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Hyperholomorphy of multiplicative inverse functions of bicomplex variables

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Abstract In this paper, we give some algebraic properties of bicomplex numbers and some different forms of bicomplex conjugations, based on the expression of the usual complex conjugation. We provide the multiplicative inverse forms and hyperholomorphy of bicomplex-valued multiplicative inverse functions in bicomplex numbers.

Keywords bicomplex number · conjugation · Cauchy-Riemann system · holomorphic function · multiplicative inverse functions

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1 Introduction

The set of bicomplex numbers which are complex numbers with complex coefficients, is denoted by

$$\mathbf{BC} := \{Z = z_1 + z_2 j \mid z_1 = x_0 + x_1 i,\ z_2 = x_2 + x_3 i \in \mathbf{C}\},$$

where $x_r \in \mathbf{R}$ $(r = 0,1,2,3)$, and both i and j are the imaginary units, that is,

$$ij = ji,\ \ i^2 = j^2 = -1, \tag{1.1}$$

$\mathbf{C}$ is the set of complex numbers with the imaginary unit i and $\mathbf{R}$ is the set of real numbers. Segre [9] introduced bicomplex numbers in 1892. Dragoni [3] and Spampinato [10,11] studied the origin of a function theory for bicomplex number system and developed the first theory of differentiability in $\mathbf{BC}$. The equation (1.1) shows that the commutative law of the product in $\mathbf{BC}$ can be satisfied. In quaternions, applying the differential operator into the left-side of the function makes the different result comparing with the case of the right-side of the function. In bicomplex numbers, however, the differential operators can be applied into the function regardless of the order of the calculation. Furthermore, Price [8] extended previous work for bicomplex number system on hyperfunction theory, built on the space of the differentiability of functions defined by several bicomplex variables. Charak et al. [2] introduced the extended bicomplex plane and the concept of the normality of a family of bicomplex meromorphic functions and holomorphic functions on bicomplex domains. Elizarrarás et al. [1,7] showed that the derivative of any holomorphic function in the sense of bicomplex numbers. Lavoie et al. [6] developed the analysis of bicomplex holomorphic functions, based on general theory of functional analysis with the bicomplex scalars and the quaternionic scalars. Kim and Shon [4] researched elementary functions in bicomplex number systems and properties of regular functions with values in bicomplex settings. Kim [5] investigated the differentiability of multidual functions and the notion of the hyperholomorphicity to multidual-valued functions which extend holomorphic functions to the higher multidual generalized Clifford analysis.

This paper in addition to other papers referred above, shows that some functions with bicomplex numbers have hyperholomorphy and these functions are induced from the commutative law of the product in **BC**. Then, more functions with the hyperholomorphy can be applied into the expansion of the power series. In this paper, we introduce basic definitions and notations of bicomplex number system and give some different forms of bicomplex conjugations, generalizing the usual complex conjugation. By using the bicomplex conjugations, we propose the multiplicative inverse forms of bicomplex-valued functions and we see that these functions satisfies the conditions of hyperholomorphy in bicomplex numbers.

2 Preliminaries

For two bicomplex numbers $Z = z_1 + z_2 j$ and $W = w_1 + w_2 j$, where z_r and w_r are complex numbers with the imaginary i, we give the addition and multiplication over **BC** such that

$$Z + W = (z_1 + w_1) + (z_2 + w_2) j$$

and

$$ZW = (z_1 w_1 - z_2 w_2) + (z_1 w_2 + z_2 w_1) j,$$

respectively. From the rule of the multiplication over **BC**, the bicomplex conjugate of Z, denoted by $\overline{Z}$, is

$$\overline{Z} = \overline{z_1} - \overline{z_2} j = x_0 - x_1 i - x_2 j + x_3 i j.$$

