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On the error term for some arithmetic functions in different number fields

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Abstract Lü and Ma [5] established asymptotic formulae for some arithmetic functions in different number fields whose discriminants relatively prime. In this paper we obtain the sharper bound for the mean square of the error term for the asymptotic formula they proved.

Keywords Divisor function · number fields · Riemann zeta function

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1 Introduction

In number theory there are many arithmetic functions which play important role in arithmetics, number theory and discrete mathematics. We often try to study the asymptotic behavior, since their behavior is very irregular. One of the most famous examples is the k-dimensional divisor problem, which studies the behavior of the mean value of $d_k(n)$. Here $d_k(n)$ denotes the number of representations of n as a product of k natural numbers. The Dirichlet series associated to $d_k(n)$ is $\zeta(s)^k$ where $\zeta(s)$ is Riemann zeta function. Dirichlet first obtained that for $k \geq 2$, $\sum_{n\leq x} d_k(n) = xQ_k(\log x) + \Delta_k(x), x \to \infty$, where $Q_k(t)$ is the polynomial of degree $k-1$ defined by $xQ_k(\log x) = \mathrm{Res}_{s=1}\zeta(s)^k \frac{x^s}{s}$ and where $\Delta_k(x)$ is the error term. Subsequently the problem of l-th power moments of $\Delta_k(x)$ has been extensively studied by many well-known mathematicians, for example Huxley [1], Ivić [2] and Kolesnik [4]. Later, this problem has been generalised to various number fields.

Let E be an algebraic number field of finite degree d over the rational field $\mathbb{Q}$. Let $\mathfrak{a}$ denote an integral ideal of the field E, and $N\mathfrak{a}$ the absolute norm of $\mathfrak{a}$. Suppose $a_E(n)$ denotes the number of integral ideals in E with norm n. The k-dimensional divisor problem in the field E is to study the mean value of the arithmetic function

$$d_k^E(n) = \sum_{N(\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_k)=n} 1 = \sum_{n=n_1n_2\cdots n_k} a_E(n_1)a_E(n_2)\cdots a_E(n_k).$$

In 2014, Lü and Ma [5] studied the asymptotic behavior of the product function of several multi-dimensional divisor functions involving different number fields. Let $E_j/\mathbb{Q}$ be number fields of degrees d_j, and whose discriminants D_j are relatively prime. The Dirichlet series associated to the problem $d_{k_1}^{E_1}(n)d_{k_2}^{E_2}(n)\cdots d_{k_l}^{E_l}(n)$ is

$$L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots,E_l}(s) = \sum_{n=1}^{\infty} \frac{d_{k_1}^{E_1}(n)d_{k_2}^{E_2}(n)\cdots d_{k_l}^{E_l}(n)}{n^s}. \tag{1.1}$$

In [5] it is proved that the function $L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots,E_l}(s)$ has a meromorphic continuation to $\mathfrak{Res} > \frac{1}{2}$ with a unique pole at $s = 1$ of order m and

$$L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots E_l}(s) = \zeta_{E_1E_2\cdots E_l}^m(s)U(s), \tag{1.2}$$

where $m = k_1k_2\cdots k_l$ and $U(s)$ denotes a Dirichlet series, which is absolutely and uniformly convergent for $\mathfrak{Res} > \frac{1}{2}$. From (1.1), Perron's formula (see Proposition 5.54 in [3]) and Cauchy's residue theorem, it is proved that

$$\Delta(x) \ll x^{1-\frac{3}{md_1d_2\cdots d_l+6}+\varepsilon}, \tag{1.3}$$

where $m = k_1k_2\cdots k_l$ and

$$\Delta(x) := \sum_{n\le x} d_{k_1}^{E_1}(n)d_{k_2}^{E_2}(n)\cdots d_{k_l}^{E_l}(n) - xQ_m(\log x), \tag{1.4}$$

where $Q_m(t)$ is a polynomial of degree $m-1$.

