An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 1

A new look on Baire spaces

Zbigniew Duszyński

Abstract The classical version of Baire category theorem was proved independently in 1897 by W. Osgood [*Non-uniform convergence and the integration of series term by term*, Amer. J. Math., **19** (1897), 155–190] and in 1899 by R. Baire [*Sur les fonctions de variable reeles*, Annali di Mat. Pura et Appl., **3** (1899), 65]. Our paper offers several characterizations and properties of Baire spaces with respect to the class of semi-open sets. Generalizations of Baire Category Theorem in this context are established.

Keywords semi-open · semi-closed · nowhere dense sets · Baire category theorem

Mathematics Subject Classification (2010) 54C08

1 Preliminaries

Topological spaces are denoted by (X, τ) while subspaces by (S, τ_S), $S \subset X$. The ***interior*** and the ***closure*** of S in the space (X, τ) will be denoted by $\mathrm{int}_\tau(S)$ (or $\mathrm{int}\,(S)$) and $\mathrm{cl}_\tau(S)$ (or $\mathrm{cl}\,(S)$) respectively. A subset S of (X, τ) is said to be ***semi-open*** [8] (resp. ***semi-closed*** [3], α***-open*** [10]) if $S \subset \mathrm{cl}\,(\mathrm{int}\,(S))$ (resp. $S \supset \mathrm{int}\,(\mathrm{cl}\,(S))$, $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S)))$). Levine [8] defined semi-open subsets as follows: S is semi-open in (X, τ) if there exists an $O \in \tau$ such that $O \subset S \subset \mathrm{cl}\,(O)$. The collection of all semi-open (resp. semi-closed, closed, α-open) subsets of (S, τ) is denoted by $\mathrm{SO}\,(X, \tau)$ (resp. $\mathrm{SC}\,(X, \tau)$, $\mathrm{c}\,(X, \tau)$, τ^α). For every space (X, τ) the family τ^α forms a topology on X [10]. The following inclusions hold for each (X, τ): $\tau \subset \tau^\alpha \subset \mathrm{SO}\,(X, \tau)$ and dually $\mathrm{c}\,(X, \tau) \subset \mathrm{c}\,(X, \tau^\alpha) \subset \mathrm{SC}\,(X, \tau)$. The union of all $U \in \mathrm{SO}\,(X, \tau)$ with $U \subset S$ is called the ***semi-interior*** [3] of S in (X, τ) and denoted as $\mathrm{sint}\,(S)$. The intersection of all $F \in \mathrm{SC}\,(X, \tau)$ with $S \subset F$ is called the ***semi-closure*** [3] of S in (X, τ) and is denoted as $\mathrm{scl}\,(S)$ (or $\mathrm{scl}_\tau(S)$). The union of any subfamily $\{U_i\}_{i \in \mathfrak{I}} \subset \mathrm{SO}\,(X, \tau)$ belongs to $\mathrm{SO}\,(X, \tau)$ as well [8, Theorem 2]. The following characterizations of semi-open and semi-closed sets will be useful: ($\mathfrak{a}$) $S \in \mathrm{SO}\,(X, \tau)$ iff $S = \mathrm{sint}\,(S)$ [3, Theorem 1.4(1)], ($\mathfrak{b}$) $S \in \mathrm{SC}\,(X, \tau)$ iff $S = \mathrm{scl}\,(S)$ [3, Theorem 1.4(2)], ($\mathfrak{c}$)

