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A new class of functions suggested by the generalized hypergeometric function

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Abstract We introduce here an entire function of order zero, representing a rapidly convergent power series which extends the generalized hypergeometric function ${}_pF_q[z]$. We introduce the infinite order hyper-Bessel operators and thereby obtain its differential equation and the eigen function property. Some contiguous function relations are also derived. We finally show that the Ramanujan's theorem and Kummer's first formula admit extension by means of this theory.

Keywords hypergeometric function · differential equation · eigen function · Ramanujan's theorem · Kummer's first formula

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1 Introduction

It is known that the generalized hypergeometric function [9]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; z \\ b_1, b_2, \ldots, b_q; \end{matrix}\right] = \sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{1.1} $$

is an entire function of (real or) a complex variable z when $p \leq q$. Here $\forall i = 1, 2, \ldots, p,\ a_i \in \mathbf{C}$ and $\forall j = 1, 2, \ldots, q,\ b_j \in \mathbf{C} \setminus \{0, -1, -2, \ldots, \}$. The Pochhammer symbol $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.
With $p = q = 1$ the function ${}_pF_q[z]$ in (1.1) reduces to

$$ {}_1F_1\left[\begin{matrix} a; z \\ b; \end{matrix}\right] = \sum_{n\geq 0} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{1.2} $$

which is again an entire function.
In 1953, P. C. Sikkema [10, p. 6] considered the entire function of the novel

form:

$$\sum_{n\geq 1} \frac{z^n}{n!^n}. \tag{1.3}$$

With the aid of the power series considered in (1.2) and (1.3), we defined and studied certain properties of a class of new functions given by [3]

$$H\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] = \sum_{n\geq 0} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{1.4}$$

in which $a, \ell, z \in \mathbf{C}$ with $\Re(\ell) \geq 0$, and $b, c \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$. Evidently, when $\ell = 0$, the function in (1.4) reduces to the function (1.2). In context with the power series considered in (1.1), we further generalize (1.4) as follows.

Definition 1.1 *For $p, r, s \in \mathbf{N} \cup \{0\}$, $a_i, b_j, c_k \in \mathbf{C}$, with $b_j, c_k \neq 0, -1, -2, \ldots,$ $\forall i = 1, 2, \ldots, r, \forall j = 1, 2, \ldots, s,\ \forall k = 1, 2, \ldots, p$, we define the generalized ℓ-Hypergeometric function as*

$$\begin{aligned} {}_rH_s^p(\ell; z) &= {}_rH_s^p\left[\begin{matrix} a_1,\, a_2,\, \ldots,\, a_r; & & z \\ b_1,\, b_2,\, \ldots,\, b_s; & (c_1,\, c_2,\, \ldots,\, c_p : \ell); & \end{matrix}\right] \\ &= \sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n} \cdots (c_p)_n^{\ell n}} \frac{z^n}{n!}, \end{aligned} \tag{1.5}$$

where $\ell \in \mathbf{C}$ with $\Re(\ell) \geq 0$.

Remark 1.1 The function ${}_rH_s^p(\ell; z)$ reduces to the generalized hypergeometric function ${}_rF_s[z]$ when $\ell = 0$.

2 Main Results

In this section, we take

$$\frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n} \cdots (c_p)_n^{\ell n}\ n!} = \varphi_n.$$

We shall now onward refer to the function defined by (1.5) as the *generalized ℓ-H function*. Also, throughout the work we fix the range of values of i, j, k to be $i = 1, 2, \ldots, r, j = 1, 2, \ldots, s$, and $k = 1, 2, \ldots, p$.

2.1 Convergence

We first show in the following theorem that the function in (1.5) exists.

Theorem 2.1 *If* $\Re(\ell) \geq 0$ *and* $\Re\left((c_1 + c_2 + \cdots + c_p)\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$ *then the generalized* ℓ*-H function is an entire function of* z.

