that $i > j$, then $i \equiv j \bmod r$. From the definition, we can easily derive that

$$a_i(m) \equiv a_j(m), a_{i-1}(m) \equiv a_{j-1}(m), \ldots, a_{i-(j-1)}(m) \equiv a_{j-(j-1)}(m) = a_1(m).$$

Thus we get that the sequence is a simply periodic.
There is similar proof for the sequence $\{x_n(m)\}$. $\square$

We denote the periods of the sequences $\{x_n(m)\}$ and $\{a_n(m)\}$ by $l_x^{F,k}(m)$ and $l_a^r(m)$ respectively.
Then, we have the following useful results from (2.1), (2.3) and (2.4) respectively.
Corollary 3.1. Let p be a prime. Then
i. If $\gcd\left(\det H_F^k, p\right) = 1$, then $l_x^{F,k}(p) = k.\left|\left\langle H_F^k\right\rangle_p\right|$.
ii. $l_a^r(p) = \left|\langle M_s^r\rangle_p\right|$ for $s = 1, 2$. $\square$

Theorem 3.3 *Let m be a positive odd integer and let $m = \prod_{i=1}^{k} u_i^{e_i}$, $(k \geq 1)$ such that u_i 's are distinct primes. Then $l_a^r(m) = \mathrm{lcm}[l_a^r(u_1^{e_1}), l_a^r(u_2^{e_2}), \ldots, l_a^r(u_k^{e_k})]$.*

Proof. Since $l_a^r(u_i^{e_i})$ is the length of the period of the sequence $\{a_n(u_i^{e_i})\}$, the sequence $\{a_n(u_i^{e_i})\}$ repeats only after blocks of length $\alpha \cdot l_a^r(u_i^{e_i})$, $(\alpha \in N)$. Since also $l_a^r(m)$ is the length of the period $\{a_n(m)\}$, the sequence $\{a_n(u_i^{e_i})\}$ repeats after $l_a^r(m)$ terms for all values i. This implies that $l_a^r(m)$ is the form $\alpha \cdot l_a^r(u_i^{e_i})$ for all values of i. We thus prove that $l_a^r(m)$ equals the least common multiple of $l_a^r(u_i^{e_i})$'s. $\square$

Let p be a prime and let α be a positive integer. Consider

$$A_1(p^\alpha) = \left\{x^n\,(modp^\alpha) : n \in \mathbb{Z},\ x^r = -x^{\frac{r-1}{2}} - x^{\frac{r-3}{2}} - \cdots - x + 1\right\}$$

such that r is a positive odd number,
and

$$A_2(p^\alpha) = \left\{x^n\,(modp^\alpha) : n \in \mathbb{Z},\ x^r = -x^{\frac{r-2}{2}} - x^{\frac{r-4}{2}} - \cdots - x + 1\right\}$$

such that r is a positive even number.
It is important note that the sets $A_1(p^\alpha)$ and $A_2(p^\alpha)$ are cyclic groups.
Now we can give a relationship between the auxiliary equations of the r-step Fibonacci-Hurwitz sequence and the period $l_a^r(m)$ by the following Corollary.
Corollary 3.2. Let p be a prime and let $\alpha \in N$. Then, the cyclic groups $A_1(p^\alpha)$ and $A_2(p^\alpha)$ are isomorphic to the cyclic groups $\langle M_1^r\rangle_{p^\alpha}$ and $\langle M_2^r\rangle_{p^\alpha}$.
$\square$

4 The r-step Fibonacci-Hurwitz Sequences in Groups

Let G be a finite j-generator group and let

$$X = \left\{ (x_1, x_2, \ldots, x_j) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid \langle x_1, x_2, \ldots, x_j \rangle = G \right\}$$

We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.
Definition 4.1. Let $G=\langle X \rangle$ be a finitely generated group such that $X=\{x_1, x_2, \ldots, x_j\}$ and $r \geq 5$. Then we define the r-step Fibonacci-Hurwitz sequence in the group G as follows:
If $j < r$, then

$$a_n = \begin{cases} (a_{n-r}) \left(a_{n-(r-1)}\right)^{-1} \cdots \left(a_{n-\left(\frac{r+1}{2}\right)}\right)^{-1} & \text{if } r \text{ is odd,} \\ (a_{n-r}) \left(a_{n-(r-1)}\right)^{-1} \cdots \left(a_{n-\left(\frac{r}{2}+1\right)}\right)^{-1} & \text{if } r \text{ is even,} \end{cases} \quad \text{for } n > r$$

where
$$\begin{cases} a_1 = a_2 = \cdots = a_{r-j} = e, \\ a_{r-j+1} = x_1, a_{r-j+2} = x_2, \ldots, a_r = x_j. \end{cases}$$