$S \in \mathrm{SO}(X,\tau)$ iff $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{int}(S))$ [12, Lemma 2], ($\mathfrak{d}$) $S \in \mathrm{SC}(X,\tau)$ iff $\mathrm{int}(S) = \mathrm{int}(\mathrm{cl}(S))$ (dual to ($\mathfrak{c}$)), ($\mathfrak{e}$) for any $A \subset B \subset X$, $A \in \mathrm{SO}(X,\tau)$ iff $A \in \mathrm{SO}(B,\tau_B)$ [11, Theorem 5]. Also, the following equalities and lemmas for arbitrary (X,τ) and its subset S are needed in the sequel: ($\mathfrak{f}$) $\mathrm{scl}(S) = S \cup \mathrm{int}(\mathrm{cl}(S))$ [1, Theorem 1.5(a)], ($\mathfrak{g}$) $\mathrm{sint}(S) = S \cap \mathrm{cl}(\mathrm{int}(S))$ [1, Theorem 1.5(b)], ($\mathfrak{h}$) $\mathrm{int}_{\tau^\alpha}(S) = S \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$ [1, Theorem 1.5(d)], ($\mathfrak{i}$) $\mathrm{sint}(S) = X \setminus \mathrm{scl}(X \setminus S)$ [3, Theorem 1.6(1)], ($\mathfrak{j}$) if $S \in \mathrm{SO}(X,\tau)$ and $O \in \tau$, then $S \cap O \in \mathrm{SO}(X,\tau)$ [3, Theorem 1.9], ($\mathfrak{k}$) $x \in \mathrm{scl}(S)$ iff $S \cap U_x \neq \emptyset$ for every $U_x \in \mathrm{SO}(X,\tau)$ with $x \in U_x$ [13, Lemma 3]. We recall that a topological space (X,τ) is said to be ***semi-connected*** [15] (***$\mathcal{S}$-connected***, in brief) if X cannot be split into two nonempty semi-open sets in (X,τ). In the opposite case a space is called ***$\mathcal{S}$-disconnected***.

2 Results

Our startpoint is the following definition: A space (X,τ) is said to be a ***Baire space*** if the intersection of any countable family of dense open sets (so-called: ***dense family***), is still dense in (X,τ).

Lemma 2.1 *Let (X,τ) be an arbitrary topological space. For any subset $S \subset X$ the following are equivalent:*

($\mathfrak{a}$) *S is dense in (X,τ),*
($\mathfrak{b}$) $\mathrm{cl}_{\tau^\alpha}(S) = X$,
($\mathfrak{c}$) *[5, Theorem 2.4]* $\mathrm{scl}_\tau(S) = X$.

Proof. ($\mathfrak{a}$)$\Leftrightarrow$($\mathfrak{c}$) follows by [5, Theorem 2.4]. We offer a shorter proof of ($\mathfrak{c}$)$\Rightarrow$($\mathfrak{a}$) than that given in [5]: by [1, Theorem 1.5(a)] we have $X = S \cup \mathrm{int}(\mathrm{cl}(S))$. Then $X = \mathrm{cl}(S) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \subset \mathrm{cl}(S)$. ($\mathfrak{a}$)$\Rightarrow$($\mathfrak{b}$). In view of ($\mathfrak{c}$) $\mathrm{scl}_\tau(S) = X$. Since $\mathrm{scl}_\tau(S) \subset \mathrm{cl}_{\tau^\alpha}(S)$, $\mathrm{cl}_{\tau^\alpha}(S) = X$. ($\mathfrak{b}$)$\Rightarrow$($\mathfrak{a}$) is obvious since $\mathrm{cl}_{\tau^\alpha}(S) \subset \mathrm{cl}(S)$. $\square$

Theorem 2.2 *The following statements are equivalent for any space (X,τ):*

1. *(X,τ) is a Baire space,*
2. $\mathrm{cl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \tau^\alpha$,
3. $\mathrm{cl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}(X,\tau)$,
4. $\mathrm{cl}_{\tau^\alpha}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \tau$,
5. $\mathrm{cl}_{\tau^\alpha}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \tau^\alpha$,
6. $\mathrm{cl}_{\tau^\alpha}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}(X,\tau)$,
7. $\mathrm{scl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \tau$,
8. $\mathrm{scl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \tau^\alpha$, *and*
9. $\mathrm{scl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right) = X$ *for every dense family* $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}(X,\tau)$.

Proof. 1 $\Rightarrow$ 3. Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}(X,\tau)$ be an arbitrary dense family, and let for each $n \in \mathbb{N}$ a set $O_n \in \tau$ be such that $O_n \subset A_n \subset \mathrm{cl}_\tau(O_n)$.