Proof. With φ_n as stated above and from the Cauchy-Hadamard formula, we have

$$\begin{aligned}\frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{|\varphi_n|}\\ &= \lim_{n\to\infty} \sup \left| \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n \ (c_1)_n^{\ell n} (c_2)_n^{\ell n} \cdots (c_p)_n^{\ell n}} \frac{1}{n!} \right|^{\frac{1}{n}}\\ &= \lim_{n\to\infty} \sup \left| \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_s)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)} \right|^{\frac{1}{n}} \left| \frac{\Gamma(a_1+n)\Gamma(a_2+n)\cdots\Gamma(a_r+n)}{\Gamma(b_1+n)\Gamma(b_2+n)\cdots\Gamma(b_s+n)} \right|^{\frac{1}{n}}\\ &\quad \times \left| \frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_p)}{\Gamma(c_1+n)\Gamma(c_2+n)\cdots\Gamma(c_p+n)} \right|^{\ell} \frac{1}{\Gamma^{\frac{1}{n}}(n+1)}.\end{aligned}$$

Now applying the Stirling's formula [4]:

$$\Gamma(\alpha+n) \sim \sqrt{2\pi} e^{-(\alpha+n)} \ (\alpha+n)^{(\alpha+n-1/2)} \tag{2.1}$$

for large n and taking $\alpha = a_i, b_j, c_k$, in turn, we get

$$\begin{aligned}\frac{1}{R} &\sim \left| \frac{\prod\limits_{1\leq k\leq p} \Gamma(c_k)}{(2\pi)^{\frac{p}{2}}} \right|^{\ell} \lim_{n\to\infty} \sup \ \frac{\left| \prod\limits_{1\leq k\leq p} e^{-(c_k+n)} \ (c_k+n)^{c_k+n-1/2} \right|^{-\ell}}{\left| (2\pi)^{\frac{1}{2}} \ e^{-(n+1)} \ (n+1)^{n+1-1/2} \right|^{\frac{1}{n}}}\\ &\quad \times \left| \frac{\prod\limits_{1\leq j\leq s} \Gamma(b_j)}{\prod\limits_{1\leq i\leq r} \Gamma(a_i)} \right|^{\frac{1}{n}} \left| \frac{(2\pi)^{\frac{r}{2}} \prod\limits_{1\leq i\leq r} e^{-(a_i+n)} \ (a_i+n)^{a_i+n-1/2}}{(2\pi)^{\frac{s}{2}} \prod\limits_{1\leq j\leq s} e^{-(b_j+n)} \ (b_j+n)^{b_j+n-1/2}} \right|^{\frac{1}{n}}\\ &= \left| \frac{\prod\limits_{1\leq k\leq p} \Gamma^{\ell}(c_k)}{(2\pi)^{\frac{p\ell}{2}} \ e^{r-s}} \right| \lim_{n\to\infty} \sup \ \left| \frac{1}{n^{s-r+1} n^{(c_1+c_2+\cdots+c_p)\ell - \frac{\ell p}{2}}} \left(\frac{e}{n}\right)^{\ell p n} \right|\\ &= 0,\end{aligned}$$

provided that $\Re(\ell) \geq 0$ and $\Re\left((c_1 + c_2 + \cdots + c_p)\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$.
□

Remark 2.1 The series $\sum \varphi_n \ z^n$ thus converges uniformly in any compact subset of **C**.

2.2 Order of ${}_rH_s^p(\ell; z)$ function

Theorem 2.2 *If the conditions stated in Theorem 2.1 hold then the generalized ℓ-H function is an entire function of order zero.*

Proof. We use the result which states that if $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of f [2,8] is given by

$$\rho(f) = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln |a_n|^{-1}}. \tag{2.2}$$

Now

$$|\varphi_n|^{-1} = \left| \frac{\left\{ \prod\limits_{1\le i\le r} \Gamma(a_i) \right\} \left\{ \prod\limits_{1\le j\le s} \Gamma(b_j + n) \right\}}{\left\{ \prod\limits_{1\le j\le s} \Gamma(b_j) \right\} \left\{ \prod\limits_{1\le i\le r} \Gamma(a_i + n) \right\}} \right| \times \left| \frac{\left\{ \prod\limits_{1\le k\le p} \Gamma^{\ell n}(c_k + n) \right\}}{\left\{ \prod\limits_{1\le k\le p} \Gamma^{\ell n}(c_k) \right\}} \right| \Gamma(n+1).$$

Since for large r,

$$\ln \Gamma(r) \sim \left(r - \frac{1}{2}\right) \ln r - r + \frac{1}{2} \ln \sqrt{2\pi},$$

