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Some remarks on metallic Riemannian structures

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Abstract In this paper we make three remarks about metallic Riemannian manifolds. Firstly we give an integrability condition using a Codazzi type equation, then we consider another metallic structure on a metallic Riemannian manifold and we study the mixed twin metric associated to them. Finally, we show that a Riemannian foliated manifold has a natural (p, q)-metallic structure and every leaf of the foliation carries a metallic structure which is always integrable.

Keywords metallic structures · integrability · foliations

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1 Introduction

In the 60's-70's some authors introduced and studied polynomial structures on manifolds, closely related to known structures as almost product, almost complex and contact structures. They give properties and integrability conditions for such structures, [3], [11]. A polynomial structure f of degree k on a $\mathbf{C}^\infty$ manifold M is a $(1,1)$-tensor field satisfying a polynomial equation

$$f^k + a_1 f^{k-1} + ... + a_{k-1} f + a_k \cdot Id = 0,$$

where $a_1, ...a_k$ are real numbers and k is the smallest integer for which $Id, f, ..., f^k$ are dependent.

The particular cases of golden, then metallic structures were considered in the last decade. Some properties of the submanifolds of manifolds endowed with golden/metallic structure are investigated in [4], [5], [6], [7], since the integrability conditions for such structures and another properties are given in [1], [2]. The metallic structures are related to almost product structures; the class of Riemannian almost product manifolds with nonintegrable structure is considered in [8]. In this paper we continue these studies about metallic structures on Riemannian manifolds.

After a briefly introduction in particularities of metallic Riemannian manifolds, in section 2 we give an integrability condition for the metallic structure in terms of Codazzi type equation, inspired by similar conditions for almost complex or almost para-Norden structures, [9].

In section 3, from the duality between metallic and almost product structures, we investigate a (p,q)-metallic manifold (M,φ) with the almost product J induced by φ, endowed with a new distinct (p',q')-metallic structure determined by J. We define a mixed twin metric and we prove that it is decomposable when the initial metric is decomposable.

Finally, in section 4, we make some remarks about metallic structures on foliated manifolds. We prove that every leaf of a foliated Riemannian manifold is a (p,q)-metallic manifold with the metallic structure integrable.

2 Metallic Riemannian manifolds

Let (M,g) be a Riemannian n-dimensional $\mathbf{C}^\infty$-manifold. We present metallic Riemannian manifolds following [7].

A (p,q)-metallic structure on M is a polynomial structure of second degree given by a $(1,1)$-tensor field φ which satisfies the equation,

$$\varphi^2 - p\cdot\varphi - q\cdot Id = 0, \tag{2.1}$$

where Id is the identity on the vector fields space $\Gamma(TM)$ and p,q are fixed integers such that the equation $x^2-px-q=0$ has a positive irrational root $\sigma_{p,q}$.

The number $\sigma_{p,q}$ is usually named a member of the metallic family. For example, we can talk about golden structure if $p=1$, $q=1$ when the root $\sigma_{1,1}$ is exactly the golden ratio $\Phi=\frac{1+\sqrt{5}}{2}$, or about the silver structure ($p=2$, $q=1$, $\sigma_{2,1}=1+\sqrt{2}$), the bronze structure ($p=3$, $q=1$, $\sigma_{3,1}=\frac{3+\sqrt{13}}{2}$), the nickel structure ($p=1$, $q=3$, $\sigma_{1,3}=\frac{1+\sqrt{13}}{2}$). The above numbers are closely related with different mathematical domains as dynamical systems, quasicrystals, theory of Cantorial fractal-like micro-space-time.

The main properties of a (p,q)-metallic structure are given in [7]:

Proposition 2.1 *Let φ a (p,q)-metallic structure on the manifold M. Then φ is an isomorphism on the tangent space T_xM, for every $x\in M$. Its inverse is $\varphi^{-1}=\frac{1}{q}\varphi-\frac{p}{q}Id$, which is not a metallic structure but still a polynomial one. The eigenvalues of φ are $\sigma_{p,q}$ and $p-\sigma_{p,q}$.*