In this paper our aim is to prove the following results.

Theorem 1.1 *Under the previous hypothesis we have*

$$\int_1^X \Delta^2(x)dx \ll_\varepsilon X^{3-\frac{6}{md_1d_2\cdots d_l+3}+\varepsilon}.$$

Remark One can obtain by (1.3) the bound

$$\int_1^X \Delta^2(x)dx \ll_\varepsilon X^{3-\frac{6}{md_1d_2\cdots d_l+6}+\varepsilon}.$$

Then it is easy to see the result in Theorem 1.1 is sharper than the above bound.

From Theorem 1.1, when $k_1 = \cdots = k_l = 1$, we obtain the following corollary.

Corollary 1.2 *Define* $\Delta^*(x) := \sum_{n\le x} a_{E_1}(n)a_{E_2}(n)\cdots a_{E_l}(n) - c_{E_1,E_2,\cdots,E_l}x$. *Then we have*

$$\int_1^X (\Delta^*(x))^2\,dx \ll_\varepsilon X^{3-\frac{6}{d_1d_2\cdots d_l+3}+\varepsilon},$$

where $c_{E_1,E_2,\cdots,E_l}$ *is a certain constant concerned with number fields* $E_1, \cdots, E_l$.

As usual, we assume that X is a large positive number, and the letter ε denotes an arbitrary small positive number, not the same at each occurrence.

2 Proof of Theorem 1.1

In this paper we follow the notation of Lü and Ma [5] as closely as possible. Recall that $E_j/\mathbb{Q}$ are number fields of degrees d_j and whose discriminants D_j are relatively prime. As in [5], we have

$$\begin{aligned}\sum_{n\le x} d_{k_1}^{E_1}(n)\cdots d_{k_l}^{E_l}(n) &= \operatorname*{\mathbf{Res}}_{s=1} L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots,E_l}(s)\frac{x^s}{s} + O\left(\frac{x^{1+\varepsilon}}{T}\right)\\ &\quad + \frac{1}{2\pi i}\left\{\int_{\frac12+\varepsilon-iT}^{\frac12+\varepsilon+iT} + \int_{\frac12+\varepsilon+iT}^{1+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac12+\varepsilon-iT}\right\} L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots,E_l}(s)\frac{x^s}{s}ds\\ &:= xQ_m(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + I_1(x) + I_2(x) + I_3(x), \quad (2.1)\end{aligned}$$

so that the main term is $\operatorname*{\mathbf{Res}}_{s=1} L_{k_1,k_2,\cdots,k_l}^{E_1,E_2,\cdots,E_l}(s)\frac{x^s}{s}$ and this term is indeed of the form $xQ_m(\log x)$. We put

$$T = X^{\frac{3}{md_1d_2\cdots d_l+3}}. \qquad (2.2)$$

From (2.2), it is easy to get

$$\int_1^X \left(O\left(x^{1+\varepsilon}T^{-1}\right)\right)^2 dx = O\left(X^{3+\varepsilon}T^{-2}\right) \ll X^{3-\frac{6}{md_1d_2\cdots d_l+3}+\varepsilon}. \qquad (2.3)$$

By (1.4), (2.1) and (2.3), to prove Theorem 1.1 it suffices to prove the following results.

$$\int_1^X |I_i(x)|^2\,dx \ll_\varepsilon X^{3-\frac{6}{md_1d_2\cdots d_l+3}+\varepsilon},\ i=1,2,3. \qquad (2.4)$$