we further have

$$\begin{aligned}
\ln|\varphi_n|^{-1} \sim & \left|\sum_{1\le i\le r} \ln \Gamma(a_i)\right| - \left|\sum_{1\le j\le s} \ln \Gamma(b_j)\right| \\
& + \sum_{1\le j\le s} \left|\left(b_j + n - \frac{1}{2}\right)\ln(b_j+n) - (b_j+n) + \frac{1}{2}\ln\sqrt{2\pi}\right| \\
& - \sum_{1\le i\le r} \left|\left(a_i + n - \frac{1}{2}\right)\ln(a_i+n) + (a_i+n) - \frac{1}{2}\ln\sqrt{2\pi}\right| \\
& + \left|\ell n \sum_{1\le k\le p} \left[\left(c_k + n - \frac{1}{2}\right)\ln(c_k+n) - (c_k+n) + \frac{1}{2}\ln\sqrt{2\pi}\right]\right| \\
& + \left|\left(n + 1 - \frac{1}{2}\right)\ln(n+1) - (n+1) + \frac{1}{2}\ln\sqrt{2\pi}\right| \\
& - \left|\ell n \sum_{1\le k\le p} \ln \Gamma(c_k)\right|. \qquad (2.3)
\end{aligned}$$

But since

$$\lim_{n\to\infty} \frac{\ln|\varphi_n|^{-1}}{n\ln n}$$

is unbounded, it follows from (2.2) and (2.3) that

$$\rho\left({}_rH_s^p(\ell;z)\right) = \lim_{n\to\infty} \sup \frac{n\ln n}{\ln|\varphi_n|^{-1}} = 0.$$

□

Remark 2.2 It is known that [1, Theorem 1.1] "If f is entire and $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros or it is a polynomial." Thus, the generalized ℓ-H function has infinitely many zeros.

2.3 Integral Representation

An ℓ-analogue of the integral representation of ${}_pF_q[z]$ is stated here as

Theorem 2.3 *If* $a_i, b_j, c_k \in \mathbf{C}$ *with* $\Re(b_1) > \Re(a_1) > 0,\ b_j, c_k \neq 0, -1, -2, \ldots,$ *and* $\Re\left(\sum_{1\le k\le p} c_k\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$ *then*

$$\begin{aligned} & {}_rH_s^p\left[\begin{matrix} a_1, a_2, \ldots, a_r; & z \\ b_1, b_2, \ldots, b_s; (c_1, c_2, \ldots, c_p : \ell); & \end{matrix}\right] \\ &= \frac{\Gamma(b_1)}{\Gamma(a_1)\ \Gamma(b_1 - a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} \\ &\quad \times {}_{r-1}H_{s-1}^p\left[\begin{matrix} a_2, \ldots, a_r; & zt \\ b_2, \ldots, b_s; (c_1, c_2, \ldots, c_p : \ell) & \end{matrix}\right] \mathrm{dt}. \end{aligned}$$

The proof follows readily by following the technique adopted in [9, Ch.4, p.47].

2.4 Differential Equation

Unlike the derivation of the differential equation of ${}_pF_q[z]$ which is order two (finite), here the method of deriving differential equation of ${}_rH_s^p(\ell; z)$ uses the operator ${}_p\Delta_\alpha^\Theta$ as defined below, which turns out to be of infinite order. The restriction to the parameter ℓ is that $\ell \in \mathbf{N} \cup \{0\}$.

Definition 2.4 *For the function* $f(z) = \sum_{n\ge1} a_n z^n,\ z \in \mathbf{C},$ *and* $p \in \mathbf{N} \cup \{0\}$, *we define*

$$ {}_p\Delta_\alpha^\Theta\ f(z) = \begin{cases} \sum_{n\ge1} a_n (\alpha)_{n-1}^p (\Theta + \alpha - 1)^{pn} z^n, & \text{if} \quad p \in \mathbf{N} \\ f(z), & \text{if} \quad p = 0, \end{cases}, \tag{2.4}$$

where Θ *is the Euler differential operator* $\theta = z\frac{d}{dz}$ *and*

$$(\Theta + \alpha)^r = \underbrace{(\Theta + \alpha)(\Theta + \alpha) \ldots (\Theta + \alpha)}_{r \text{ times}}$$

is a special case of the hyper-Bessel differential operators (see e.g. [5–7]).

In this notation, we have

Theorem 2.5 *For* $\ell, p, r, s \in \mathbf{N} \cup \{0\}, a_i, b_j, c_k \in \mathbf{C}$ *with* $b_j, c_k \neq 0, -1, -2, \ldots,$ *the function*

$$w = {}_rH_s^p\left[\begin{matrix} a_1, a_2, \ldots, a_r; & z \\ b_1, b_2, \ldots, b_s; (c_1, c_2, \ldots, c_p : \ell); & \end{matrix}\right]$$

satisfies the equation

$$\left[\left\{\prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\left\{\prod_{1\le j\le s}(\theta+b_j-1)\right\}\theta - z\prod_{1\le i\le r}(\theta+a_i)\right] w = 0. \quad (2.5)$$

In order to prove this theorem, we first prove the following lemma in which we show that the operator on the l. h. s. of (2.5) is indeed applicable to the function w.