For the Riemannian manifold (M,g) endowed with the (p,q)-metallic structure φ, we say that the metric g is φ-compatible and that M is a Riemannian metallic manifold, if

$$g(\varphi X,Y)=g(X,\varphi Y), \tag{2.2}$$

for every $X,Y\in\Gamma(TM)$. An equivalent condition is

$$g(\varphi X,\varphi Y)=p\cdot g(X,\varphi Y)+q\cdot g(X,Y).$$

An important tool in study of metallic manifolds is the almost product structure (an $(1,1)$-tensor field J with $J^2 = Id$) induced by φ, [7]:

Proposition 2.2 *Every metallic structure φ given by* (2.1) *on M induces two almost product structures on this manifold:*

$$J_{\pm} = \pm\left(\frac{2}{2\sigma_{p,q} - p}\cdot\varphi - \frac{p}{2\sigma_{p,q} - p}\cdot Id\right). \tag{2.3}$$

Conversely, every almost product structure J on M induces two metallic structures on M given as follows:

$$\varphi_1 = \frac{p}{2}\cdot Id + \frac{2\sigma_{p,q} - p}{2}\cdot J; \quad \varphi_2 = \frac{p}{2}\cdot Id - \frac{2\sigma_{p,q} - p}{2}\cdot J. \tag{2.4}$$

In the following we will denote by J one of the associated almost product structures in (2.3), usually J_+. For a metallic manifold (M, g, φ) and the associated almost product J, it is easy to see that

$$g(JX, Y) = g(X, JY), \quad \forall X, Y \in \Gamma(TM). \tag{2.5}$$

The integrability of structures φ, J are usually expressed by the vanishing of the Nijenhuis tensors N_φ, N_J, respectively,

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y].$$

In [2] are given integrability conditions using the Tachibana operators Φ_φ, Φ_J associated to φ and J, respectively. Generally, this operator is defined on $(0,r)$-tensors ω which are pure with respect to a $(1,1)$-tensor field ψ,

$$\omega(\psi Y_1, Y_2, ..., Y_r) = \omega(Y_1, \psi Y_2, ..., Y_r) = ... = \omega(Y_1, Y_2, ..., \psi Y_r),$$

by

$$\begin{aligned}(\Phi_\psi\omega)(X, Y_1, \ldots, Y_r) &= \psi X(\omega(Y_1, \ldots, Y_r)) - X(\omega(\psi Y_1, Y_2, \ldots, Y_r)) \\ &+\omega((L_{Y_1}\psi)X, Y_2, \ldots, Y_r) + \ldots + \omega(Y_1, Y_2, \ldots, (L_{Y_r}\psi)X),\end{aligned} \tag{2.6}$$

for every vector fields $X, Y_1, \ldots, Y_r$, where L_X denotes the Lie derivative with respect to X.

When the tensor ψ is an almost product, and $\Phi_\psi\omega = 0$, the tensor ω is said to be decomposable.

By relations (2.2) and (2.5), the metric g is a $(0,2)$-tensor pure with respect to φ and J, respectively. In [2], it is proved that:

Proposition 2.3 *Let (M, g, φ) be a metallic Riemannian manifold. Then:*

a) If g is decomposable, that is $\Phi_\varphi g = 0$, then the metallic structure is integrable.

b) The condition $\Phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.

c) The dependence between Φ_J and Φ_φ is as follows:

$$\Phi_J g = \pm\frac{2}{2\sigma_{p,q} - p}\cdot\Phi_\varphi g. \tag{2.7}$$

In the following we shall consider the metallic manifold (M, g, φ) with (2.1) and the associated almost product structure

$$J = \frac{2}{2\sigma_{p,q} - p} \cdot \varphi - \frac{p}{2\sigma_{p,q} - p} \cdot Id. \tag{2.8}$$

Our fist remark is about the integrability of φ, which is equivalent to $N_\varphi = 0$.

Let ∇ be the Levi-Civita connection of g. By a direct computation, we have the relations:

$$\nabla\varphi = \frac{2\sigma_{p,q} - p}{2}\nabla J. \tag{2.9}$$

$$N_\varphi(X,Y) = (\nabla_{\varphi X}\varphi)\,Y - (\nabla_{\varphi Y}\varphi)\,X + \varphi\left((\nabla_X\varphi)\,Y\right) - \varphi\left((\nabla_Y\varphi)\,X\right). \tag{2.10}$$

Moreover, the Nihenjuis tensor N_J could be expressed, using the symmetry of ∇ and $J^2 = Id$, by

$$N_J(X,Y) = (\nabla_{JX}J)\,Y - (\nabla_{JY}J)\,X + J\left((\nabla_X J)\,Y\right) - J\left((\nabla_Y J)\,X\right).$$

The link between the Nihenjuis tensors N_J and N_φ is

$$N_J = \frac{4}{p^2 + 4q} \cdot N_\varphi, \tag{2.11}$$

which show that the metallic structure φ (2.1) is integrable if and only if the associated almost product (2.8) is integrable.

