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On split equality problem for proximal point algorithm

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Abstract In this paper, we introduce a modified Mann iterative scheme for solving a split equality proximal point algorithm problem. We also establish strong convergence of this iterative sequence to a minimizer of a convex function which is also a common fixed point of a *nonspreading-type multi-valued mapping* in Hilbert space.

Keywords Split equality · Proximal point algorithm · Fixed point · Nonexpansive mapping · Strong convergence

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1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. Let C be any nonempty subset of H. We shall denote by $CB(C)$ the family of nonempty closed bounded subsets of C and by $K(C)$ the family of nonempty compact subsets of C. Let $H(.,.)$ be the Hausdorff distance on $CB(H)$, that is,

$$H(A,B) = max\Big\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\Big\}, \quad A,B \in CB(C),$$

where $dist(a,B) = inf\{d(a,b) : b \in B\}$ is the distance from the point a to the set B.
Recall that a self-mapping T on C is said to be a single valued nonexpansive mapping if

$$||Tx - Ty|| \leq ||x - y||, \forall\ x, y \in C.$$

A multivalued mapping $T : C \to CB(H)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq ||x - y|| \ \ \forall\ x, y \in C.$$

Various kinds of nonexpansiveness are clearly explained in [5]. A point x is called a fixed point of T if $x \in Tx$. We denote by $F(T)$ the set of all fixed points of T and $T : C \to CB(C)$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq ||x - p||, \quad \forall\, x \in C, p \in F(T).$$

Recall that a single-valued mapping $T : C \to C$ is said to be *nonspreading mappings* [38] if

$$2||Tx - Ty||^2 \leq ||x - Ty||^2 + ||y - Tx||^2, \quad \forall\, x, y \in C.$$

It is easy to see that T is nonspreading if and only if

$$||Tx - Ty||^2 \leq ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle \quad \forall\, x, y \in C.$$

A mapping $T : C \to CB(C)$ is said to be *nonspreading-type multi-valued mapping* [19] if

$$2H^2(Tx, Ty) \leq d^2(x, Ty) + d^2(y, Tx), \quad \forall x, y \in C. \tag{1.1}$$

It is well known that, if T is a nonspreading-type multi-valued mapping, and $F(T) \neq \emptyset$, then T is a *quasi-nonexpansive multi-valued mapping*, that is:

$$H(Tx, Tp) \leq ||x - p||, \quad \forall\, x \in C,\ p \in F(T). \tag{1.2}$$

Then, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H^2(Tx, Tp) &\leq d^2(p, Tx) + d^2(x, Tp) \\ &\leq H^2(Tx, Tp) + ||x - p||^2. \end{aligned}$$

This implies that

$$H(Tx, Tp) \leq ||x - p||.$$

In modelling inverse problems which arise from phase retrievals and medical image reconstruction [9], Censor and Elfving [14] firstly introduced the following Split Feasibility Problem (SFP) in finite-dimensional Hilbert spaces:
Let C and Q be nonempty closed convex subsets of the Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \qquad \text{and} \qquad Ax^* \in Q. \tag{1.3}$$

It has been known that the SFP can be used in many areas such as image restoration, computer tomography, and therapy treatment planing [16, 23, 35]. Some methods have been developed to solve SFP [1, 3, 11, 12, 29, 41].

It is easy to see that $x^* \in C$ solves equation (1.3) if and only if it solve the following fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q A))x^*$$

where P_C (respectively P_Q) is the orthogonal projection from H_1 (respectively H_2) onto C (respectively Q), $\gamma > 0$ and A^* is the adjoint of A.
Recently, Moudafi [33] introduced the following new split feasibility problem, which is also called General Split Equality Problem (SEP).
Let H_1, H_2 and H_3 be real Hilbert spaces, let $C \subset H_1, Q \subset H_2$ be two closed convex sets, let $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. The SEP which was first introduced by Moudafi [33] is to find

$$x \in C, \quad y \in Q, \quad \text{such that} \quad Ax = By, \tag{1.4}$$

which allows asymmetric and partial relations between the variables x and y. Assume the SEP (1.4) is consistent (i.e., has a solution), we denote through out this paper its solution by Γ, i.e.,

$$\Gamma = \{x \in C, \ y \in Q : Ax = By\}.$$

The interest of the SEP is to cover many situations, for instance in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see e.g. [4]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [13,37,39] for further details on SEP).
In order to solve SEP (1.4) Moudafi [33] introduced the following alternating CQ algorithm (ACQA):

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.5}$$

where $\gamma_k, \beta_k \in \Big(\varepsilon, \min\{\frac{1}{\gamma_A}, \frac{1}{\gamma_B}\} - \varepsilon\Big)$; γ_A and γ_B are the spectral radius of A^*A and B^*B, respectively. The algorithm (1.5) is exactly the CQ algorithm proposed by Byrne [9] when $B = I, \beta_k = 1$. When the SEP (1.4) has a solution, it can be seen as the following convex minimization problem

$$\min_{x \in C, y \in Q} \frac{1}{2}||Ax - By||^2. \tag{1.6}$$

The classical projection gradient algorithm (PGA) (or the projection algorithm) for (1.6) is