For the sake of brevity, we put

$$\left[\left\{\prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\left\{\prod_{1\le j\le s}(\theta+b_j-1)\right\}\theta\right] = {}_{(b)}\Lambda^{\theta}_{(c:\ell)}.$$

In this notation, we have

Lemma 2.6 *If* $\ell \in \mathbf{N}\cup\{0\}$, $w = \sum\limits_{n\ge 0}\varphi_n z^n$ *and*

$${}_{(b)}\Lambda^{\theta}_{(c:\ell)} w = \sum_{n\ge 0} f_n((a,r),(b,s),(c,p:\ell);z),$$

then the operator ${}_{(b)}\Lambda^{\theta}_{(c:\ell)}$ *is applicable to the generalized* ℓ*-H function provided that the series*

$$\sum_{n\ge 0}\varphi_n\ f_n((a,r),(b,s),(c,p:\ell);z)$$

converges (cf. [10, Definition 11, p.20]).

Proof. With $w = \sum\limits_{n\ge 0}\varphi_n z^n$, and $n!\ \varphi_n = A_n$, we have

$$\begin{aligned}
&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\
&= \left\{\prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_s-1)\}\sum_{n\ge 0}A_n\ \frac{\theta z^n}{n!} \\
&= \left\{\prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_{s-1}-1)\} \\
&\quad\times\sum_{n\ge 1}A_n\ \frac{(\theta+b_s-1)z^n}{(n-1)!} \\
&= \left\{\prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_{s-1}-1)\} \\
&\quad\times\sum_{n\ge 1}A_n\ \frac{(n+b_s-1)z^n}{(n-1)!}.
\end{aligned}$$

By applying the operator $(\theta + b_j - 1)$ for $j = 1, 2, \ldots, s-1$, and proceeding as above, we obtain

$$
\begin{aligned}
&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\
&= \left\{ \prod_{1\le k\le p} {}_{\ell}\Delta^{\theta}_{c_k} \right\} \sum_{n\ge 1} \left\{ \prod_{1\le j\le s} (b_j + n - 1) \right\} A_n \, \frac{z^n}{(n-1)!} \\
&= \left\{ \prod_{1\le k\le p-1} {}_{\ell}\Delta^{\theta}_{c_k} \right\} {}_{\ell}\Delta^{\theta}_{c_p} \left(\sum_{n\ge 1} \left\{ \prod_{1\le j\le s} (b_j + n - 1) \right\} A_n \, \frac{z^n}{(n-1)!} \right) \\
&= \left\{ \prod_{1\le k\le p-1} {}_{\ell}\Delta^{\theta}_{c_k} \right\} \sum_{n\ge 1} \left\{ \prod_{1\le j\le s} (b_j + n - 1) \right\} A_n \, \frac{(c_p)^{\ell}_{n-1}}{(n-1)!} \\
&\quad \times (\theta + c_p - 1)^{\ell n} \, z^n.
\end{aligned} \tag{2.6}
$$

Noticing that for $n \in \mathbf{N}$,

$$
(\theta + c_k - 1)^{\ell n} \, z^n = (n + c_k - 1)^{\ell n} \, z^n, \tag{2.7}
$$

the substitution of (2.7) in (2.6) gives

$$
\begin{aligned}
&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\
&= \left\{ \prod_{1\le k\le p-1} {}_{\ell}\Delta^{\theta}_{c_k} \right\} \sum_{n\ge 1} \left\{ \prod_{1\le j\le s} (b_j + n - 1) \right\} A_n \, \frac{(c_p)^{\ell}_{n-1}}{(n-1)!} \\
&\quad \times (c_p + n - 1)^{\ell n} z^n.
\end{aligned} \tag{2.8}
$$