Clearly, $\mathrm{cl}_\tau(A_n) = \mathrm{cl}_\tau(O_n) = X$. Thus, by hypothesis, $X = \mathrm{cl}_\tau\left(\bigcap_{n\in\mathbb{N}} O_n\right) \subset \mathrm{cl}_\tau\left(\bigcap_{n\in\mathbb{N}} A_n\right)$. *3* $\Rightarrow$ *2* and *2* $\Rightarrow$ *1* are obvious, while equivalences *1* $\Leftrightarrow$ *4* $\Leftrightarrow$ *7*, *2* $\Leftrightarrow$ *5* $\Leftrightarrow$ *8*, and *3* $\Leftrightarrow$ *6* $\Leftrightarrow$ *9* follow directly from Lemma 2.1. $\square$

Corollary 2.3 *(X, τ) is Baire if and only if (X, τ^α) is Baire.*

Proof. Apply *1* $\Leftrightarrow$ *5* of Theorem 2.2. $\square$

Lemma 2.4 *In any space (X, τ) we have for each nonempty $S \subset X$:*

($\mathfrak{a}$) $\mathrm{sint}_\tau(S) \neq \emptyset$ *if and only if* $\mathrm{int}_\tau(S) \neq \emptyset$,
($\mathfrak{b}$) $\mathrm{int}_{\tau^\alpha}(S) \neq \emptyset$ *if and only if* $\mathrm{int}_\tau(S) \neq \emptyset$.

Proof. ($\mathfrak{a}$) If $\mathrm{sint}_\tau(S) = \emptyset$ then $\mathrm{int}_\tau(S) = \emptyset$ (as $\mathrm{int}_\tau(S) \subset \mathrm{sint}_\tau(S)$). For the case $\mathrm{int}_\tau(S) = \emptyset$ use [1, Theorem 1.5(b)]. ($\mathfrak{b}$) For $\mathrm{int}_{\tau^\alpha}(S) = \emptyset$, $\mathrm{int}_\tau(S) = \emptyset$ (as $\mathrm{int}_\tau(S) \subset \mathrm{int}_{\tau^\alpha}(S)$). If $\mathrm{int}_\tau(S) = \emptyset$ then by [1, Theorem 1.5(d)], $\mathrm{int}_{\tau^\alpha}(S) = \emptyset$. $\square$

The next two theorems slightly improve the well-known Baire's result.

Theorem 2.5 *For a Baire space (X, τ) and for every 1st category subset $F \subset X$, $\mathrm{sint}_\tau(F) = \emptyset$ (and $\mathrm{int}_{\tau^\alpha}(F) = \emptyset$).*

Proof. By Lemma 2.4. $\square$

Theorem 2.6 (*Baire Category Theorem*) *Let (X, τ) be a nonempty Baire space. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SC}\,(X, \tau)$ is a family covering X, then at least one A_n must contain a nonempty member of τ; i.e., $\mathrm{int}_\tau(A_n) \neq \emptyset$ for at least one n.*

Proof. Since $X = \bigcup_{n\in\mathbb{N}} A_n$, where each $A_n \in \mathrm{SC}\,(X, \tau)$, then $\bigcap_{n\in\mathbb{N}}(X \setminus A_n) = \emptyset$ where each $X \setminus A_n \in \mathrm{SO}\,(X, \tau)$. Not every set $X \setminus A_n$ is dense (X, τ), since in the opposite case, by Theorem 2.2(*3*), $X = \mathrm{cl}\left(\bigcap_{n\in\mathbb{N}}(X \setminus A_n)\right) = \emptyset$. Consequently, $\mathrm{scl}_\tau(X \setminus A_{n_0}) \subsetneq X$ for some $n_0 \in \mathbb{N}$. Thus $\mathrm{sint}_\tau(A_{n_0}) = X \setminus \mathrm{scl}_\tau(X \setminus A_{n_0}) \neq \emptyset$ and by Lemma 2.4($\mathfrak{a}$), $\mathrm{int}_\tau(A_{n_0}) \neq \emptyset$. $\square$

Corollary 2.7 *Let (X, τ) be a nonempty Baire space. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{c}\,(X, \tau^\alpha)$ covers X, then $\mathrm{int}_\tau(A_{n_0}) \neq \emptyset$ for a certain $n_0 \in \mathbb{N}$.*

Observe that the result of Corollary 2.7 can be obtained directly by the classical version of Baire category theorem applied to the space (X, τ^α) (see Corollary 2.3 and Lemma 2.4($\mathfrak{b}$)).

By [6, Theorem 1.13(ii)] it is known that (X, τ) is a Baire space if and only if its every nonempty open subset is of the 2nd category. We offer another characterizations of Baire spaces.