We give the norm $\sqrt{R(Z)}$ of Z on **BC**, referring the usual norm on n-dimensional complex space $\mathbf{C}^n$, as follows:

$$\sqrt{R(Z)} := \sqrt{|z_1|^2 + |z_2|^2}.$$

Let the modulus $N(Z)$ of Z be

$$N(Z) := Z\overline{Z} = R(Z) + I(Z) j,$$

where $R(Z) = z_1\overline{z_1} + z_2\overline{z_2}$ is a real part of $N(Z)$ and $I(Z) = z_2\overline{z_1} - z_1\overline{z_2}$ is a complex part (with j) of $N(Z)$. For any $Z \in \mathbf{BC}$, we let the inverse element of Z, denoted by Z^{-1}, be

$$Z^{-1} = \frac{\overline{Z}}{N(Z)} = \frac{\overline{z_1}R(Z) - \overline{z_2}I(Z)}{R(Z)^2 + I(Z)^2} - \frac{\overline{z_1}I(Z) + \overline{z_2}R(Z)}{R(Z)^2 + I(Z)^2}\, j.$$

Since we have

$$\begin{aligned} R(Z)^2 + I(Z)^2 &= (\sum_{r=0}^{3} x_r^2)^2 - 4(x_0x_3 - x_1x_2)^2 \\ &= (\sum_{r=0}^{3} x_r^2 + 2x_0x_3 2x_1x_2)(\sum_{r=0}^{3} x_r^2 - 2x_0x_3 + 2x_1x_2) \\ &= \{(x_0 + x_3)^2 + (x_1 - x_2)^2\}\{(x_0 - x_3)^2 + (x_1 + x_2)^2\}, \end{aligned}$$

for the existence of the inverse element Z^{-1} of Z, we preclude the following case $(x_0 = -x_3,\, x_1 = x_2)$ or $(x_0 = x_3,\, x_1 = -x_2)$. We also let the norm $\sqrt{R(Z)}$ of Z. So, we provide the pseudo-inverse element of Z, denoted by $Z^{-1}_{|.|}$, such that

$$Z^{-1}_{|.|} = \frac{\overline{Z}}{R(Z)} = \frac{\overline{z_1}}{z_1\overline{z_1} + z_2\overline{z_2}} - \frac{\overline{z_2}}{z_1\overline{z_1} + z_2\overline{z_2}}\, j,$$

where $x_r \neq 0$ $(r = 0, 1, 2, 3)$.

Proposition 2.1 *For a bicomplex number $Z = z_1 + z_2 j$, from setting the bicomplex conjugate of Z, the following inequality*

$$R(Z)R(\overline{Z}) \leq R(Z\overline{Z}) \tag{2.1}$$

holds, by the definition of $R(Z)$.

Proof. By each definition of $R(Z)$ and $\overline{Z}$ of Z, we have

$$Z\overline{Z} = (x_0^2 + x_1^2 + x_2^2 + x_3^2) + 2(x_0x_3 i - x_1x_2 i)\, j$$

$$\begin{aligned} R(Z\overline{Z}) &= (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 + 4(x_0x_3 i - x_1x_2 i)(-x_0x_3 i + x_1x_2 i) \\ &= (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 + 4(x_0x_3 - x_1x_2)^2 \end{aligned}$$

and

$$R(Z) = R(\overline{Z}) = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

Since we have

$$R(Z)R(\overline{Z}) - R(Z\overline{Z}) = 4(x_0x_3 - x_1x_2)^2 \geq 0,$$

we obtain the inequality (2.1). $\square$

3 Hyperholomorphic functions of bicomplex variables

In this section, as mentioned in the introduction, we show that some functions with bicomplex numbers have hyperholomorphy and these functions are induced from the commutative law of the product in **BC**. Then, we can deal with more functions with the hyperholomorphy which can be applied into the expansion of the power series. Let U be an open set in **BC**. Consider a bicomplex-valued function $F : U \subset \mathbf{BC} \to \mathbf{BC}$ such that

$$F(Z) = F(z_1 + z_2 j) = f_1(z_1, z_2) + f_2(z_1, z_2) j,$$

where $f_r : \mathbf{C}^2 \to \mathbf{C}$ $(r = 1, 2)$ are complex-valued functions, which have the forms $f_1 = f_1(z_1, z_2) = u_0 + u_1 i$ and $f_2 = f_2(z_1, z_2) = u_2 + u_3 i$ with letting real-valued functions $u_l : \mathbf{R}^4 \to \mathbf{R}$ be $u_l = u_l(x_0, x_1, x_2, x_3)$ $(l = 0, 1, 2, 3)$. We give a bicomplex Cauchy-Riemann differential operator

$$\overline{D} := \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + ij\frac{\partial}{\partial x_3}\Big) = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}.$$