Proceeding similarly by applying ${}_{\ell}\Delta^{\theta}_{c_k}$ for $k = 1, 2, \ldots, p-1$, leads us to

$$
\begin{aligned}
{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w &= \sum_{n\ge 1} \frac{(a_1)_n \, (a_2)_n \cdots (a_r)_n}{(b_1)_{n-1} \, (b_2)_{n-1} \cdots (b_s)_{n-1}} \\
&\quad \times \frac{z^n}{(c_1)^{\ell n-\ell}_{n-1} \, (c_2)^{\ell n-\ell}_{n-1} \cdots (c_p)^{\ell n-\ell}_{n-1} \, (n-1)!} && (2.9) \\
&= \sum_{n\ge 0} \frac{(a_1)_{n+1} \, (a_2)_{n+1} \cdots (a_r)_{n+1}}{(b_1)_n \, (b_2)_n \cdots (b_s)_n \, (c_1)^{\ell n}_n \, (c_2)^{\ell n}_n \cdots (c_p)^{\ell n}_n} \frac{z^{n+1}}{n!} && (2.10) \\
&= \sum_{n\ge 0} f_n((a,r),(b,s),(c,p:\ell);z) \text{ (say).}
\end{aligned}
$$

To complete the proof of the lemma it remains to show that the series

$$\sum_{n\geq 0}\varphi_n\ f_n((a,r),(b,s),(c,p:\ell);z) = \sum_{n\geq 0}\frac{\left\{\prod\limits_{1\leq i\leq r}(a_i)_n^2(a_i)_{n+1}\right\}}{\left\{\prod\limits_{1\leq j\leq s}(b_j)_n^2\right\}\ \left\{\prod\limits_{1\leq k\leq p}(c_k)_n^{2\ell n}\right\}}\frac{z^{n+1}}{(n!)^2}$$

is convergent.
For that take

$$\mu_n = \frac{\left\{\prod\limits_{1\leq i\leq r}(a_i)_n^2(a_i+n)\right\}}{\left\{\prod\limits_{1\leq j\leq s}(b_j)_n^2\right\}\ \left\{\prod\limits_{1\leq k\leq p}(c_k)_n^{2\ell n}\right\}}\ \frac{1}{(n!)^2}$$

$$= \frac{\left\{\prod\limits_{1\leq j\leq s}\Gamma^2(b_j)\right\}}{\left\{\prod\limits_{1\leq i\leq r}\Gamma^2(a_i)\right\}}\frac{\left\{\prod\limits_{1\leq i\leq r}(a_i+n)\ \Gamma^2(a_i+n)\right\}\left\{\prod\limits_{1\leq k\leq p}\Gamma^{2\ell n}(c_k)\right\}}{\left\{\prod\limits_{1\leq j\leq s}\Gamma^2(b_j+n)\right\}\left\{\prod\limits_{1\leq k\leq p}\Gamma^{2\ell n}(c_k+n)\right\}\Gamma^2(n+1)}.$$

Now in view of the Stirling's asymptotic formula (2.1) for large n, we have

$$|\mu_n|^{\frac{1}{n}} \sim \left|\frac{\prod\limits_{1\leq j\leq s}\Gamma^2(b_j)}{\prod\limits_{1\leq i\leq r}\Gamma^2(a_i)}\right|^{\frac{1}{n}}\ \frac{\left|\prod\limits_{1\leq i\leq r}\left\{e^{-(a_i+n)}(a_i+n)^{a_i+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{\frac{2}{n}}}{\left|\prod\limits_{1\leq j\leq s}\left\{e^{-(b_j+n)}(b_j+n)^{b_j+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{\frac{2}{n}}}$$
$$\times\frac{\left|\prod\limits_{1\leq i\leq r}(a_i+n)^{\frac{1}{n}}\right|\ \left|\prod\limits_{1\leq k\leq p}\Gamma^{2\ell}(c_k)\right|\ \left|e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}\sqrt{2\pi}\right|^{-\frac{2}{n}}}{\left|\prod\limits_{1\leq k\leq p}\left\{e^{-(c_k+n)}(c_k+n)^{c_k+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{2\ell}}.$$

Hence from the Cauchy-Hadamard formula,

$$\lim_{n\to\infty}\sup\ |\mu_n|^{\frac{1}{n}}\sim\lim_{n\to\infty}\sup\left\{\prod_{1\leq k\leq p}\left|\Gamma^{2\ell}(c_k)\right|\right\}\frac{\left|(2\pi)^{-\ell p}\ e^{2(s-r)}\right|\ \left|\frac{e}{n}\right|^{2n\ell p}}{\left|n^{2(c_1+c_2+\cdots+c_p)\ell-\ell p+2(s-r+1)}\right|}$$
$$= 0,$$

provided that $\Re(\ell) \geq 0$ and $\Re\left(2(c_1 + c_2 + \cdots + c_p)\ell - \ell p + 2(s - r + 1)\right) > 0$. This completes the proof of the lemma. □