Theorem 2.8 *The following statements are equivalent for any space (X, τ):*

($\mathfrak{a}$) (X, τ) is a Baire space;
($\mathfrak{b}$) for every subfamily $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SC}\,(X, \tau)$, where $\mathrm{int}\,(A_n) = \emptyset$, $n \in \mathbb{N}$, the union $\bigcup_{n\in\mathbb{N}} A_n$ has no interior point in (X, τ);

($\mathfrak{c}$) *every nonempty set* $S \in \mathrm{SO}\,(X,\tau)$ *is of the 2nd category in* (X,τ).

Proof. $(\mathfrak{a}) \Rightarrow (\mathfrak{b})$. Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SC}\,(X,\tau)$, where $\mathrm{int}\,(A_n) = \emptyset$, $n \in \mathbb{N}$. By the dual to [12, Lemma 2] we get that $\mathrm{int}_\tau(\mathrm{cl}_\tau(A_n)) = \emptyset$. So, $\bigcup_{n\in\mathbb{N}} A_n$ is of the 1st category. Thus $\mathrm{int}_\tau\big(\bigcup_{n\in\mathbb{N}} A_n\big) = \emptyset$. $(\mathfrak{b}) \Rightarrow (\mathfrak{c})$. Let $\emptyset \neq S \in \mathrm{SO}\,(X,\tau)$. Then for a certain nonempty $O_s \in \tau$, $O_s \subset S$ (see [8] and [3, Remark 1.2]). By [6, Theorem 1.13,(iv)$\Rightarrow$(i)], O_s is of the 2nd category. Thus S is of the 2nd category. $(\mathfrak{c}) \Rightarrow (\mathfrak{a})$. Obvious. $\square$

It is known that each open subset of a Baire space is also Baire (as a subspace): see [4, p.256, Problem 2] and [6, Proposition 1.14].

Theorem 2.9 *If* (X,τ) *is a Baire space, then for each* $G \in \mathrm{SO}\,(X,\tau)$, (G,τ_G) *is also Baire.*

Proof. Let $G \in \mathrm{SO}\,(X,\tau)$ and let $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}\,(G,\tau_G)$ be any dense family. By [11, Theorem 5], $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SO}\,(X,\tau)$. Put $S_n = A_n \cup (X \setminus \mathrm{cl}\,(G)) \in \mathrm{SO}\,(X,\tau)$ for each $n \in \mathbb{N}$. First, we shall show that $\mathrm{cl}_\tau(S_n) = X$ for each $n \in \mathbb{N}$. We have $\mathrm{cl}\,(S_n) \supset \mathrm{cl}_{\tau_G}(A_n) \cup \mathrm{cl}_\tau\big(X \setminus \mathrm{cl}_\tau(G)\big) = G \cup \mathrm{cl}_\tau\big(X \setminus \mathrm{cl}_\tau(G)\big)$. So, $\mathrm{cl}_\tau\big[G \cup \mathrm{cl}_\tau\big(X \setminus \mathrm{cl}_\tau(G)\big)\big] \subset \mathrm{cl}_\tau(S_n) \subset X$. Then $\mathrm{cl}_\tau\big[\mathrm{cl}_\tau(G) \cup (X \setminus \mathrm{cl}_\tau(G))\big] = X \subset \mathrm{cl}_\tau(S_n) \subset X$. Now, since (X,τ) is Baire, $\mathrm{cl}_\tau\big(\bigcap_{n\in\mathbb{N}} S_n\big) = X$ (Theorem 2.2(*3*)); i.e.,

$$X = (X \setminus \mathrm{cl}_\tau(G)) \cup \bigcap_{n\in\mathbb{N}} A_n.$$

Hence $G = G \cap \bigcap_{n\in\mathbb{N}} A_n$ and consequently $\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(G))) = \mathrm{cl}_\tau\big(\bigcap_{n\in\mathbb{N}} A_n\big)$. Using [12, Lemma 2] one gets $G \subset \mathrm{cl}_\tau\big(\bigcap_{n\in\mathbb{N}} A_n\big)$. Therefore $\mathrm{cl}_{\tau_G}\big(\bigcap_{n\in\mathbb{N}} A_n\big) = G$. $\square$

Recall that a collection of subsets of a space (X,τ) is called ***pseudo-cover*** [6, p.13] if the union of its elements is dense in (X,τ). A pseudo-cover is said to be ***semi-open*** (resp. ***open***) if all its members are semi-open (resp. open).