The following condition for integrability of the metallic structure is inspired by the Codazzi type equation, [9].

Proposition 2.4 *Let (M, g, φ) be a (p, q)-metallic manifold. The metallic structure φ is integrable if it satisfies a Codazzi type equation,*

$$(d^\nabla\varphi)(X,Y) \coloneqq (\nabla_X\varphi)\,Y - (\nabla_Y\varphi)\,X = 0, \quad \forall X, Y \in \Gamma(TM), \tag{2.12}$$

where d^∇ is the exterior covariant derivative.

Proof. If (2.12) is true, then the Nihenjuis tensor (2.10) becomes

$$N_\varphi(X,Y) = (\nabla_{\varphi X}\varphi)\,Y - (\nabla_{\varphi Y}\varphi)\,X.$$

Replacing X, Y in (2.12) by φX, Y, then by φY, X, respectively and using (2.1), we obtain

$$\begin{aligned}
(\nabla_{\varphi X}\varphi)\,Y = (\nabla_Y\varphi)\,\varphi X &= \nabla_Y(\varphi^2 X) - \varphi(\nabla_Y(\varphi X)) = \\
&= p\nabla_Y(\varphi X) + q\nabla_Y X - \varphi(\nabla_Y(\varphi X)), \\
(\nabla_{\varphi Y}\varphi)\,X = (\nabla_X\varphi)\,\varphi Y &= \nabla_X(\varphi^2 Y) - \varphi(\nabla_X(\varphi Y)) = \\
&= p\nabla_X(\varphi Y) + q\nabla_X Y - \varphi(\nabla_X(\varphi Y)).
\end{aligned}$$

Relation (2.12) gives us

$$\varphi[X,Y] = \varphi(\nabla_X Y - \nabla_Y X) = \nabla_X(\varphi Y) - \nabla_Y(\varphi X),$$

$$\varphi^2[X,Y] = \varphi(\nabla_X(\varphi Y)) - \varphi(\nabla_Y(\varphi X)).$$

We compute

$$N_\varphi(X,Y) = -p\varphi[X,Y] - q[X,Y] + \varphi^2[X,Y] = 0,$$

from (2.1), hence φ is integrable.

□

3 Multiple metallic structures

Our second remark is referring to the duality between metallic structures and almost product structures, Proposition 2.2.

Let (M, g, φ) be a Riemannian (p, q)-metallic manifold and J given by (2.8), the almost product structure induced by φ on M.

We fix two integers $p' \neq p$, $q' \neq q$ such that the positive root $\sigma_{p',q'}$ of the equation $x^2 - p'x - q' = 0$ is irrational.

From Proposition 2.2, we can introduce on the (p, q)-metallic manifold M a (p', q')-metallic structure φ' by relation

$$\varphi' = \frac{p'}{2} \cdot Id + \frac{2\sigma_{p',q'} - p'}{2} \cdot J. \tag{3.1}$$

The above relation and (2.5) show that the Riemannian metric g is φ'-compatible,

$$g(\varphi' X, Y) = g(X, \varphi' Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.2}$$

By direct computation, from relations (2.8) and (3.1) we obtain that

Proposition 3.1 *The link between the initial metallic structure and the new one is*

$$\varphi' = \frac{\sigma_{p,q} \cdot p' - \sigma_{p',q'} \cdot p}{2\sigma_{p,q} - p} \cdot Id + \frac{2\sigma_{p',q'} - p'}{2\sigma_{p,q} - p} \cdot \varphi. \tag{3.3}$$

Obviously, using now the relation (2.8) for the metallic structure φ', we obtain that this new metallic structure induces the same almost product J on M.