Proof. (of Theorem 2.5)
From (2.10) we have

$$
\begin{aligned}
{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w &= \sum_{n\geq 0} A_n \left\{ \prod_{1\leq i\leq r} (a_i + n) \right\} \frac{z^{n+1}}{n!} \\
&= z \sum_{n\geq 0} A_n \left\{ \prod_{1\leq i\leq r} (\theta + a_i) \right\} \frac{z^n}{n!} \\
&= z \left\{ \prod_{1\leq i\leq r} (\theta + a_i) \right\} w.
\end{aligned}
$$

Thus the theorem. □

2.5 Contiguous function relations

In parallel to the theory of contiguous functions of ${}_pF_q[z]$ [9, p. 82], we define the functions that are contiguous to ${}_rH^p_s(\ell; z)$ and obtain the relations amongst them as follows. Put

$$
{}_rH^p_s = {}_rH^p_s \left[\begin{matrix} a_1, a_2, \ldots, a_r; & z \\ b_1, b_2, \ldots, b_s; (c_1, c_2, \ldots, c_p : \ell); & \end{matrix} \right];
$$

and define

$$
\begin{aligned}
{}_rH^p_s(a_i+) &= {}_rH^p_s \left[\begin{matrix} a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, \quad a_r; & z \\ b_1, b_2, \ldots, \quad b_s; \quad (c_1, \quad c_2, \quad \ldots, c_p : \ell); & \end{matrix} \right], \\
{}_rH^p_s(a_i-) &= {}_rH^p_s \left[\begin{matrix} a_1, a_2, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, \quad a_r; & z \\ b_1, b_2, \ldots, \quad b_s; \quad (c_1, \quad c_2, \quad \ldots, c_p : \ell); & \end{matrix} \right],
\end{aligned}
$$

and similarly, ${}_rH^p_s(b_j+)$, ${}_rH^p_s(b_j-)$, ${}_rH^p_s(c_k+)$, and ${}_rH^p_s(c_k-)$ as the functions contiguous to ${}_rH^p_s(\ell; z)$.
We symbolize the finite products as follows.

$$
\prod_{1\leq i\leq r} a_i = A, \quad \prod_{1\leq j\leq s} b_j = B, \quad \prod_{1\leq k\leq p} (c_k)^{\ell} = C.
$$

Now, in view of

$$
{}_rH^p_s = \sum_{n\geq 0} \varphi_n \ z^n,
$$

we find at once that

$$
\begin{aligned}
{}_rH_s^p(a_i+) &= \sum_{n\geq 0} \tfrac{a_i+n}{a_i}\ \varphi_n\ z^n, & {}_rH_s^p(a_i-) &= \sum_{n\geq 0} \tfrac{a_i-1}{a_i+n-1}\ \varphi_n\ z^n,\\
{}_rH_s^p(b_j+) &= \sum_{n\geq 0} \tfrac{b_j}{b_j+n}\ \varphi_n\ z^n, & {}_rH_s^p(b_j-) &= \sum_{n\geq 0} \tfrac{b_j+n-1}{b_j-1}\ \varphi_n\ z^n,\\
{}_rH_s^p(c_k+) &= \sum_{n\geq 0} \tfrac{(c_k)^{\ell n}}{(c_k+n)^{\ell n}}\ \varphi_n\ z^n, & {}_rH_s^p(c_k-) &= \sum_{n\geq 0} \tfrac{(c_k+n-1)^{\ell n}}{(c_k-1)^{\ell n}}\ \varphi_n\ z^n.
\end{aligned}
$$

By making use of these, we obtain contiguous functions relations in

Theorem 2.7 *For $j = 2, 3, \ldots, s$ there hold the contiguous functions relations:*

$$(a_1 - b_j + 1)\ {}_rH_s^p = a_1\ {}_rH_s^p(a_1+) - (b_j - 1)\ {}_rH_s^p(b_j-). \tag{2.11}$$

Whereas for $\ell \in \mathbf{N} \cup \{0\}$, there hold the following extended contiguous function relations.

$$
\begin{aligned}
a_i\ {}_rH_s^p &= a_i\ {}_rH_s^p(a_i+)\\
&\quad -\frac{A}{B\ C}\ z\ {}_rH_{s+\ell}^p\left[\begin{matrix}(a)+1; \\ (b)+1, ((c)+1)^\ell; ((c)+1:\ell);\end{matrix}\ \frac{z}{C}\right],
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
(b_j-1)\ {}_rH_s^p &= (b_j-1)\ {}_rH_s^p(b_j-) - \frac{A}{B\ C}\ z\\
&\quad \times {}_rH_{s+\ell}^p\left[\begin{matrix}(a)+1; \\ (b)+1, ((c)+1)^\ell; ((c)+1:\ell);\end{matrix}\ \frac{z}{C}\right],
\end{aligned} \tag{2.13}
$$

in which $(\alpha)+1$ stands for the array of the parameters $\alpha_1+1, \alpha_2+1, \ldots, \alpha_m+1$.