Theorem 2.10 *If there exists a semi-open pseudo-cover of* (X,τ) *whose all members are Baire subspaces of* (X,τ), *then* (X,τ) *is a Baire space.*

Proof. Let $\{S_n\}_{n\in\mathbb{N}} \subset \tau$ be any dense family in (X,τ), and let $\{G_i\}_{i\in I\subset X} \subset \mathrm{SO}\,(X,\tau)$ be a pseudo-cover of X such that each space (G_i,τ_{G_i}) is Baire, $i \in I$. By [12, Lemma 1], $\{S_n \cap G_{i_0}\}_{n\in\mathbb{N}} \subset \mathrm{SO}\,(X,\tau)$ for each fixed $i_0 \in Y$. Applying [11, Theorem 5] we obtain $\{S_n \cap G_{i_0}\}_{n\in\mathbb{N}} \subset \mathrm{SO}\,(G_{i_0},\tau_{G_{i_0}})$. We shall show that $\mathrm{scl}_{\tau_{G_{i_0}}}(S_n \cap G_{i_0}) = G_{i_0}$ for each $n \in \mathbb{N}$. Take into consideration any $x \in G_{i_0}$ and $U_x \in \mathrm{SO}\,(G_{i_0},\tau_{G_{i_0}})$ with $x \in U_x$. By [11, Theorem 5] each $U_x \in \mathrm{SO}\,(X,\tau)$ and since $\mathrm{scl}_\tau(S_n) = X$, $n \in \mathbb{N}$, we infer from [13, Lemma 3] that $U_x \cap S_n \neq \emptyset$ for each $x \in G_{i_0}$ and $n \in \mathbb{N}$. So, $U_x \cap (S_n \cap G_{i_0}) \neq \emptyset$. Hence, we get $x \in \mathrm{scl}_{\tau_{G_{i_0}}}(S_n \cap G_{i_0})$; i.e., $\mathrm{scl}_{\tau_{G_{i_0}}}(S_n \cap G_{i_0}) = G_{i_0}$. By hypothesis and Theorem 2.2, $\mathrm{cl}_{\tau_{G_{i_0}}}\left(\bigcap_{n\in\mathbb{N}}(S_n \cap G_{i_0})\right) = G_{i_0}$. Now, we calculate as follows:

$$X = \mathrm{cl}_\tau\left(\bigcup_{i\in I} G_i\right) = \mathrm{cl}_\tau\left[\bigcup_{i\in I}\mathrm{cl}_{\tau_{G_i}}\left(\bigcap_{n\in\mathbb{N}}(S_n \cap G_i)\right)\right] \subset$$
$$\subset \mathrm{cl}_\tau\left[\bigcup_{i\in I}\left(G_i \cap \bigcap_{n\in\mathbb{N}} S_n\right)\right] =\subset \mathrm{cl}_\tau\left[\bigcap_{n\in\mathbb{N}} S_n \cap \bigcup_{i\in I} G_i\right] \subset \mathrm{cl}_\tau\left(\bigcap_{n\in\mathbb{N}} S_n\right).$$

Therefore, (X,τ) is a Baire space. □

Corollary 2.11 *The following statements are equivalent for any space (X,τ):*

($\mathfrak{a}$) (X,τ) is a Baire space;
($\mathfrak{b}$) for each $G \in \mathrm{SO}(X,\tau)$, (G,τ_G) is Baire;
($\mathfrak{c}$) (X,τ) has a semi-open pseudo-cover, whose members are Baire subspaces of (X,τ).

The reader is advised to compare this corollary with [6, Theorem 1.21] (and [6, Corollary 1.22]). Using also [6, Proposition 1.14] one gets a result similar to our Corollary 2.11:

Corollary 2.12 *The following statements are equivalent for a space (X,τ):*

($\mathfrak{a}$) (X,τ) is a Baire space;
($\mathfrak{b}$) for each $G \in \tau$, the subspace (G,τ_G) is Baire;
($\mathfrak{c}$) (X,τ) has an open pseudo-cover, whose members are Baire subspaces of (X,τ).

Proof. Directly by Theorems 2.9 and 2.10. □

Theorem 2.13 (*Baire category theorem for open subsets*) *Let (X,τ) be a Baire space. If $G \in \tau$ admits a cover $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{c}(X,\tau)$, then at least one A_n must have a nonvoid interior.*

First we shall prove a lemma.