We can also see that

Proposition 3.2 *For a Riemannian metallic manifold (M, g, φ) with the almost product J, any metallic structure induced by J on M is integrable if and only if the initial metallic structure is integrable.*

Proof. Using relation (2.11) for the Nihenjuis tensors N_J, N_φ and $N_{\varphi'}$, respectively, we have

$$N_\varphi = \frac{p^2+4q}{p'^2+4q'} N_{\varphi'},$$

which proves the statement. □

In order to see the link between the Tachibana operators associated to φ and φ', respectively, we can use relations (2.7), (2.9), or the expression of Φ_φ with respect to the Levi-Civita connection of g. Let ∇ be the Levi-Civita connection of g. Then, from the definition of the Tachibana operator and ∇g=0, we have

$$\Phi_\varphi g(X,Y,Z) = g((\nabla_Y\varphi)X,Z) - g((\nabla_X\varphi)Y,Z) + g((\nabla_Z\varphi)X,Y). \quad (3.4)$$

From (3.3), by direct computation, we obtain

$$\nabla\varphi' = \frac{2\sigma_{p',q'} - p'}{2\sigma_{p,q} - p}\nabla\varphi, \quad (3.5)$$

and replacing this in (3.4), it results

$$\Phi_{\varphi'} g = \frac{2\sigma_{p',q'} - p'}{2\sigma_{p,q} - p}\Phi_\varphi g \quad (3.6)$$

The metric g is satisfying $\Phi_\varphi g = 0$ if and only if $\Phi_{\varphi'} g = 0$.

Inspired by the twin Riemannian metallic metric associated to g by φ, [2], we can define a twin metric associated to g by the pair of above metallic structures (φ, φ'):

$$\tilde{g}(X,Y) = g(\varphi X, \varphi' Y), \quad \forall X, Y \in \Gamma(TM), \quad (3.7)$$

and we call it a *mixed twin metric*.

Proposition 3.3 *The mixed twin metric is pure with respect to φ and φ', respectively.*

Proof. According to (2.2), (3.2), and using the following relation obtained from (3.3),

$$\varphi\varphi' = \varphi'\varphi = \frac{\sigma_{p,q}\cdot p' + \sigma_{p',q'}\cdot p - pp'}{2\sigma_{p,q} - p}\cdot\varphi + q\frac{2\sigma_{p',q'} - p'}{2\sigma_{p,q} - p}\cdot Id,$$

it results that

$$\tilde{g}(\varphi X, Y) = g(\varphi^2 X, \varphi' Y) = g(\varphi X, \varphi\varphi' Y) = g(\varphi X, \varphi'\varphi Y) = \tilde{g}(X, \varphi Y),$$

$$\tilde{g}(\varphi' X, Y) = g(\varphi\varphi' X, \varphi' Y) = g(\varphi'\varphi X, \varphi' Y) = g(\varphi X, \varphi'\varphi' Y) = \tilde{g}(X, \varphi' Y),$$

hence the mixed twin metric is pure with respect to φ and φ', respectively. □

Proposition 3.4 *If the Riemannian metric g is decomposable (i.e. $\Phi_J g = 0$), then the Levi-Civita connection of g is a metric connection with respect to the mixed twin metric. Moreover, the mixed twin metric is also decomposable.*

Proof. Taking into account $\nabla g = 0$, we compute

$$(\nabla_X \tilde{g})(Y,Z) = g((\nabla_X \varphi)Y, \varphi Z) + g(\varphi Y, (\nabla_X \varphi')Z).$$

Using now the equivalence between $\Phi_J g = 0$, $\Phi_\varphi g = 0$ and $\nabla\varphi = 0$, (see Proposition 2.3), and relation (3.5), the decomposability of g gives us $\nabla\tilde{g} = 0$. Hence, a similar expression with (3.4) could be given for $\Phi_\varphi \tilde{g}$:

$$(\Phi_\varphi \tilde{g})(X,Y,Z) = \tilde{g}((\nabla_Y \varphi)X, Z) - \tilde{g}((\nabla_X \varphi)Y, Z) + \tilde{g}((\nabla_Z \varphi)X, Y),$$

which shows that the decomposability of g implies the vanishing of $\Phi_\varphi \tilde{g}$. Then we also have $\Phi_J \tilde{g} = 0$ which means that $\tilde{g}$ is also a decomposable metric. □

4 Metallic structures on foliated manifolds

In this section we study a particular case of metallic manifolds. Let (M, g) be a Riemannian n-dimensional manifold endowed with a r-codimensional foliation $\mathcal{F}$. More about foliated manifolds could be find in [10], [12].