If $j \geq r$, then

$$a_n = \begin{cases} (a_{n-r}) \left(a_{n-(r-1)}\right)^{-1} \cdots \left(a_{n-\left(\frac{r+1}{2}\right)}\right)^{-1} & \text{if } r \text{ is odd,} \\ (a_{n-r}) \left(a_{n-(r-1)}\right)^{-1} \cdots \left(a_{n-\left(\frac{r}{2}+1\right)}\right)^{-1} & \text{if } r \text{ is even,} \end{cases} \quad \text{for } n > j$$

where $a_k = x_k$ for $1 \leq k \leq j$.
For a j-tuple $(x_1, x_2, \ldots, x_j) \in X$, the r-step Fibonacci-Hurwitz sequence in G is denoted by $FH^r_{(x_1, x_2, \ldots, x_j)}(G)$.

Theorem 4.1 *A r-step Fibonacci-Hurwitz sequence in a finite group is simply periodic.*

Proof. Suppose that n is the order of G. Since there n^r distinct r-tuples of elements of G, at least one of the r-tuples appears twice in the sequence $FH^r_{(x_1, x_2, \ldots, x_j)}(G)$. Because of the repeating, the sequence is periodic. Since the sequence $FH^r_{(x_1, x_2, \ldots, x_j)}(G)$ is periodic, there exist natural numbers u and v, with $u \geq v$, such that

$$a_{u+1} = a_{v+1}, \quad a_{u+2} = a_{v+2}, \quad \ldots, \quad a_{u+r} = a_{v+r}.$$

By the definition of the sequence $FH^r_{(x_1, x_2, \ldots, x_j)}(G)$, we know that

$$a_{n-r} = \begin{cases} (a_n) \left(a_{n-\left(\frac{r+1}{2}\right)}\right) \cdots \left(a_{n-(r-1)}\right) & \text{if } r \text{ is odd,} \\ (a_n) \left(a_{n-\left(\frac{r}{2}+1\right)}\right) \cdots \left(a_{n-(r-1)}\right) & \text{if } r \text{ is even} \end{cases}$$

for the cases $j < r$ and $j \geq r$. Therefore, we obtian $a_u = a_v$, and hence,

$$a_{u-(v-1)} = a_{v-(v-1)} = a_1,\ a_{u-(v-2)} = a_{v-(v-2)} = a_2,\ \ldots,\ a_{u-(v-r)} = a_{v-(v-r)} = a_r$$

which implies that $FH^r_{(x_1,x_2,\ldots,x_j)}(G)$ is a simply periodic sequence. □

Let $PFH^r_{(x_1,x_2,\ldots,x_j)}(G)$ denote the period of the sequence $FH^r_{(x_1,x_2,\ldots,x_j)}(G)$. From the definition of the sequence $FH^r_{(x_1,x_2,\ldots,x_j)}(G)$ it is clear that the period of this sequence in a finite group depends on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_j$ are made.
We will now address the periods of the r-step Fibonacci-Hurwitz sequences in the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ which are metacyclic groups of order 2^m. The groups SD_{2^m} and $M\_m(2)$ are defined as follows:

$$SD_{2^m} = \left\langle x, y \middle| \, x^{2^{m-1}} = y^2 = e,\ y^{-1}xy = x^{-1+2^{m-2}} \right\rangle$$

and

$$M\_m(2) = \left\langle x, y \middle| \, x^{2^{m-1}} = y^2 = e,\ y^{-1}xy = x^{1+2^{m-2}} \right\rangle$$

where m is a integer such that $m \geq 4$.
For more information on these groups see [10–12].

Theorem 4.2 *For generating pairs (x, y) and (y, x), the periods of the r-step Fi-bonacci-Hurwitz sequences in the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ are $l^r_a(2^{m-1})$.*

Proof. Let us consider the r-step Fibonacci-Hurwitz sequence in the semidihedral group SD_{2^m} for generating pair (x, y). We prove this by direct calculation. The sequence $FH^r_{(x,y)}(SD_{2^m})$ is in the following form:

$$\begin{gathered}
a_1 = e,\ a_2 = e,\ \ldots,\ a_{r-2} = e,\ a_{r-1} = x,\ a_r = y, \ldots, \\
a_{2\cdot l^r_a(2)+1} = x^{2^{r-3}\cdot\lambda_1},\ a_{2\cdot l^r_a(2)+2} = x^{\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_2}, \ldots, \\
{}_{2\cdot l^r_a(2)+r-3} = x^{\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_{r-3}},\ a_{2\cdot l^r_a(2)+r-2} = x^{2^{r-3}\cdot\lambda_{r-2}}, \\
a_{2\cdot l^r_a(2)+r-1} = x^{4\cdot\lambda_{r-1}+1},\ a_{2\cdot l^r_a(2)+r-1} = yx^{\left(2^{r-1}-4(r-2)\right)\cdot\lambda_r}, \ldots, \\
a_{4\cdot l^r_a(2)+1} = x^{\left(2^{r-3}\cdot\lambda_1\right)\cdot 2},\ a_{4\cdot l^r_a(2)+2} = x^{\left(\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_2\right)\cdot 2}, \ldots, \\
a_{4\cdot l^r_a(2)+r-3} = x^{\left(\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_{r-3}\right)\cdot 2},\ a_{4\cdot l^r_a(2)+r-2} = x^{\left(2^{r-3}\cdot\lambda_{r-2}\right)\cdot 2}, \\
a_{4\cdot l^r_a(2)+r-1} = x^{8\cdot\lambda_{r-1}+1},\ a_{4\cdot l^r_a(2)+r-1} = yx^{\left(\left(2^{r-1}-4(r-2)\right)\cdot\lambda_r\right)\cdot 2}, \ldots.
\end{gathered}$$

Using the above, the sequence becomes:

$$\begin{gathered}
a_{2\cdot i\cdot l^r_a(2)+1} = x^{\left(2^{r-3}\cdot\lambda_1\right)\cdot i},\ a_{2\cdot i\cdot l^r_a(2)+2} = x^{\left(\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_2\right)\cdot i},\ \ldots, \\
a_{2\cdot i\cdot l^r_a(2)+r-3} = x^{\left(\left(2^{r-3}+2^{n-1}-4\right)\cdot\lambda_{r-3}\right)\cdot i},\ a_{2\cdot i\cdot l^r_a(2)+r-2} = x^{\left(2^{r-3}\cdot\lambda_{r-2}\right)\cdot i}, \\
a_{2\cdot i\cdot l^r_a(2)+r-1} = x^{4\cdot i\cdot\lambda_{r-1}+1},\ a_{2\cdot i\cdot l^r_a(2)+r-1} = yx^{\left(\left(2^{r-1}-4(r-2)\right)\cdot\lambda_r\right)\cdot i}, \ldots,
\end{gathered}$$

where $i \geq 0$ and $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive odd numbers suct that $\gcd(\lambda_1, \lambda_2, \ldots, \lambda_r) = 1$. So we need the smallest integer i such that $4i = 2^{m-1} \cdot k$ for $k \in N$. If we choose $i = 2^{m-3}$ we obtain $PFH^r_{(x,y)}(SD_{2^m}) = 2^{m-2} \cdot l^r_a(2) = l^r_a(2^{m-1})$.
There are similar proofs for the sequences $FH^r_{(y,x)}(SD_{2^m})$, $FH^r_{(x,y)}(M\_m(2))$ and $FH^r_{(y,x)}(M\_m(2))$. $\square$

5 Conclusions

This paper develops properties of recurrence sequences defined from circulant matrices obtained from the characteristic polynomial of the k-step Fibonacci sequence. The study of these sequences modulo m yields cyclic groups from the generating matrices. Finally, we extend the r-step Fibonacci -Hurwitz to groups and then, we obtain the periods of the r-step Fibonacci-Hurwitz sequences in the semidihedral group SD_{2^m} and the modular maximal-cyclic group $M\_m(2)$ for $m \geq 4$ as applications of the results obtained.

6 Further Questions

There are many open problems in this area. Below are a few of them:

1. Does there exist a relationship among the r-step Fibonacci sequence and the considered sequences in this paper?
2. Does there exist formulas for calculating the periods $l^{F,k}_x(m)$ and $l^r_a(m)$?
3. Does there exist a relationship among the periods $l^{F,k}_x(m)$ and $l^r_a(m)$?
4. Let us consider infinite groups such that the r-step Fibonacci-Hurwitz sequences in these groups are periodic. To find these periods it would be useful to have a program. This would possibly rely on using the Knuth-Bendix method, see [18].

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Authors

Ömür Deveci (Corresponding author),
Kafkas University,
Department of Mathematics,
Faculty of Science and Letters,
36100 Kars, Turkey,
E-mail: odeveci36@hotmail.com

Yeşim Aküzüm,
Kafkas University,
Department of Mathematics,
Faculty of Science and Letters,
36100 Kars, Turkey,
E-mail: yesim_036@hotmail.com

Güventürk Uğurlu,
Kafkas University,
Department of Physics,
Faculty of Science and Letters,
36100 Kars, Turkey,
E-mail: ugurlu@kafkas.edu.tr