Proof. The function relations (2.11) may be obtained as follows. Choose the parameter a_1 from the set $\{a_i; i = 1, 2, \ldots, r\}$ of numerator parameters and consider

$$a_1\ {}_rH_s^p(a_1+) - a_i\ {}_rH_s^p(a_i+),$$

where $i \neq 1$. Then from the above definitions,

$$
\begin{aligned}
a_1\ {}_rH_s^p(a_1+) - a_i\ {}_rH_s^p(a_i+) &= a_1 \sum_{n\geq 0} \frac{a_1+n}{a_1}\ \varphi_n\ z^n - a_i \sum_{n\geq 0} \frac{a_i+n}{a_i}\ \varphi_n\ z^n\\
&= (a_1 - a_i)\ {}_rH_s^p.
\end{aligned} \tag{2.14}
$$

Taking $\theta = z\frac{\mathrm{d}}{\mathrm{dz}}$, we find that

$$
\begin{aligned}
(\theta + a_i)\ {}_rH_s^p &= a_i \sum_{n\geq 0} \frac{a_i+n}{a_i}\ \varphi_n\ z^n\\
&= a_i\ {}_rH_s^p(a_i+),
\end{aligned} \tag{2.15}
$$

and

$$(\theta + b_j - 1)\ {}_rH_s^p = (b_j - 1) \sum_{n\geq 0} \frac{b_j + n - 1}{b_j - 1}\ \varphi_n\ z^n$$
$$= (b_j - 1)\ {}_rH_s^p(b_j-). \tag{2.16}$$

From (2.14), (2.15), (2.16), we arrive at the contiguous functions relations (2.11).
If the parameter a_1 is replaced by a_m with $m \neq i$ in (2.14) then (2.11) gives rise to a set of contiguous functions relations:

$$(a_m - b_j + 1)\ {}_rH_s^p = a_m\ {}_rH_s^p(a_m+) - (b_j - 1)\ {}_rH_s^p(b_j-).$$

Now if $\ell \in \mathbf{N} \cup \{0\}$ then

$$\begin{aligned}
\theta\ {}_rH_s^p &= \sum_{n\geq 0} \varphi_n\ \theta z^n \\
&= \sum_{n\geq 0} \frac{\left\{\prod\limits_{1\leq i\leq r} (a_i)_{n+1}\right\}}{\left\{\prod\limits_{1\leq j\leq s} (b_j)_{n+1}\right\} \left\{\prod\limits_{1\leq k\leq p} (c_k)_{n+1}\right\}^{\ell n+\ell}} \frac{z^{n+1}}{n!} \\
&= z \sum_{n\geq 0} \frac{\left\{\prod\limits_{1\leq i\leq r} (a_i)_{n+1}\right\}}{\left\{\prod\limits_{1\leq j\leq s} (b_j)_{n+1}\right\} \left\{\prod\limits_{1\leq k\leq p} (c_k)_n\ (c_k + n)\right\}^{\ell n+\ell}} \frac{z^n}{n!} \\
&= z \sum_{n\geq 0} \frac{\left\{\prod\limits_{1\leq i\leq r} (a_i)_{n+1}\right\} \left\{\prod\limits_{1\leq k\leq p} \left(\frac{c_k}{c_k+n}\right)\right\}^{\ell n}}{\left\{\prod\limits_{1\leq j\leq s} (b_j)_{n+1}\right\} \left\{\prod\limits_{1\leq k\leq p} (c_k)_{n+1}^{\ell}\ (c_k)_n^{\ell n}\right\} \left\{\prod\limits_{1\leq k\leq p} (c_k)^{\ell}\right\}^n} \frac{z^n}{n!} \\
&= \frac{A}{B\ C}\ z\ {}_rH_{s+\ell}^p \left[\begin{matrix} (a) + 1; \\ (b) + 1, ((c) + 1)^{\ell}; ((c) + 1 : \ell); \end{matrix} \ \frac{z}{C}\right].
\end{aligned} \tag{2.17}$$

Whence (2.12) and (2.13) follow by eliminating θ from (2.14), (2.17) and (2.16), (2.17) respectively. □

2.6 Eigen function property

We next introduce an operator with respect to which the generalized ℓ-H function becomes an eigen function.