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.7}$$

where γ_k, is the stepsize at the iteration k, is chosen in the interval $\left(\varepsilon, \frac{2}{\gamma_A+\gamma_B}-\varepsilon\right)$. For more details about the PGA, see [24]. Byrne and Moudafi [10] obtain the algorithm (1.7) by discussing the projected Landweber algorithm for the problem (1.6) in a product space. It is easy to see that the alternating CQ algorithm (1.5) is sequential but the algorithm (1.7) is simultaneous. When $B = I$, the algorithm (1.7) resembles closely, but is not equivalent to the original CQ algorithm proposed by Byrne [9], but it does solve the same problem (see [10] for more details).
In order to avoid using the projection, recently, Moudafi [34] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. If $C = F(T)$ and $Q = F(S)$, then the split equality feasibility problem (1.4) reduces to finding

$$x \in F(T) \quad \text{and} \quad y \in F(S) \quad \text{such that} \quad Ax = By, \tag{1.8}$$

which is called *split equality fixed point problem* (SEFPP). Denote by Γ the solution set of SEFPP (1.8).
If $H_2 = H_3$ and $B = I$, then (SEFPP) (1.8) reduces to the split fixed-point problem (SFPP) introduced by Censor and Segal [17]:

$$x \in F(T), \quad \text{such that} \quad Ax \in F(S). \tag{1.9}$$

To solve (1.9), Censor and Segal [17] proposed and proved, in finite-dimensional spaces, the convergence of the sequence $\{x_n\}$ generated by the following algorithm:

$$x_{n+1} = T(x_n + \gamma A^t(S-I)AX_n), \quad n \geq 1$$

where $\gamma \in (2/\lambda)$ with λ being the largest eigenvalue of the matrix At A (A^t stands for matrix transposition).
We remark here that the SEFPP generalizes the SFPP which is at the core of the modelling of many inverse problems in various areas of physical sciences and has been used to model significant real world inverse problem in sensor networks, in radiation therapy treatment planning, in resolution enhancement, in watermarking, in data compression, in magnetic resonance imaging, in holography, in colour imaging, in optics and neural networks and in graph matching (for more details, see, for example, [15]). SEFPP also has some other important applications different areas of applied mathematics, such as fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstruction (see, for example, [9, 13, 20]).

Recently Moudafi [34] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.8):

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{1.10}$$

He studied the weak convergence of the sequence generated by scheme (1.10) under the condition that T and S are firmly quasi-nonexpansive mappings. In 2015, Che and Li [21] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.8):

$$\begin{cases} u_n = x_n - \gamma_k A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n \\ v_n = y_n + \beta_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n y_n + (1-\alpha_n)Sv_n. \end{cases} \tag{1.11}$$

They also established the weak convergence of the scheme (1.11) under the condition that the operators T and S are quasi-nonexpansive mapping.

Let $f : H \to H$ be a proper convex and lower semi-continuous function. One of the major problems in optimization in Hilbert space H is to find $x \in H$ such that

$$f(x) = \min_{y \in H} f(y).$$

We denote by $\arg\min_{y\in H} f(y)$ the set of minimizers of f in H.

On the other hand, a popular method in convex minimization is the Proximal Point Algorithm (PPA) which was introduced by Martinet [32] in 1970. In 1976, Rockafellar [36] studied the convergence of PPA for finding a solution of the unconstrained convex minimization problem in H as follows:
Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \arg\min_{y\in H}\Big(f(y) + \frac{1}{2\lambda_n}||y - x_n||^2\Big), \end{cases}$$

where $\lambda_n > 0$, and $n \geq 1$. It has been shown that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\{x_n\}$ weakly converges to the minimizer of f [36]. Rockafellar [36] also raised a question as to whether the PPA always converges strongly and it was answered in negative by Güler [25]. In 2000, Kamimura-Takahashi [28] combined the PPA with Halperns algorithm [26], so that the strong convergence is guaranteed.
In the recent years, the problem of finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for a single-valued mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, (see [5, 7, 18, 22, 31]) and the references therein.

In 2016, Chang et al.[19] introduced the following iterative scheme:

$$\begin{cases} y_n = \arg\min_{y\in C}\Big(f(y) + \frac{1}{2\lambda_n}||y - x_n||^2\Big), \\ z_n = (1-\beta_n)x_n + \beta_n w_n, \quad w_n \in Ty_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n, \quad v_n \in Tw_n \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and C is a nonempty closed convex subset of a Hilbert space H, $g : C \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function, and $T : C \to K(C)$ is a nonspreading multivalued mapping. They proved weak convergence and strong convergence theorems for minimizers of proper convex and lower semi-continuous functions and fixed points of nonspreading multivalued mappings in Hilbert spaces, using (1.12).

Furthermore, it is well known that fixed point theory for multi-valued mappings has many useful applications in various fields, in particular game theory and mathematical economics. As far as we know, all the recent and important results regarding approximation of solutions to SEFPP in the literature have not been discussed with respect to PPA.

In this paper, we consider the following pair of proximal point algorithm and fixed points problems called Split Equality Proximal Point Algorithm Problems (SEPPAP).

Definition 1.1 *Let $f : H_1 \to (-\infty, \infty]$ and $g : H_2 \to (-\infty, \infty]$ be a proper convex and lower semi-continuous functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Then the SEPPAP is to find $x^* \in C$ and $y^* \in Q$ such that*

$$f(x^*) = \min_{u \in C} f(u), \quad g(y^*) = \min_{v \in Q} g(v) \quad \text{and} \quad Ax^* = By^* \tag{1.13}$$

The set of solutions of (1.13) is denoted by SEPPAP(f, g).
If $H_2 = H_3$ and $B = I$, then SEPPAP (1.13) reduces to the Split Feasibility Proximal Point Algorithm Problems (SFPPAP) is to find $x^* \in C$ and $y^* \in Q$ such that

$$f(x^*) = \min_{u \in C} f(u), \quad g(y^*) = \min_{v \in Q} g(v) \quad \text{and} \quad y^* \in Ax^* \tag{1.14}$$

The set of solutions of (1.14) is denoted by SFPPAP(f, g).