Definition 3.1 *Let U be an open set in* **BC**. *A bicomplex-valued function $F = f_1 + f_2 j$ is said to be hyperholomorphic on U, if*
(i) f_r $(r = 1, 2)$ *are continuously differentiable functions and*
(ii) $\overline{D}F = 0$ *on U.*

From the differential operator, we obtain that hyperholomorphy of F is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial f_2}{\partial \overline{z_2}} \quad \text{and} \quad \frac{\partial f_1}{\partial \overline{z_2}} = -\frac{\partial f_2}{\partial \overline{z_1}}. \tag{3.1}$$

The above equations are called a corresponding bicomplex Cauchy-Riemann system on **BC**.

Definition 3.2 *Let $C^1(U)$ be the class that consists of all differentiable functions whose derivative is continuous. A bicomplex-valued function $F \in C^1(U)$ is said to be bihyperholomorphic, if F is invertible and F and F^1 are both hyperholomorphic, where*

$$F^{-1} = \frac{\overline{F}}{N(F)} = \frac{\overline{f_1}R(F) - \overline{f_2}I(F)}{R(F)^2 + I(F)^2} - \frac{\overline{f_1}I(F) + \overline{f_2}R(F)}{R(F)^2 + I(F)^2} j,$$

$R(F) = f_1\overline{f_1} + f_2\overline{f_2}$ and $I(F) = f_2\overline{f_1} - f_1\overline{f_2}$. Furthermore, since we have

$$\begin{aligned} R(F)^2 + I(F)^2 &= (\sum_{r=0}^{3} u_r^2)^2 - 4(u_0u_3 - u_1u_2)^2 \\ &= (\sum_{r=0}^{3} u_r^2 + 2u_0u_3 - 2u_1u_2)(\sum_{r=0}^{3} u_r^2 - 2u_0u_3 + 2u_1u_2) \\ &= \{(u_0 + u_3)^2 + (u_1 - u_2)^2\}\{(u_0 - u_3)^2 + (u_1 + u_2)^2\}, \end{aligned}$$

by excluding the cases

$$(u_0 = -u_3, u_1 = u_2) \quad or \quad (u_0 = u_3, u_1 = -u_2),$$

the existence of F^{-1} *is guaranteed. Specially, if* F *and* $F_{|.|}^{-1}$ *are hyperholomorphic, where*

$$F_{|.|}^{-1} = \frac{\overline{F}}{R(F)} = \frac{\overline{f_1}}{f_1\overline{f_1} + f_2\overline{f_2}} + \frac{-\overline{f_2}}{f_1\overline{f_1} + f_2\overline{f_2}} j,$$

satisfying $u_r \neq 0$ $(r = 0,1,2,3)$*, a bicomplex-valued function* F *is said to be pseudo bihyperholomorphic on* U*.*

Theorem 3.3 *Let* U *be an open set in* **BC***. If a function* F *is hyperholomorphic on* U*, then the multiplicative inverse function* F^{-1} *is hyperholomorphic on* U*.*

Proof. Since $R(F)$ and $I(F)$ are non-zero, we consider the following equation:

$$\begin{aligned}
\overline{D}F^{-1} &= \overline{D}\Big(\frac{\overline{f_1}R(F) - \overline{f_2}I(F)}{R(F)^2 + I(F)^2} - \frac{\overline{f_1}I(F) + \overline{f_2}R(F)}{R(F)^2 + I(F)^2} j\Big) \\
&= \overline{D}\Big(\frac{f_1(\overline{f_1}^2 + \overline{f_2}^2)}{R(F)^2 + I(F)^2} - \frac{f_2(\overline{f_1}^2 + \overline{f_2}^2)}{R(F)^2 + I(F)^2} j\Big) \\
&= \overline{D}\Big(\frac{f_1}{f_1^2 + f_2^2} - \frac{f_2}{f_1^2 + f_2^2} j\Big) \\
&= \Big\{\frac{\partial}{\partial \overline{z_1}}\Big(\frac{f_1}{f_1^2 + f_2^2}\Big) - \frac{\partial}{\partial \overline{z_2}}\Big(\frac{-f_2}{f_1^2 + f_2^2}\Big)\Big\} \\
&\quad + \Big\{\frac{\partial}{\partial \overline{z_1}}\Big(\frac{-f_2}{f_1^2 + f_2^2}\Big) + \frac{\partial}{\partial \overline{z_2}}\Big(\frac{f_1}{f_1^2 + f_2^2}\Big)\Big\} j.
\end{aligned}$$