Lemma 2.14 *If $S \in \mathrm{SO}(X,\tau)$, then $\mathrm{int}_{\mathrm{cl}_\tau(S)}(\mathrm{cl}_\tau(S) \setminus S) = \emptyset$.*

Proof. Applying [12, Lemma 2] we have $\mathrm{cl}_\tau(S) \setminus S \subset \mathrm{cl}_\tau(S) \setminus \mathrm{int}_\tau(S) = \mathrm{cl}_\tau(\mathrm{int}_\tau(S)) \setminus \mathrm{int}_\tau(S)$. Thus $\mathrm{cl}_\tau(S) \setminus S$ is nowhere dense in (X,τ). Suppose now $\mathrm{int}_{\mathrm{cl}_\tau(S)}(\mathrm{cl}_\tau(S) \setminus S) \neq \emptyset$. Hence $\emptyset \neq O \cap \mathrm{cl}_\tau(S) \subset \mathrm{cl}_\tau(S) \setminus S$ for a certain $O \in \tau$. But, by $O \cap \mathrm{cl}_\tau(S) \subset \mathrm{cl}_\tau(O \cap S)$, $O \cap S$ must be nonempty. Since $O \cap S = (O \cap \mathrm{cl}_\tau(S)) \cap S = \emptyset$, we get a contradiction. □

Proof of Theorem 2.13. Since $\mathrm{cl}_\tau(G) \in \mathrm{SO}(X,\tau)$, by Theorem 2.9 $\left(\mathrm{cl}_\tau(G), \tau_{\mathrm{cl}_\tau(G)}\right)$ is a Baire subspace of (X,τ). We have $\mathrm{cl}(G) = (\mathrm{cl}(G) \setminus G) \cup \bigcup_{n\in\mathbb{N}} A_n$. Obviously, the sets $\mathrm{cl}(G) \setminus G$ and A_n for each $n \in \mathbb{N}$ are closed in $\left(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)}\right)$. Moreover, $\mathrm{cl}(G) \setminus G$ is nowhere dense in $\left(\mathrm{cl}(G), \tau_{\mathrm{cl}(G)}\right)$ (Lemma 2.14). Thus, for a certain $n_0 \in \mathbb{N}$, $\mathrm{int}_{\mathrm{cl}(G)}(A_{n_0}) \neq \emptyset$, that is, $\emptyset \neq U \cap \mathrm{cl}(G) \subset A_{n_0}$ and consequently $\mathrm{int}_\tau(A_{n_0}) \neq \emptyset$. □

Corollary 2.15 *Let G be an open subset of a complete metric space (X, ϱ). Then:* ($\mathfrak{a}$) *G contains a closed subset F with $\operatorname{int}_\varrho(F) \neq \emptyset$,* ($\mathfrak{b}$) *$(X, \varrho)$ is $\mathcal{S}$-disconnected if $G \neq X$.*

Proof. ($\mathfrak{a}$) By Theorem 2.13 (each $G \in \tau$ is $\mathcal{F}_\sigma$ in (X, ϱ)). ($\mathfrak{b}$) It is enough to observe that $\emptyset \neq \operatorname{cl}_\varrho(\operatorname{int}_\varrho(F)) \subset F$. □

It is known that in a Baire space (X, τ) every 1st category set $S \in \mathrm{c}\,(X, \tau)$ is nowhere dense. This property holds on for $S \in \mathrm{SC}\,(X, \tau)$.

Lemma 2.16 *For any subset S of a space (X, τ),* $\operatorname{cl}(\operatorname{sint}(S)) = \operatorname{cl}(\operatorname{int}(S))$.

Proof. By [1, Theorem 1.5(b)] we have $\operatorname{sint}(S) = S \cap \operatorname{cl}(\operatorname{int}(S))$. So,

$$\operatorname{cl}(\operatorname{int}(S)) \subset \operatorname{cl}\big(S \cap \operatorname{cl}(\operatorname{int}(S))\big) \subset \operatorname{cl}(\operatorname{int}(S)).\square$$

Corollary 2.17 *A subset S of (X, τ) is nowhere dense if and only if* $\operatorname{cl}(X \setminus \operatorname{scl}(S) = X$.