Question. Can we construct an iterative scheme and prove strong convergence theorem using the iterative scheme for approximation of solution for split equality proximal point problem and common fixed point of multi-valued mappings in real Hilbert spaces?
Motivated by the above results, our aim in this paper is to introduce a new iterative algorithm for solving SEPPAP. We establish a strong convergence theorem in Hilbert space for the new algorithm for minimizer of proper convex and lower semi-continuous functions and common fixed points of *nonspreading-type multi-valued mapping.* Our contribution complements the results of Chang et al.[19] in the direction of split equality fixed point problem. Our result also extend the results of Che and Li [21], Moudafi [34] and Zhao [42] from single-valued mappings and weak convergence to multi-valued mappings and strong convergence, respectively.

2 Preliminaries

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence respectively. We shall make use of the following well-known results.

Lemma 2.1 *Let H be a real Hilbert space, $x, y, z \in H$ and $\lambda \in [0,1]$. Then*

$$||\lambda(x-z)+(1-\lambda)(y-z)||^2 = \lambda||x-z||^2+(1-\lambda)||y-z||^2-\lambda(1-\lambda)||x-y||^2.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following holds*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle \quad \forall\ x, y \in H.$$

Let $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of g in a real Hilbert space H as follows:

$$J_\lambda x = \arg\min_{u\in H}\Big[g(u) + \frac{1}{2\lambda}||u-x||^2\Big]$$

for all $x \in H$. It was shown in [25] that the set of fixed points of the resolvent associated with g coincides with the set of minimizers of g. Also, the resolvent J_λ of g is nonexpansive for all $\lambda > 0$, (see the proof in [27, Lemma 4], under metric space with nonpositive curvature in the sense of Alexandrov which is more general than Hilbert space).

Lemma 2.3 (The resolvent identity, ([27, Lemma 1], see also [6])) *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in H$ and $\lambda > \mu > 0$, we have the following identity holds:*

$$J_\lambda x = J_\mu\Big(\frac{\lambda-\mu}{\lambda}J_\lambda x + \frac{\mu}{\lambda}x\Big).$$

Since Hilbert space is a special type of metric space with convex structure, then the following lemma follows from [2] (also, see for example [19, Lemma 2.2]).

Lemma 2.4 (Sub-differential Inquality) *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x-y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 \leq g(y) - g(J_\lambda x). \quad (2.1)$$

Lemma 2.5 *Let C be a nonempty closed convex subset of real Hilbert space H. Then*

1. If $T : C \to CB(C)$ is a nonspreading-type multivalued mapping and $F(T) \neq \emptyset$, the following conclusions hols:
(a) $F(T)$ is closed,

(b) If, in addition T satisfies the condition $Tp = \{p\}$, $\forall\, p \in F(T)$ is convex.

2. *Let $T : C \to K(C)$ be a nonspreading-type multi-valued mapping. If $x, y \in C$ and $u \in Tx$, then there exists $v \in Ty$ such that*

$$H^2(Tx, Ty) \leq ||x - y||^2 + 2\langle x - u, y - v\rangle. \tag{2.2}$$

3. (Demicloseness principle) *If $\{x_n\}$ is a bounded sequence in C such that $x_n \rightharpoonup p$ and $x_n - y_n \to 0$ for some $y_n \in Tx_n$, then $p \in Tp$.*

Lemma 2.6 [8] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive single-valued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ with $x_x - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.7 [30] *If $\{a_n\}$ is a sequence of real numbers and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$, then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1},$$

for all sufficiently large numbers $k \in N$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.8 [40] *If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\, \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$ and
$\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3 Main result

Theorem 3.1 *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $f : C \to (-\infty, \infty]$ and $g : Q \to (-\infty, \infty]$ be a proper convex and lower semi-continuous functions. Let $T : C \to K(C)$ and $S : Q \to K(Q)$ be nonspreading-type multi-valued mappings such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with its Adjoint A^* and B^* respectively. Let $(x_1, y_1) \in C \times Q$ and $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} u_n = arg\min\limits_{u \in C}[f(u) + \frac{1}{2\lambda_n}||u - x_n||^2], \\ v_n = arg\min\limits_{v \in Q}[g(v) + \frac{1}{2\lambda_n}||v - x_n||^2], \\ w_n = (1 - \alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n)), \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n\bar{w}_n \quad \bar{w}_n \in Tw_n, \\ z_n = (1 - \alpha_n)(v_n + \rho_n B^*(Au_n - Bv_n)), \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n\bar{z}_n \quad \bar{z}_n \in Sz_n, \end{cases} \tag{3.1}$$

where γ_A and γ_B stand for the spectral radii of A^*A and B^*B respectively, $\{\rho_n\}$ is a positive sequence such that $\rho_n \in \left(\epsilon, \frac{2}{\gamma_A+\gamma_B} - \epsilon\right)$ (for $\epsilon > 0$ small enough) $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in (0,1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \le \beta_n \le b < 1$,*(for some* $a, b \in (0,1)$*).*

If $\Gamma := (F(T) \times F(S)) \cap \mathrm{SEPPAP}(f,g) \neq \emptyset$, then $\{(x_n, y_n)\}$ converges strongly to (p,q) in the solution set of problem (1.13)*.*