By arranging the above terms, we have

$$\begin{aligned}
\overline{D}F^{-1} = \frac{1}{(f_1^2 + f_2^2)^2}\Big\{&\Big(-f_1^2 \frac{\partial f_1}{\partial \overline{z_1}} + f_2^2 \frac{\partial f_1}{\partial \overline{z_1}} - 2f_1 f_2 \frac{\partial f_2}{\partial \overline{z_1}} + f_1^2 \frac{\partial f_2}{\partial \overline{z_2}} \\
&- f_2^2 \frac{\partial f_2}{\partial \overline{z_2}} - 2f_1 f_2 \frac{\partial f_1}{\partial \overline{z_2}}\Big) + \Big(-f_1^2 \frac{\partial f_2}{\partial \overline{z_1}} + f_2^2 \frac{\partial f_2}{\partial \overline{z_1}} + 2f_1 f_2 \frac{\partial f_1}{\partial \overline{z_1}} \\
&- f_1^2 \frac{\partial f_1}{\partial \overline{z_2}} + f_2^2 \frac{\partial f_1}{\partial \overline{z_2}} - 2f_1 f_2 \frac{\partial f_2}{\partial \overline{z_2}}\Big) j\Big\}.
\end{aligned}$$

By applying the corresponding Cauchy-Riemann equations (3.1), we get

$$\begin{aligned}
\overline{D}F^{-1} = \frac{1}{(f_1^2 + f_2^2)^2}\Big\{&f_1^2\Big(-\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\Big) + f_2^2\Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) \\
&- 2f_1 f_2\Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big) - f_1^2\Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big) j + f_2^2\Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big) j \\
&+ 2f_1 f_2\Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) j\Big\} = 0.
\end{aligned}$$

Therefore, by the definition of a hyperholomorphic function on U, the result is obtained. □

Example 3.1 Let U be an open set in **BC**. A function

$$F(Z) = f_1(z_1, z_2) + f_2(z_1, z_2)j = z_1 + z_2 j$$

is hyperholomorphic on U since we have for $r = 1, 2$,

$$\frac{\partial f_r}{\partial \overline{z_r}} = \frac{\partial z_r}{\partial \overline{z_r}} = 0$$

and

$$\frac{\partial f_1}{\partial \overline{z_2}} = \frac{\partial z_1}{\partial \overline{z_2}} = 0, \quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial z_2}{\partial \overline{z_1}} = 0,$$

that is, F satisfies the equations in (3.1). Thus, from Theorem 3.3, the multiplicative inverse function

$$\begin{aligned} F^{-1}(Z) &= \frac{\overline{f_1}R(F) - \overline{f_2}I(F)}{R(F)^2 + I(F)^2} - \frac{\overline{f_1}I(F) + \overline{f_2}R(F)}{R(F)^2 + I(F)^2} j \\ &= \frac{\overline{z_1}R(F) - \overline{z_2}I(F)}{R(F)^2 + I(F)^2} - \frac{\overline{z_1}I(F) + \overline{z_2}R(F)}{R(F)^2 + I(F)^2} j \end{aligned}$$

of F, where $R(F) = z_1\overline{z_1} + z_2\overline{z_2}$ and $I(F) = z_2\overline{z_1} - z_1\overline{z_2}$, is also hyperholomorphic on U. In fact, we have

$$\overline{z_1}R(F)\overline{z_2}I(F) = z_1\overline{z_1}^2 + z_1\overline{z_2}^2, \quad \overline{z_1}I(F) + \overline{z_2}R(F) = z_2\overline{z_1}^2 + z_2\overline{z_2}^2,$$