Proof. By the use of Lemma 2.16 it is enough to observe that

$$\operatorname{cl}(X \setminus \operatorname{scl}(S)) = \operatorname{cl}(\operatorname{sint}(X \setminus S)) = \operatorname{cl}(\operatorname{int}(X \setminus S)) = \operatorname{cl}(X \setminus \operatorname{cl}(S)).\square$$

Theorem 2.18 *In a Baire space (X, τ) every 1st category set $S \in \mathrm{SC}\,(X, \tau)$ is nowhere dense.*

Proof. Let $S = \bigcup_{n \in \mathbb{N}} S_n \in \mathrm{SC}\,(X, \tau)$, where the sets S_n, $n \in \mathbb{N}$, are nowhere dense. Obviously, all the sets $X \setminus \operatorname{cl}(S_n) \in \tau$ are dense in (X, τ). Hence, by our assumption,

$$X = \operatorname{cl}\left(\bigcap_{n \in \mathbb{N}} (X \setminus \operatorname{cl}(S_n))\right) = \operatorname{cl}\left(X \setminus \bigcup_{n \in \mathbb{N}} \operatorname{cl}(S_n)\right) \subset \operatorname{cl}(X \setminus S).$$

But $S \in \mathrm{SC}\,(X, \tau)$, whence by [3, Theorem 1.4(2)] and Corollary 2.17, S is nowhere dense. □

Remark 2.1 It is worth being noticed that in any topological space (X, τ) the notions of *boundarity* and *nowhere density* are equivalent among semi-closed subsets. Indeed, by [12, Lemma 2] one has $\operatorname{cl}(X \setminus S) = \operatorname{cl}(\operatorname{int}(X \setminus S)) = \operatorname{cl}(X \setminus \operatorname{cl}(S))$.

A subset S of a space (X, τ) will be called $\boldsymbol{s}$-$\mathcal{G}_\delta$ if $S = \bigcap_{n \in \mathbb{N}} U_n$ for some $\{U_n\}_{n \in \mathbb{N}} \subset \mathrm{SO}\,(X, \tau)$.

Theorem 2.19 (*compare with [6, Proposition 1.17]*) *Let Y be a subset of a Baire space (X, τ). Then Y is of 1st category if and only if $X \setminus Y$ contains a dense s-$\mathcal{G}_\delta$ subset of (X, τ).*

Proof. $\mathfrak{Necessity}$. Suppose $Y = \bigcup_{i\in\mathbb{N}} N_i$, where for each $i \in \mathbb{N}$ the sets N_i are nowhere dense. Since the sets $X \setminus \mathrm{scl}_\tau(N_i)$ are semi-open in (X,τ), $A = \bigcap_{i\in\mathbb{N}}(X \setminus \mathrm{scl}_\tau(N_i))$ is a dense s-$\mathcal{G}_\delta$ subset of (X,τ), because (X,τ) is a Baire space (Theorem 2.2(*3*)). Moreover, $A \subset X \setminus Y$.

$\mathfrak{Sufficiency}$. Let $A = \bigcap_{i\in\mathbb{N}} U_i \subset X \setminus Y$ be a dense s-$\mathcal{G}_\delta$ subset of (X,τ). Then, for each $i \in \mathbb{N}$ we have $\mathrm{cl}_\tau(U_i) = X$ and applying Corollary 2.17, it follows that the sets $X \setminus U_i$ are nowhere dense in (X,τ). Thus, $Y \subset X \setminus A$ is of 1st category in (X,τ). □

A space (X,τ) is called ***locally indiscrete*** [9] if each point $x \in X$ has an indiscrete neighbourhood A_x, that is, such a set $A_x \subset X$, $A_x \ni x$, that either $A_x \subset U$ or $A_x \cap U = \emptyset$ for each $U \in \tau$. The following characterizations of these spaces were found by Janković [7, Theorem 3.3]: (X,τ) is locally indiscrete $\Leftrightarrow$ (b) every open (closed) subset is closed (open) in (X,τ) $\Leftrightarrow$ (c) $\mathrm{cl}\,(\{x\}) \in \tau$ for each $x \in X$.

Remark 2.2 The following easily follows by [7, Theorem 3.3(b)]: every locally indiscrete space is Baire.