Proof. Let $(x^*, y^*) \in \Gamma$. Then we have $x^* \in Tx^*, f(x^*) \le f(u)$ for all $u \in C$ and $y^* \in Sy^*, g(y^*) \le g(v)$ for all $v \in Q$. It follows from (2.1), that

$$f(x^*) + \frac{1}{2\lambda_n}||x^* - x^*||^2 \le f(u) + \frac{1}{2\lambda_n}||u - x^*||^2, \ \ \forall\ u \in C$$

and

$$g(y^*) + \frac{1}{2\lambda_n}||y^* - y^*||^2 \le f(v) + \frac{1}{2\lambda_n}||v - y^*||^2, \ \ \forall\ v \in Q.$$

Hence $x^* = J_{\lambda_n}x^*$ and $y^* = J_{\lambda_n}y^*$ for all $n \ge 1$. Since $u_n = J_{\lambda_n}x_n$ and $v_n = J_{\lambda_n}y_n$, it follows by nonexpansive of J_{λ_n} that

$$||u_n - x^*|| = ||J_{\lambda_n}x_n - J_{\lambda_n}x^*|| \le ||x_n - x^*|| \tag{3.2}$$

and

$$||v_n - y^*|| = ||J_{\lambda_n}y_n - J_{\lambda_n}y^*|| \le ||y_n - x^*||. \tag{3.3}$$

Letting

$$w_n = (1-\alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n)) \tag{3.4}$$

and

$$z_n = (1-\alpha_n)(v_n + \rho_n B^*(Au_n - Bv_n)). \tag{3.5}$$

From (3.1), we obtain

$$\begin{aligned}||w_n - x^*||^2 &= ||(1-\alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n)) - x^*||^2\\ &\le (1-\alpha_n)||u_n - x^* - \rho_n A^*(Au_n - Bv_n)||^2 + \alpha_n||x^*||^2\\ &\le (1-\alpha_n)||u_n - x^*||^2 + \alpha_n||x^*||^2\\ &\quad +(1-\alpha_n)[\rho_n^2||A^*(Au_n - Bv_n)||^2\\ &\quad -2\rho_n\langle A^*(Au_n - Bv_n), u_n - x^*\rangle].\\ &\le (1-\alpha_n)||u_n - x^*||^2 + \alpha_n||x^*||^2\\ &\quad +(1-\alpha_n)[\rho_n^2||A^*(Au_n - Bv_n)||^2\\ &\quad -2\rho_n\langle Au_n - Bv_n, Au_n - Ax^*\rangle].\\ &\le (1-\alpha_n)||u_n - x^*||^2 + \alpha_n||x^*||^2\\ &\quad +(1-\alpha_n)\Big[\rho_n^2||A^*(Au_n - Bv_n)||^2 - \rho_n||Au_n - Ax^*||^2\\ &\quad -\rho_n||Au_n - Bv_n||^2 + \rho_n||Bv_n - Ax^*||^2\Big]. \qquad (3.6)\end{aligned}$$

But

$$\begin{aligned}||A^*(Au_n - Bv_n)||^2 &= \langle A^*(Au_n - Bv_n), A^*(Au_n - Bv_n)\rangle\\ &= \langle (Au_n - Bv_n), AA^*(Au_n - Bv_n)\rangle\\ &\le \gamma_A||Au_n - Bv_n||^2. \qquad (3.7)\end{aligned}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned}||w_n - x^*||^2 &\le (1-\alpha_n)||u_n - x^*||^2 + \alpha_n||x^*||^2\\ &\quad (1-\alpha_n)\Big[-\rho_n(1-\rho_n\gamma_A)||Au_n - Bv_n||^2\\ &\quad -\rho_n||Au_n - Ax^*||^2 + \rho_n||Bv_n - Ax^*||^2\Big]\\ &\le (1-\alpha_n)||u_n - x^*||^2 + \alpha_n||x^*||^2\\ &\quad +(1-\alpha_n)\Big[-\rho_n(1-\rho_n\gamma_A)||Au_n - Bv_n||^2\\ &\quad -\rho_n||Au_n - Ax^*||^2 + \rho_n||Bv_n - Ax^*||^2\Big]. \qquad (3.8)\end{aligned}$$

Similarly by following the same argument, we can show that

$$\begin{aligned}||z_n - y^*||^2 &\le (1-\alpha_n)||v_n - x^*||^2 + \alpha_n||y^*||^2\\ &\quad +(1-\alpha_n)\Big[-\rho_n(1-\rho_n\gamma_B)||Au_n - Bv_n||^2\\ &\quad -\rho_n||Bv_n - By^*||^2 + \rho_n||Au_n - By^*||^2\Big]. \qquad (3.9)\end{aligned}$$

Adding (3.8) and (3.9), by using the assumption that $Ax^* = By^*$, from (3.2) and (3.3), we obtain

$$\begin{aligned} ||w_n - x^*||^2 + ||z_n - y^*||^2 &\leq (1-\alpha_n)[||u_n - x^*||^2 + ||v_n - y^*||^2] \\ &\quad +\alpha_n[||x^*||^2 + ||y^*||^2] - \rho_n[2 - \rho_n(\gamma_A + \gamma_B)]||Au_n - Bv_n||^2 \quad (3.10) \\ &\leq (1-\alpha_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\alpha_n[||x^*||^2 + ||y^*||^2] - \rho_n[2 - \rho_n(\gamma_A + \gamma_B)]||Au_n - Bv_n||^2. \quad (3.11) \end{aligned}$$