The structural bundle is denoted by $T\mathcal{F}$ and the transversal bundle, $T^\perp\mathcal{F}$, is orthogonal on $T\mathcal{F}$ by g. Obviously, there is the decomposition $TM = T\mathcal{F} \oplus T^\perp\mathcal{F}$. If v, h are projections of TM on $T\mathcal{F}$ and $T^\perp\mathcal{F}$, respectively, then the $(1,1)$ tensor fields

$$I = v - h,$$

satisfies $I^2 = Id$, so it is an almost product structure on M.

Let us consider φ a (p,q)-metallic structure associated to I (see relation (2.4)), by

$$\varphi = \frac{p}{2} \cdot Id + \frac{2\sigma_{p,q} - p}{2} \cdot I. \tag{4.1}$$

By direct calculus, we have

$$g(IX, Y) = g(vX - hX, vY + hY) = g(vX, vY) + g(hX, -hY) = g(X, IY),$$

$\forall\, X, Y \in \Gamma(TM)$, which, using (4.1), gives us

$$g(\varphi X, Y) = g(X, \varphi Y), \forall X, Y \in \Gamma(TM).$$

We obtained that

Proposition 4.1 *Let p, q be two fixed integers such that the equation $x^2 - px - q = 0$ has a positive irrational root $\sigma_{p,q}$. Every Riemannian foliated manifold has a natural (p,q)-metallic Riemannian manifold structure given by* (4.1).

It is known, [7], that on a Riemannian metallic manifold (M, g, φ), there are two orthogonal distributions (we shall call them metallic distributions), defined by projections

$$l = \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} \cdot Id - \frac{1}{2\sigma_{p,q} - p} \cdot \varphi, \quad m = \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} \cdot Id + \frac{1}{2\sigma_{p,q} - p} \cdot \varphi,$$

$$\mathcal{D}_l = Ker(m), \quad \mathcal{D}_m = Ker(l).$$

Proposition 4.2 *Let $(M, g, \mathcal{F})$ be a Riemannian foliated manifold and φ the natural (p, q)-metallic structure on it. The metallic distributions $\mathcal{D}_m$ and $\mathcal{D}_l$ are exactly the structural and the transversal distributions of foliation $\mathcal{F}$, respectively.*

Proof. For the metallic structure φ given in (4.1), the projectors l and m become $l = h$, $m = v$, so the metallic distributions are

$$\mathcal{D}_l = Ker(v), \quad \mathcal{D}_m = Ker(h),$$

which proves the statement. □

Proposition 4.3 *The metallic structure on a foliated manifold is integrable if the transversal distribution of foliation $\mathcal{F}$ is integrable.*

Proof. Taking into account the link (2.11) between the Nijenhuis tensors N_φ and N_I, we shall prove that the almost product structure I is integrable if the transversal distribution of foliation $\mathcal{F}$ is also integrable. From the integrability of structural distribution, the Nijenhuis tensor N_I associated to almost product structure I is

$$N_I(X, Y) = 2h[vX, vY] + 2v[hX, hY] = 2v[hX, hY], \tag{4.2}$$

The tensor N_I vanishes if and only if $[X, Y] \in \Gamma(T^\perp\mathcal{F})$ for all $X, Y \in \Gamma(T^\perp\mathcal{F})$, which means that transversal distribution is integrable, too. □

Remark 4.1 a) The restriction of the Nijenhuis tensor N_φ to the structural bundle vanishes. Indeed, if $X, Y \in \Gamma(TM)$, then $hX = 0$, $hY = 0$ and in (4.2) we obtain $N_I(X, Y) = 0$, which implies $N_\varphi(X, Y) = 0$.

b) Generally speaking, in a metallic Riemannian manifold, the metallic structure is integrable if and only if both metallic distributions $\mathcal{D}_m$ and $\mathcal{D}_l$ are integrable. Indeed,

$$\mathcal{D}_m = \{X \in \Gamma(TM) | \quad \varphi X = \sigma_{p,q} X\} = \{X \in \Gamma(TM) | \quad JX = X\},$$

$$\mathcal{D}_l = \{X \in \Gamma(TM) | \quad \varphi X = (p - \sigma_{p,q}) X\} = \{X \in \Gamma(TM) | \quad JX = -X\},$$

where J is the associated almost product of φ.