$$R(F)^2 + I(F)^2 = z_1^2\overline{z_1}^2 + z_2^2\overline{z_2}^2 + z_2^2\overline{z_1}^2 + z_1^2\overline{z_2}^2,$$

$$\frac{\partial}{\partial \overline{z_1}}(\overline{z_1}R(F) - \overline{z_2}I(F)) = 2z_1\overline{z_1}, \quad \frac{\partial}{\partial \overline{z_2}}(\overline{z_1}R(F) - \overline{z_2}I(F)) = 2z_1\overline{z_2},$$

$$\frac{\partial}{\partial \overline{z_1}}(\overline{z_1}I(F) + \overline{z_2}R(F)) = 2\overline{z_1}z_2, \quad \frac{\partial}{\partial \overline{z_2}}(\overline{z_1}I(F) + \overline{z_2}R(F)) = 2z_2\overline{z_2},$$

$$\frac{\partial}{\partial \overline{z_1}}(R(F)^2 + I(F)^2) = 2z_1^2\overline{z_1} + 2z_2^2\overline{z_1}$$

and

$$\frac{\partial}{\partial \overline{z_2}}(R(F)^2 + I(F)^2) = 2z_2^2\overline{z_2} + 2z_1^2\overline{z_2}.$$

So, by the rules of calculations of differential operators in complex analysis, we get

$$\begin{aligned} \frac{\partial}{\partial \overline{z_1}}\left(\frac{\overline{z_1}R(F) - \overline{z_2}I(F)}{R(F)^2 + I(F)^2}\right) &= \frac{2z_1\overline{z_1}(z_1^2\overline{z_1}^2 + z_2^2\overline{z_2}^2 + z_2^2\overline{z_1}^2 + z_1^2\overline{z_2}^2)}{(R(F)^2 + I(F)^2)^2} \\ &\quad - \frac{(z_1\overline{z_1}^2 + z_1\overline{z_2}^2)(2z_1^2\overline{z_1} + 2z_2^2\overline{z_1})}{(R(F)^2 + I(F)^2)^2} = 0, \end{aligned}$$

$$\frac{\partial}{\partial \overline{z_2}}\left(-\frac{\overline{z_1}I(F)-\overline{z_2}R(F)}{R(F)^2+I(F)^2}\right)=\frac{2z_2\overline{z_2}(z_1^2\overline{z_1}^2+z_2^2\overline{z_2}^2+z_2^2\overline{z_1}^2+z_1^2\overline{z_2}^2)}{(R(F)^2+I(F)^2)^2}-\frac{(z_2\overline{z_1}^2+z_2\overline{z_2}^2)(2z_2^2\overline{z_2}+2z_1^2\overline{z_2})}{(R(F)^2+I(F)^2)^2}=0,$$

$$\frac{\partial}{\partial \overline{z_2}}\left(\frac{\overline{z_1}R(F)-\overline{z_2}I(F)}{R(F)^2+I(F)^2}\right)=\frac{2z_1\overline{z_1}(z_1^2\overline{z_1}^2+z_2^2\overline{z_2}^2+z_2^2\overline{z_1}^2+z_1^2\overline{z_2}^2)}{(R(F)^2+I(F)^2)^2}-\frac{(z_1\overline{z_1}^2+z_1\overline{z_2}^2)(2z_2^2\overline{z_2}+2z_1^2\overline{z_2})}{(R(F)^2+I(F)^2)^2}=0,$$

and

$$\frac{\partial}{\partial \overline{z_1}}\left(-\frac{\overline{z_1}I(F)-\overline{z_2}R(F)}{R(F)^2+I(F)^2}\right)=\frac{2z_2\overline{z_1}(z_1^2\overline{z_1}^2+z_2^2\overline{z_2}^2+z_2^2\overline{z_1}^2+z_1^2\overline{z_2}^2)}{(R(F)^2+I(F)^2)^2}-\frac{(z_2\overline{z_1}^2+z_2\overline{z_2}^2)(2z_1^2\overline{z_1}+2z_2^2\overline{z_1})}{(R(F)^2+I(F)^2)^2}=0.$$

Thus, the function F^{-1} satisfies the equations in (3.1), that is, the function F^{-1} is hyperholomorphic on U.