Now from (3.1) and (3.11), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 & \\ +||y_{n+1} - y^*||^2 &\leq (1-\beta_n)||(x_n - x^*) + \beta_n(\bar{w}_n - x^*)||^2 \\ &\quad +||(1-\beta_n)(y_n - y^*) + \beta_n(\bar{z}_n - y^*)||^2 \\ &\leq (1-\beta_n)||x_n - x^*||^2 + \beta_n H^2(w_n, Tx^*) \\ &\quad +(1-\beta_n)||y_n - x^*||^2 + \beta_n H^2(Sz_n, Sx^*) \\ &\leq (1-\beta_n)||x_n - x^*||^2 + \beta_n||w_n - x^*||^2 \\ &\quad +(1-\beta_n)||y_n - x^*||^2 + \beta_n||z_n - x^*||^2 \\ &\leq (1-\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\beta_n[||w_n - x^*||^2 + ||z_n - y^*||^2] \\ &\leq (1-\alpha_n\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\alpha_n\beta_n[||x^*||^2 + ||y^*||^2] \\ &\quad -\rho_n\beta_n[2 - \rho_n(\gamma_A + \gamma_B)]||Au_n - Bv_n||^2 \quad (3.12) \\ &\leq (1-\alpha_n\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\alpha_n\beta_n[||x^*||^2 + ||y^*||^2] \\ &\leq \max\{[||x_n - x^*||^2 + ||y_n - y^*||^2], [||x^*||^2 + ||y^*||^2]\} \\ &\quad \vdots \\ &\leq \max\{[||x_1 - x^*||^2 + ||y_1 - y^*||^2], [||x^*||^2 + ||y^*||^2]\}. \end{aligned}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded. Hence $\{u_n\}$ and $\{v_n\}$ are also bounded. Now from (3.12), we obtain

$$\begin{aligned} &\rho_n\beta_n[2-\rho_n(\gamma_A+\gamma_B)]||Au_n - Bv_n||^2 \leq \alpha_n\beta_n[||x^*||^2 + ||y^*||^2] \\ &+(1-\alpha_n\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] - [||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2]. \end{aligned} \quad (3.13)$$

We have now the following two cases.

Case 1. Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is monotonically decreasing sequence. Then from (3.13), we obtain

$$\rho_n\beta_n[2 - \rho_n(\gamma_A + \gamma_B)]||Au_n - Bv_n||^2 \to 0 \quad \text{as} \quad n \to \infty$$

since $\beta_n > 0, \rho_n > 0$ and $[2 - \rho_n(\gamma_A + \gamma_B)] > 0$, then

$$\lim_{n\to\infty} ||Au_n - Bv_n||^2 = 0$$

which implies

$$\lim_{n\to\infty} ||Au_n - Bv_n|| = 0. \tag{3.14}$$

From (3.4) and (3.5), we respectively get

$$||u_n - w_n|| \leq \rho_n ||A^*|| ||Au_n - Bv_n|| + \alpha_n ||u_n - \rho_n A^*(Au_n - Bv_n)||$$

and

$$||v_n - z_n|| \leq \rho_n ||A^*|| ||Au_n - Bv_n|| + \alpha_n ||v_n + \rho_n A^*(Au_n - Bv_n)||.$$

From (3.14) and condition (C1), we obtain

$$\lim_{n\to\infty} ||u_n - w_n|| = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty} ||v_n - z_n|| = 0. \tag{3.16}$$

Using Lemma 2.1 in (3.1), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= (1-\beta_n)||x_n - x^*||^2 + \beta_n ||\bar{w}_n - x^*||^2 \\ &\quad -\beta_n(1-\beta_n)||x_n - \bar{w}_n||^2 \\ &\leq (1-\beta_n)||x_n - x^*||^2 + \beta_n H^2(Tw_n, Tx^*) \\ &\quad -\beta_n(1-\beta_n)||x_n - \bar{w}_n||^2 \\ &\leq (1-\beta_n)||x_n - x^*||^2 + \beta_n ||w_n - x^*||^2 \\ &\quad -\beta_n(1-\beta_n)||x_n - \bar{w}_n||^2. \end{aligned} \tag{3.17}$$

Following similar argument, we can show that

$$\begin{aligned} ||y_{n+1} - x^*||^2 &\leq (1-\beta_n)||y_n - x^*||^2 + \beta_n ||z_n - y^*||^2 \\ &\quad -\beta_n(1-\beta_n)||y_n - \bar{z}_n||^2. \end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18), we get

$$\begin{aligned} &||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 \leq (1-\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &+\beta_n[||w_n - x^*||^2 + ||z_n - y^*||^2] - \beta_n(1-\beta_n)[||x_n - \bar{w}_n||^2 + ||y_n - \bar{z}_n||^2] \end{aligned}$$

and from (3.11), we obtain

$$\begin{aligned} &||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 \leq (1-\alpha_n\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\alpha_n\beta_n[||x^*||^2 + ||y^*||^2] - \beta_n(1-\beta_n)[||x_n - \bar{w}_n||^2 + ||y_n - \bar{z}_n||^2] \\ &\leq [||x_n - x^*||^2 + ||y_n - y^*||^2] + \alpha_n\beta_n[||x^*||^2 + ||y^*||^2] \\ &\quad -\beta_n(1-\beta_n)[||x_n - \bar{w}_n||^2 + ||y_n - \bar{z}_n||^2]. \end{aligned}$$

Hence

$$\beta_n(1-\beta_n)[||x_n - \bar{w}_n||^2 + ||y_n - \bar{z}_n||^2] \leq \alpha_n\beta_n[||x^*||^2 + ||y^*||^2]$$
$$+[||x_n - x^*||^2 + ||y_n - y^*||^2] - [||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2].$$