Theorem 2.20 *Assume that (X,τ) is not locally indiscrete, T_1-space. If (X,τ) is Baire, then there is a dense (in (X,τ)) set $X_0 \in \tau$ such that (X_0, τ_{X_0}) is also Baire.*

Proof. By [7, Theorem 3.3(c)] there exists a point $x_0 \in X$ with $\mathrm{cl}_\tau(\{x_0\}) \notin \tau$. Since (X,τ) is T_1, $\{x_0\} \notin \tau$. Put $X_0 = X \setminus \{x_0\}$. X_0 is dense in (X,τ), because $x_0 \in \mathrm{cl}_\tau(X_0)$. By [6, Proposition 1.14] (or Theorem 2.9) (X_0, τ_{X_0}) is a Baire space. □

Making use of [6, Theorem 1.15] one gets

Corollary 2.21 *Assume (X,τ) is not locally indiscrete, T_1-space. Then, (X,τ) is Baire if and only if there exists a dense $X_0 \in \tau$ such that (X_0, τ_{X_0}) is Baire.*

The concluding theorem generalizes the implication $(\mathfrak{a}) \Rightarrow (\mathfrak{c})$ of Theorem 2.8 ((X,τ) is of the 2nd category if it is of the 2nd category in itself).

Theorem 2.22 (*compare with [6, Proposition 1.15]*) *Assume (X,τ) is of the 2nd category. Every dense s-$\mathcal{G}_\delta$ subset of (X,τ) is of the 2nd category as well.*

Proof. If $S = \bigcap_{n\in\mathbb{N}} U_n$ is a dense s-$\mathcal{G}_\delta$ subset of (X,τ), then in view of Corollary 2.17 we have that each $X \setminus U_n$, $n \in \mathbb{N}$, is nowhere dense in (X,τ). Thus, if S were of the 1st category, as $X \setminus S = \bigcup_{n\in\mathbb{N}}(X \setminus U_n)$, (X,τ) would have been so. □

References

1. ANDRIJEVIĆ, D. – *Semi-preopen sets*, Mat. Vesnik, **38** (1986), no. 1, 24–32.
2. BAIRE, R. – *Sur les fonctions de variable réeles*, Annali di Matematica Pura ed Applicata, **3** (1899), no. 1, 65.
3. CROSSLEY, S.G.; HILDEBRAND, S.K. – *Semi-closure*, Texas J. Sci., **22** (1971), no. 2-3, 99–112.
4. DUGUNDJI, J. – *Topology*, Allyn and Bacon, Inc., Boston, Mass, 1966, xvi+447 pp.
5. HAMLETT, T.R. – *Semi-continuous functions*, Math. Chronicle, **4** (1976), no. 2-3, 101–107.
6. HAWORTH, R.C.; MCCOY, R.A. – *Baire spaces*, Dissertationes Math. (Rozprawy Mat.), PWN, Warszawa, **141** (1977), 73 pp.
7. JANKOVIĆ, D.S. – *On locally irreducible spaces*, Ann. Soc. Sci. Bruxelles, Sér. I **97** (1983), no. 2, 59–72.
8. LEVINE, N. – *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.
9. NIEMINEN, T. – *On ultrapseudocompact and related spaces*, Ann. Acad. Sci. Fennicae, Ser. A I Math, **3** (1977), no. 2, 185–205.
10. NJÅSTAD, O. – *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.
11. NOIRI, T. – *Remarks on semi-open mappings*, Bull. Calcutta Math. Soc., **65** (1973), no. 4, 197–201.
12. NOIRI, T. – *On semi-continuous mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **54** (1973), 210–214.
13. NOIRI, T. – *A note on semi-continuous mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **55** (1973), 400–403 (1974).
14. OSGOOD, W.F. – *Non-uniform convergence and the integration of series term by term*, Amer. J. Math., **19** (1897), no. 2, 155–190.
15. PIPITONE, V.; RUSSO, G. – *Spazi semiconnessi e spazi semiaperti* (Italian), Rend. Circ. Mat. Palermo (2), **24** (1975), no. 3, 273–285.

Received: 4.V.2014 / Accepted: 10.II.2015

AUTHOR

ZBIGNIEW DUSZYŃSKI,
Institute of Mathematics,
Casimirus the Great University,
Plac Weyssenhoffa 11,
85-072 Bydgoszcz, Poland,
E-mail: imath.duzb@gmail.com