Theorem 3.4 *Let U be an open set in* **BC**. *A hyperholomorphic function F is pseudo-bihyperholomorphic on U if and only if the following equations are satisfied for F:*

$$\begin{cases} -\overline{f_1}\dfrac{\partial f_1}{\partial \overline{z_1}}+f_1\dfrac{\partial \overline{f_2}}{\partial \overline{z_2}}-f_2\dfrac{\partial \overline{f_2}}{\partial \overline{z_1}}=\overline{f_2}\dfrac{\partial f_1}{\partial \overline{z_1}}-f_2\dfrac{\partial \overline{f_1}}{\partial \overline{z_1}}+f_1\dfrac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\ f_1\dfrac{\partial \overline{f_2}}{\partial \overline{z_1}}-\overline{f_1}\dfrac{\partial f_2}{\partial \overline{z_1}}+f_2\dfrac{\partial \overline{f_2}}{\partial \overline{z_2}}=\overline{f_2}\dfrac{\partial f_2}{\partial z_1}+f_2\dfrac{\partial \overline{f_1}}{\partial \overline{z_2}}+f_1\dfrac{\partial \overline{f_1}}{\partial \overline{z_1}}. \end{cases} \tag{3.2}$$

Proof. Consider the following equation:

$$\begin{aligned} DF_{|.|}^{-1} &= \overline{D}\left(\frac{\overline{f_1}}{R(F)}+\frac{-\overline{f_2}}{R(F)}j\right) \\ &= \frac{\partial}{\partial \overline{z_1}}\left(\frac{\overline{f_1}}{R(F)}\right)-\frac{\partial}{\partial \overline{z_2}}\left(\frac{\overline{f_2}}{R(F)}\right)+\frac{\partial}{\partial \overline{z_1}}\left(\frac{-\overline{f_2}}{R(F)}\right)+\frac{\partial}{\partial \overline{z_2}}\left(\frac{\overline{f_1}}{R(F)}\right)j. \end{aligned}$$

Since F is hyperholomorphic on U, we have

$$\begin{aligned} &\frac{\partial}{\partial \overline{z_1}}(\overline{f_1}R(F))-\frac{\partial}{\partial \overline{z_2}}\left(\frac{-\overline{f_2}}{R(F)}\right) \\ &= \frac{1}{R(F)^2}\left(-\overline{f_1}^2\frac{\partial f_1}{\partial \overline{z_1}}+f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}}-\overline{f_1}f_2\frac{\partial \overline{f_2}}{\partial \overline{z_1}}-\overline{f_2}^2\frac{\partial f_2}{\partial \overline{z_2}}+f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_2}}-f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}}\right) \\ &= -(\overline{f_1}^2+\overline{f_2}^2)\frac{\partial f_1}{\partial \overline{z_1}}+f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_2}}-\overline{f_1}f_2\frac{\partial \overline{f_2}}{\partial \overline{z_1}}+f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}}-f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}} \end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial \overline{z_1}}\Big(\frac{-\overline{f_2}}{R(F)}\Big) + \frac{\partial}{\partial \overline{z_2}}\Big(\frac{\overline{f_1}}{R(F)}\Big) \\
&= \frac{1}{R(F)^2}\Big(\overline{f_2}^2 \frac{\partial f_2}{\partial \overline{z_1}} - f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_1}} + f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}} + f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}} - \overline{f_1} f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} \overline{f_1}^2 \frac{\partial f_1}{\partial \overline{z_2}}\Big) \\
&= -f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_1}} + (\overline{f_1}^2 + \overline{f_2}^2)\frac{\partial f_2}{\partial \overline{z_1}} - \overline{f_1} f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} + f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}} + f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}}.
\end{aligned}$$