Since $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is monotone decreasing sequence and $\beta_n(1-\beta_n) > 0$, then

$$\lim_{n\to\infty}[||x_n - \bar{w}_n||^2 + ||y_n - \bar{z}_n||^2] = 0.$$

Hence

$$\lim_{n\to\infty}||x_n - \bar{w}_n|| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty}||y_n - \bar{z}_n|| = 0. \tag{3.20}$$

From the sub-differential inequality Lemma 2.4, we have

$$\frac{1}{2\lambda_n}||u_n - x^*||^2 - \frac{1}{2\lambda_n}||x_n - x^*||^2 + \frac{1}{2\lambda_n}||x_n - u_n||^2 \leq f(x^*) - f(u_n)$$

and

$$\frac{1}{2\lambda_n}||v_n - y^*||^2 - \frac{1}{2\lambda_n}||y_n - y^*||^2 + \frac{1}{2\lambda_n}||y_n - v_n||^2 \leq g(y^*) - f(v_n).$$

Since $f(x^*) \leq f(u_n)$ and $g(y^*) \leq g(v_n)$ for all $n \geq 1$, we obtain respectively

$$||x_n - u_n||^2 \leq ||x_n - x^*||^2 - ||u_n - x^*||^2 \tag{3.21}$$

and

$$||y_n - v_n||^2 \leq ||y_n - y^*||^2 - ||v_n - y^*||^2. \tag{3.22}$$

Adding (3.21) and (3.22), we have

$$\begin{aligned} ||x_n - u_n||^2 + ||y_n - v_n||^2 &\leq ||x_n - x^*||^2 + ||y_n - y^*||^2 \\ &\quad -[||u_n - x^*||^2 + ||v_n - y^*||^2]. \end{aligned} \tag{3.23}$$

From (3.1) and (3.10), we get

$$\begin{aligned} ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq (1-\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\beta_n[||w_n - x^*||^2 + ||z_n - y^*||^2] \\ &\leq (1-\beta_n)[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &\quad +\beta_n[(1-\alpha_n)[||u_n - x^*||^2 + ||v_n - y^*||^2] \\ &\quad +\alpha_n[||x^*||^2 + ||y^*||^2]]. \end{aligned}$$

Therefore

$$\begin{aligned}||x_n - x^*||^2 + ||y_n - y^*||^2 \leq \frac{1}{\beta_n}\Big(&||x_n - x^*||^2 + ||y_n - y^*||^2 \\ &-[||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2]\Big) \\ &+(1-\alpha_n)[||u_n - x^*||^2 + ||v_n - y^*||^2] \\ &+\alpha_n[||x^*||^2 + ||y^*||^2]. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\begin{aligned}||x_n - u_n||^2 + ||y_n - v_n||^2 \leq \frac{1}{\beta_n}\Big(&[||x_n - x^*||^2 + ||y_n - y^*||^2] \\ &-[||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2]\Big) \\ &+\alpha_n\left([||x^*||^2 + ||y^*||^2] - [||u_n - x^*||^2 \right. \\ &\left. +||v_n - y^*||^2]\right)\end{aligned}$$

by monotone decreasing of $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$, we get

$$||x_n - u_n||^2 + ||y_n - v_n||^2 \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} ||x_n - u_n|| = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{3.26}$$

From (3.15) and (3.25), we obtain

$$\lim_{n\to\infty} ||x_n - w_n|| = 0 \tag{3.27}$$

and from (3.16) and (3.26), we obtain

$$\lim_{n\to\infty} ||y_n - z_n|| = 0. \tag{3.28}$$

Also from (3.19) and (3.27)

$$\lim_{n\to\infty} ||w_n - \bar{w}_n|| = 0 \tag{3.29}$$

and from (3.20) and (3.28), we get

$$\lim_{n\to\infty} ||z_n - \bar{z}_n|| = 0. \tag{3.30}$$

Now, from (2) of Lemma 2.5, for each $w_n,\ x_n$ and $\bar{w}_n \in Tw_n$ there exists $r_n \in Tx_n$ such that

$$H^2(Tw_n, Tx_n) \leq ||w_n - x_n||^2 + 2\langle w_n - \bar{w}_n, x_n - r_n\rangle \quad n \geq 1.$$

It follows that

$$\begin{aligned} d(x_n, Tx_n) &\leq ||x_n - \bar{w}_n|| + d(\bar{w}_n, Tx_n) \\ &\leq ||x_n - \bar{w}_n|| + H(Tw_n, Tx_n) \\ &\leq \sqrt{||w_n - x_n||^2 + 2\langle w_n - \bar{w}_n, x_n - r_n\rangle} \\ &\quad + ||w_n - \bar{w}_n||^2. \end{aligned}$$

Hence, from (3.27) and (3.29), we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.31}$$

Similarly, by using (3.28) and (3.30), we can show that

$$\lim_{n\to\infty} d(y_n, Sy_n) = 0. \tag{3.32}$$

Now, from Lemma 2.3 and nonexpansive of J_{λ_n} (see [27]) and $\lambda_{n\geq\lambda>0}$, we obtain

$$\begin{aligned} ||x_n - J_\lambda x_n|| &\leq ||x_n - u_n|| + ||u_n - J_\lambda x_n|| \\ &= ||x_n - u_n|| + ||J_{\lambda_n} x_n - J_\lambda x_n|| \\ &= ||x_n - u_n|| + ||J_\lambda\Big(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n\Big) - J_\lambda x_n|| \\ &\leq ||x_n - u_n|| + ||\Big(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n\Big) - x_n|| \\ &= ||x_n - u_n|| + \Big(1 - \frac{\lambda}{\lambda_n}\Big)||J_{\lambda_n} x_n - x_n|| \\ &= \Big(2 - \frac{\lambda}{\lambda_n}\Big)||u_n - x_n||. \end{aligned}$$