For every $X \in \mathcal{D}_m$, $Y \in \mathcal{D}_l$ we have $N_J(X, Y) = 0$, while for every $X, Y \in \mathcal{D}_m$, $N_J(X, Y) = 0$ if $J[X, Y] = [X, Y]$ and for every $X, Y \in \mathcal{D}_l$, $N_J(X, Y) = 0$ if $J[X, Y] = -[X, Y]$. The result comes taking into account relation (2.11).

Now, we consider an atlas with the adapted local charts $(U,(x^a,x^u))$, $a=1,\ldots,r$, $u=r+1,\ldots,n$ on M such that the leaves are locally defined by $x^a=$ *constant*. In the following the indices $a,b,c,\ldots$ will take values from $1,2,\ldots,r$ and $u,v,w,\ldots$ will take values from $r+1,r+2,\ldots,n$.

The structural bundle $T\mathcal{F}$ is locally spanned by $\left\{\frac{\partial}{\partial x^u}\right\}$. The transversal bundle, $T^{\perp}\mathcal{F}$, orthogonal on $T\mathcal{F}$ by g, is locally spanned by

$$\frac{\delta}{\delta x^a}=\frac{\partial}{\partial x^a}-t_a^u\frac{\partial}{\partial x^u}, \tag{4.3}$$

where the smooth functions t_a^u are defined by orthogonality condition $g\left(\frac{\delta}{\delta x^a},\frac{\partial}{\partial x^u}\right)=0$, $\forall a,u$.

The almost product I and the metallic structure φ have the local expressions:

$$I\left(\frac{\delta}{\delta x^a}\right)=-\frac{\delta}{\delta x^a},\quad I\left(\frac{\partial}{\partial x^u}\right)=\frac{\partial}{\partial x^u},$$
$$\varphi\left(\frac{\delta}{\delta x^a}\right)=(p-\sigma_{p,q})\frac{\delta}{\delta x^a},\quad \varphi\left(\frac{\partial}{\partial x^u}\right)=\sigma_{p,q}\frac{\partial}{\partial x^u}. \tag{4.4}$$

In [7] are studied submanifolds of a Riemannian metallic manifolds (M,g,φ). It is proved ([7], Proposition 4.3), that a submanifold N of M is an invariant submanifold with respect to φ (i.e. $\varphi(T_xN)\subseteq T_xN$ for every $x\in N$) if and only if the induced structure $(\varphi_N;g_N)$ on N is a metallic Riemannian structure whenever φ_N is non-trivial.

A leaf L of the foliation $\mathcal{F}$ on M is a submanifold $(n-r)$-dimensional, locally defined in a plaque $U\cap L$ by $x^a=$ *constant*. It is well-known that the Riemannian metric g induces a Riemannian metric g_L on the leaf L. The metallic structure φ on M induces an $(1,1)$-tensor field φ_L on L by

$$\varphi_L(X)=v(\varphi(i_*X)),\quad \forall X\in\Gamma(TL),$$

where $i:L\to M$ is the canonical inclusion. Locally, φ_L verifies

$$\varphi_L(\frac{\partial}{\partial x^u})=\sigma_{p,q}\frac{\partial}{\partial x^u},$$

Moreover, for every $X\in\Gamma(TL)$,

$$\varphi_L^2(X)=\sigma_{p,q}^2X=p\cdot\sigma_{p,q}X+qX=(p\cdot\varphi_L+q\cdot Id)X,$$

which shows that φ_L is a (p,q)-metallic structure on L.

By (4.4), every leaf of foliation $\mathcal{F}$ is an invariant submanifold with respect to metallic structure φ. From Remark 4.1, the restriction of the Nijenhuis tensor N_φ at structural bundle $T\mathcal{F}$ vanishes, hence we obtained:

Proposition 4.4 *Let $(M,g,\mathcal{F},\varphi)$ a Riemannian foliated manifold endowed with a (p,q)-metallic structure. Every leaf L of M is a (p,q)-metallic manifold and the metallic structure φ_L on it is integrable.*

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