By definition of a pseudo-bihyperholomorphic function on U and calculating terms in the above equations, we obtain the equations in (3.2).
Conversely, suppose a hyperholomorphic function F satisfies the equations in (3.2). Then we have

$$\begin{aligned}
\overline{D}F_{|\cdot|}^{-1} = &\Big\{-(\overline{f_1}^2 + \overline{f_2}^2)\frac{\partial f_1}{\partial \overline{z_1}} + f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_2}} - \overline{f_1} f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_1}} + f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}} - f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}}\Big\} \\
&+\Big\{f_1\overline{f_1}\frac{\partial \overline{f_2}}{\partial \overline{z_1}} + (\overline{f_1}^2 + \overline{f_2}^2)\frac{\partial f_2}{\partial \overline{z_1}} - \overline{f_1} f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} + f_2\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_2}} + f_1\overline{f_2}\frac{\partial \overline{f_1}}{\partial \overline{z_1}}\Big\} j.
\end{aligned}$$

Applying the equations in (3.1), we obtain the equation $\overline{D}F_{|\cdot|}^{-1} = 0$. Therefore, F is pseudo-bihyperholomorphic on U. □

Example 3.2 Let U be an open set in **BC**. For a function $F(Z) = f_1 + f_2 j = z_1 + z_1 j$, that is, $f_1(z_1, z_2) = z_1$ and $f_2(z_1, z_2) = z_1$, we have

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial z_1}{\partial \overline{z_1}} = 0, \quad \frac{\partial \overline{f_2}}{\partial \overline{z_2}} = \frac{\partial \overline{z_1}}{\partial \overline{z_2}} = 0, \quad \frac{\partial \overline{f_2}}{\partial \overline{z_1}} = \frac{\partial \overline{z_1}}{\partial \overline{z_1}} = 1,$$

$$\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = \frac{\partial \overline{z_1}}{\partial \overline{z_1}} = 1, \quad \frac{\partial \overline{f_1}}{\partial \overline{z_2}} = \frac{\partial \overline{z_1}}{\partial \overline{z_2}} = 0, \quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial z_1}{\partial \overline{z_1}} = 0$$

and

$$\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = \frac{\partial \overline{z_1}}{\partial \overline{z_1}} = 1.$$

Hence, the first equation of (3.2) is

$$-f_2\frac{\partial \overline{f_2}}{\partial \overline{z_1}} = -z_1 \quad \text{and} \quad -f_2\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = -z_1$$

and the second equation of (3.2) is

$$f_1\frac{\partial \overline{f_2}}{\partial \overline{z_1}} = z_1 \quad \text{and} \quad f_1\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = z_1.$$

Thus, the function F satisfies the equations in (3.2) and from Theorem 3.4, we obtain that F is pseudo-bihyperholomorphic on U.

Example 3.3 Let U be an open set in **BC**. For a function $F(Z) = f_1 + f_2 j = z_2 + z_2 j$, we have

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial z_2}{\partial \overline{z_1}} = 0, \quad \frac{\partial \overline{f_2}}{\partial \overline{z_2}} = \frac{\partial \overline{z_2}}{\partial \overline{z_2}} = 1, \quad \frac{\partial \overline{f_2}}{\partial \overline{z_1}} = \frac{\partial \overline{z_2}}{\partial \overline{z_1}} = 0,$$

$$\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = \frac{\partial \overline{z_2}}{\partial \overline{z_1}} = 1, \quad \frac{\partial \overline{f_1}}{\partial \overline{z_2}} = \frac{\partial \overline{z_2}}{\partial \overline{z_2}} = 1, \quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial z_2}{\partial \overline{z_1}} = 0$$

and

$$\frac{\partial \overline{f_1}}{\partial \overline{z_1}} = \frac{\partial \overline{z_2}}{\partial \overline{z_1}} = 0.$$

Hence, the first equation of (3.2) is

$$f_1 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} = z_2 \quad \text{and} \quad f_1 \frac{\partial \overline{f_1}}{\partial \overline{z_2}} = z_2$$

and the second equation of (3.2) is

$$f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} = z_2 \quad \text{and} \quad f_1 \frac{\partial \overline{f_1}}{\partial \overline{z_1}} = z_2.$$

Therefore, the function F satisfies the equations in (3.2) and from Theorem 3.4, we obtain that F is pseudo-bihyperholomorphic on U.

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