From this together with (3.25), we get

$$\lim_{n\to\infty} ||x_n - J_\lambda x_n|| = 0. \tag{3.33}$$

By similar argument, with (3.26), we can show that

$$\lim_{n\to\infty} ||y_n - J_\lambda y_n|| = 0. \tag{3.34}$$

Since $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ are bounded, then exists $p \in H_1$ such that $x_n \rightharpoonup p$ by demiclosedness of T, (see (3) of Lemma 2.5) and (3.31), we obtain $p \in F(T)$. From (3.33) and the fact that J_λ is nonexpansive [27] , we get $p \in F(J_\lambda) = \arg\min_{u\in C} f(u)$. Also there exists $q \in H_2$ such that $y_n \rightharpoonup q$, by demiclosedness of S and (3.32), we obtain $q \in F(S)$, by nonexpansiveness of J_λ and (3.34), we obtain $q \in F(J_\lambda) = \arg\min_{v\in Q} g(v)$. On the other hand from (3.25) and (3.26), we get $u_n \rightharpoonup p$ and $v_n \rightharpoonup q$ respectively. Since A and B are bounded linear operators, we have $Au_n \rightharpoonup Ap$ and $Bv_n \rightharpoonup Bq$ respectively. Also by the weakly semi-continuity of the norm, we get

$$||Ap - Bq|| \leq \liminf_{n\to\infty} ||Au_n - Bv_n|| = 0.$$

Hence $(p, q) \in \Gamma$. Finally, we show that $(x_n, y_n) \to (p, q)$. Using Lemma 2.2 in (3.2), we obtain

$$\begin{aligned} ||w_n - p||^2 &= ||(1-\alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n)) - p||^2 \\ &= ||(1-\alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n) - p) - \alpha_n p||^2 \\ &\leq (1-\alpha_n)^2 ||u_n - \rho_n A^*(Au_n - Bv_n) - p||^2 + 2\langle -p, w_n - p\rangle \\ &\leq (1-\alpha_n)||u_n - p||^2 + 2\alpha_n \langle -p, w_n - p\rangle \end{aligned}$$

and

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\beta_n)x_n + \beta_n \bar{w}_n - p||^2 \\ &\leq (1-\beta_n)||x_n - p||^2 + \beta_n ||\bar{w}_n - p||^2 \\ &\leq (1-\beta_n)||x_n - p||^2 + \beta_n H^2(Tw_n, Tp) \\ &\leq (1-\beta_n)||x_n - p||^2 + ||w_n - p||^2 \\ &\leq (1-\alpha_n\beta_n)||x_n - p||^2 + 2\alpha_n\beta_n \langle -p, w_n - p\rangle. \end{aligned} \tag{3.35}$$

Similarly, by following the same argument, we can show that

$$||y_{n+1} - q||^2 \leq (1-\alpha_n\beta_n)||y_n - q||^2 + 2\alpha_n\beta_n \langle -q, z_n - q\rangle. \tag{3.36}$$

Adding (3.35) and (3.36), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 + ||y_{n+1} - p||^2 &\leq (1-\alpha_n\beta_n)[||x_n - p||^2 + ||y_n - q||^2] \\ &\quad + 2\alpha_n\beta_n[\langle -p, w_n - p\rangle + \langle -q, z_n - q\rangle]. \end{aligned} \tag{3.37}$$

Since $x_n \rightharpoonup p$ and $y_n \rightharpoonup q$, from (3.27) and (3.28), we get

$$\langle -p, w_n - p\rangle + \langle -q, z_n - q\rangle \to 0$$

as $n \to \infty$. Using Lemma 2.8 in (3.37), we obtain

$$||x_n - p||^2 + ||y_n - q||^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\lim_{n\to\infty} ||x_n - p|| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} ||y_n - p|| = 0.$$

Therefore, $(x_n, y_n) \to (p, q) \in \Gamma$.

Case 2. Assume that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is not a monotonically decreasing sequence. Letting $\Upsilon_n := ||x_n - p||^2 + ||y_n - q||^2$ and $\tau : N \to N$ be a mapping defined for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k \in N : k \leq n, \quad \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Then, $\{\tau(n)\}$ is a non decreasing sequence with $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}, \quad \text{for} \quad n \geq n_0.$$

Now from (3.13), we have

$$\begin{aligned}
&\rho_{\tau(n)}\beta_{\tau(n)}[2 - \rho_{\tau(n)}(\gamma_A + \gamma_B)]||Au_{\tau(n)} - Bv_{\tau(n)}||^2 \\
&\leq \alpha_{\tau(n)}\beta_{\tau(n)}[||x^*||^2 + ||y^*||^2] \\
&\quad +(1 - \alpha_{\tau(n)}\beta_{\tau(n)})[||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\
&\quad -[||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2].
\end{aligned}$$

Hence

$$||Au_{\tau(n)} - Bv_{\tau(n)}||^2 \to \quad \text{as} \quad \tau(n) \to \infty.$$