For $I_1(x)$, we get

$$\begin{aligned}
&\int_1^X |I_1(x)|^2\,dx \\
&= \frac{1}{4\pi^2}\int_{-T}^{T}\int_{-T}^{T}\frac{L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_1\right)\overline{L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_2\right)}}{(\frac{1}{2}+\varepsilon+it_1)(\frac{1}{2}+\varepsilon-it_2)} \\
&\qquad\qquad \times\left(\int_1^X x^{1+2\varepsilon+i(t_1-t_2)}dx\right)dt_1dt_2 \\
&\ll X^{2+2\varepsilon}\int_{-T}^{T}dt_1\int_{-T}^{T}\frac{|L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_1\right)||L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_2\right)|}{(1+|t_1|)(1+|t_2|)(1+|t_1-t_2|)}dt_2 \\
&\ll X^{2+2\varepsilon}\int_{-T}^{T}\frac{|L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_1\right)|^2}{(1+|t_1|)^2}dt_1\int_{-T}^{T}\frac{dt_2}{1+|t_1-t_2|}.
\end{aligned}$$

To go further, we have

$$\int_{-T}^{T}\frac{dt_2}{1+|t_1-t_2|}\ll\int_{t_1-1}^{t_1+1}dt_2+\left(\int_{t_1+1}^{T}+\int_{-T}^{t_1-1}\right)\frac{dt_2}{|t_1-t_2|}\ll\log(2T), \tag{2.5}$$

and by Lemma 2.4 in [5] and the Phragmén-Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [3]),

$$\zeta_{E_1,E_2,\cdots E_l}(\sigma+it)\ll(1+|t|)^{\frac{d_1d_2\cdots d_l}{3}(1-\sigma)+\varepsilon},\ \text{for } \frac{1}{2}\le\sigma\le1+\varepsilon. \tag{2.6}$$

Hence by (1.2), (2.5) and (2.6),

$$\begin{aligned}
\int_1^X I_1^2(x)dx &\ll X^{2+2\varepsilon}\log(2T)\int_{-T}^{T}\frac{|L_{k_1,\cdots,k_l}^{E_1,\cdots,E_l}\left(\frac{1}{2}+\varepsilon+it_1\right)|^2}{(1+|t_1|)^2}dt_1 \\
&\ll X^{2+3\varepsilon}+X^{2+3\varepsilon}\int_1^T\left|\zeta_{k_1,\cdots,k_l}^m\left(\frac{1}{2}+\varepsilon+it\right)\right|^2t^{-2}dt \\
&\ll X^{2+3\varepsilon}+X^{2+3\varepsilon}\int_1^T\left(t^{\frac{md_1d_2\cdots d_l}{6}+\varepsilon}\right)^2t^{-2}dt \\
&\ll X^{2+3\varepsilon}+X^{2+5\varepsilon}T^{\frac{md_1d_2\cdots d_l}{3}-1} \\
&\ll X^{3-\frac{6}{md_1d_2\cdots d_l+3}+\varepsilon}.
\end{aligned}\tag{2.7}$$

Now let us consider $I_2(x)$ and $I_3(x)$. For $j = 2, 3$, by (2.6), we have

$$
\begin{aligned}
I_j(x) &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma} |\zeta^m_{k_1,\cdots,k_l}(\sigma+iT)| T^{-1} d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} \left(\frac{x}{T^{md_1d_2\cdots d_l/3}}\right)^{\sigma} T^{\frac{md_1d_2\cdots d_l}{3}-1+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{\frac{md_1d_2\cdots d_l}{6}-1+\varepsilon},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\int_1^X |I_j(x)|^2\, dx &\ll \int_1^X \left(\frac{x^{1+\varepsilon}}{T}\right)^2 dx + \int_1^X \left(x^{\frac{1}{2}+\varepsilon} T^{\frac{md_1d_2\cdots d_l}{6}-1+\varepsilon}\right)^2 dx \\
&\ll X^{3+\varepsilon}T^{-2} + X^{2+2\varepsilon} T^{\frac{md_1d_2\cdots d_l}{3}-2+2\varepsilon} \\
&\ll X^{3-\frac{6}{md_1d_2\cdots d_l+3}+\varepsilon}.
\end{aligned}
\tag{2.8}
$$

From (2.3), (2.7) and (2.8), the inequalities (2.4) are proved. That is, the proof of Theorem 1.1 is completed.

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