Following arguments similar to those in the proof of Case 1, we get $\langle -p, w_{\tau(n)} - p\rangle + \langle -q, z_{\tau(n)} - q\rangle \to 0$. Also from inequality (3.37), we obtain that,

$$\begin{aligned}
||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - p||^2 \leq &(1 - \alpha_{\tau(n)}\beta_{\tau(n)})[||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\
&+2\alpha_{\tau(n)}\beta_{\tau(n)}[\langle -p, w_{\tau(n)} - p\rangle \\
&+\langle -q, z_{\tau(n)} - q\rangle].
\end{aligned} \tag{3.38}$$

which implies that

$$\begin{aligned}
&\alpha_{\tau(n)}\beta_{\tau(n)}[||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\
&\leq [||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\
&\quad -[||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - p||^2] \\
&\quad +2\alpha_{\tau(n)}\beta_{\tau(n)}[\langle -p, w_{\tau(n)} - p\rangle + \langle -q, z_{\tau(n)} - q\rangle]
\end{aligned}$$

since $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$ and $\alpha_{\tau(n)}\beta_{\tau(n)} > 0$, then

$$||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2 \leq 2[\langle -p, w_{\tau(n)} - p\rangle + \langle -q, z_{\tau(n)} - q\rangle] \to 0$$

as $n \to \infty$. This together with (3.37) implies that

$$||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - p||^2 \to 0 \quad \text{as} \quad n \to \infty$$

Thus

$$\lim_{n\to\infty} \Upsilon_{\tau(n)} = \lim_{n\to\infty} \Upsilon_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, we have $\Upsilon_n \leq \Upsilon_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e., $\tau(n) < n$), because $\Upsilon_j > \Upsilon_{j+1}$ for $\tau(n) \leq j \leq n$. It then follows that for all $n \geq n_0$, we have

$$0 \leq \Upsilon_n \leq \max\{\Upsilon_{\tau(n)}, \Upsilon_{\tau(n)+1}\} = \Upsilon_{\tau(n)+1}.$$

This implies $\lim_{n\to\infty} \Upsilon_n = 0$, and hence (x_n, y_n) converges strongly to $(p,q) \in \Gamma$. This completes the proof. □

When T and S are single-valued nonspreading mappings in Theorem 3.1, we obtain the following corollary.

Corollary 3.2 *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $f : C \to (-\infty, \infty]$ and $g : Q \to (-\infty, \infty]$ be a proper convex and lower semi-continuous functions. Let $T : C \to C$ and $S : Q \to Q$ be nonspreading mappings such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with its Adjoint A^* and B^* respectively. Let $(x_1, y_1) \in C \times Q$ and $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} u_n = arg\min\limits_{u\in C}[f(u) + \frac{1}{2\lambda_n}||u - x_n||^2] \\ v_n = arg\min\limits_{v\in Q}[g(v) + \frac{1}{2\lambda_n}||v - x_n||^2] \\ w_n = (1-\alpha_n)(u_n - \rho_n A^*(Au_n - Bv_n)) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T w_n \\ z_n = (1-\alpha_n)(v_n + \rho_n B^*(Au_n - Bv_n)) \\ y_{n+1} = (1-\beta_n)y_n + \beta_n S z_n \end{cases} \tag{3.39}$$

where γ_A and γ_B stand for the spectral radii of A^*A and B^*B respectively, $\{\rho_n\}$ is a positive sequence such that $\rho_n \in \left(\epsilon, \frac{2}{\gamma_A+\gamma_B} - \epsilon\right)$ (for $\epsilon > 0$ small enough) $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are sequences in (0,1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^\infty \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$,*(for some* $a, b \in (0,1)$*)*

If $\Gamma := (F(T) \times F(S)) \cap \mathrm{SEPPAP}(f,g) \neq \emptyset$, then $\{(x_n, y_n)\}$ converges strongly to (p,q) in the solution set of problem (1.13).

When $B = I$ and $H_2 = H_3$ in Theorem 3.1, we have the following corollary.

Corollary 3.3 *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $f : C \to (-\infty, \infty]$ and $g : Q \to (-\infty, \infty]$ be a proper convex and lower semi-continuous functions. Let $T : C \to K(C)$ and $S : Q \to K(Q)$ be nonspreading-type multivalued mappings such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, let $A : H_1 \to H_2$ and be two bounded linear operators with its Adjoint A^* and B^* respectively. Let $(x_1, y_1) \in C \times Q$ and $\{(x_n, y_n)\}$ be a sequence generated by*

$$\begin{cases} u_n = arg\min_{u\in C}[f(u) + \frac{1}{2\lambda_n}||u - x_n||^2] \\ v_n = arg\min_{v\in Q}[g(v) + \frac{1}{2\lambda_n}||v - x_n||^2] \\ w_n = (1-\alpha_n)(u_n - \rho_n A^*(Au_n - v_n)) \\ x_{n+1} = (1-\beta_n)x_n + \beta_n \bar{w}_n \quad \bar{w}_n \in Tw_n \\ z_n = (1-\alpha_n)(v_n + \rho_n B^*(Au_n - v_n)) \\ y_{n+1} = (1-\beta_n)y_n + \beta_n \bar{z}_n \quad \bar{z}_n \in Sz_n \end{cases} \tag{3.40}$$

where γ_A and stand for the spectral radii of A^*A, $\{\rho_n\}$ is a positive sequence such that $\rho_n \in \left(\epsilon, \frac{2}{\gamma_A} - \epsilon\right)$ (for $\epsilon > 0$ small enough) $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are sequences in (0,1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^\infty \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$,*(for some $a, b \in (0,1)$)*

If $\Gamma := (F(T) \times F(S)) \cap \mathrm{SFPPAP}(f, g) \neq \emptyset$, then $\{(x_n, y_n)\}$ converges strongly to (p, q) in the solution set of problem (1.14).

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