Definition 2.8 *Let* $f(z)=\sum\limits_{n\geq 0} a_n z^n,\ 0\neq |z|<R, R>0,\ l,m,n\in\mathbf{N},$ *and* $\alpha_i,\beta_j,\gamma_k\in\mathbf{C}$ *with* $\Re(\alpha_i)\geq 0,$ *and as before* $\theta = z\,\frac{\mathrm{d}}{\mathrm{dz}}$. *Define the operator*

$$
\begin{aligned}
&{}_{(\alpha,l)}\mathcal{H}^{(\gamma,n)}_{(\beta,m)} f(z)\\
&=\left[\left\{\prod_{1\leq i\leq l} I_{\alpha_i}\right\} z^{-1}\left\{\prod_{1\leq k\leq n}\ {}_{\ell}\Delta^{\theta}_{\gamma_k}\right\}\left\{\prod_{1\leq j\leq m}(\theta+\beta_j-1)\right\}\theta\right] f(z), \quad (2.18)
\end{aligned}
$$

where

$$
I_{\alpha}(f(z)) = z^{-\alpha}\int_0^z t^{\alpha-1} f(t)\ \mathrm{dt} \tag{2.19}
$$

and ${}_{\ell}\Delta^{\theta}_{\gamma_k}$ *is as defined in (2.4).*

This operator enables us to derive the eigen function property which is proved in

Theorem 2.9 *If* $z\neq 0,$ *and* $\Re(\alpha_i)\geq 0, \forall i=1,2,\ldots,r,$ *then the function*

$$
w = {}_rH^p_s\left[\begin{matrix} a_1,\,a_2,\,\ldots,\,a_r; & \lambda z\\ b_1,\,b_2,\,\ldots,\,b_s;\,(c_1,\,c_2,\,\ldots,\,c_p:\ell); & \end{matrix}\right]
$$

is an eigen function with respect to the operator ${}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)}$ *defined in (2.8).*

$$
\text{That is, }\ {}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)}\left({}_rH^p_s(\ell;\lambda z)\right) = \lambda\ {}_rH^p_s(\ell;\lambda z),\ \lambda\in\mathbf{C}. \tag{2.20}
$$

Note 2.10 Once again in view of the Lemma 2.6, the applicability of the operator ${}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)}$ to the generalized ℓ-H function follows.

Proof. We refer to (2.9) and put

$$
B_n = \frac{(a_1)_n\ (a_2)_n\cdots(a_r)_n\ \lambda^n}{(b_1)_{n-1}\ (b_2)_{n-1}\cdots(b_s)_{n-1}\ (c_1)^{\ell n-\ell}_{n-1}\ (c_2)^{\ell n-\ell}_{n-1}\cdots(c_p)^{\ell n-\ell}_{n-1}\ (n-1)!}
$$

to get

$$
\begin{aligned}
&{}_{(a,r)}\mathcal{H}_{(b,s)}^{(c,p)}\left({}_rH_s^p(\ell;\lambda z)\right)\\
&= \left[\left\{\prod_{1\le i\le r} I_{a_i}\right\} z^{-1}\left\{\prod_{1\le k\le p} {}_\ell\Delta_{c_k}^{\theta}\right\}\left\{\prod_{1\le j\le s}(\theta+b_j-1)\right\}\theta\right]\sum_{n\ge 0}\varphi_n\ \lambda^n\ z^n\\
&= \left\{\prod_{1\le i\le r} I_{a_i}\right\} z^{-1}\sum_{n\ge 1} B_n\ z^n\\
&= \left\{\prod_{1\le i\le r-1} I_{a_i}\right\} I_{a_r}\sum_{n\ge 1} B_n\ z^{n-1}\\
&= \left\{\prod_{1\le i\le r-1} I_{a_i}\right\}\ z^{-a_r}\int_0^z t^{a_r-1}\ \sum_{n\ge 1} B_n\ t^{n-1}\\
&= \left\{\prod_{1\le i\le r-1} I_{a_i}\right\}\ \sum_{n\ge 1}\frac{B_n}{a_r+n-1}\ z^{n-1}.
\end{aligned}
$$

Applying in this manner the operator I_{α_i} for $i = 1, 2, \ldots, r-1$, we finally obtain

$$
\begin{aligned}
{}_{(a,r)}\mathcal{H}_{(b,s)}^{(c,p)}\left({}_rH_s^p(\ell;\lambda z)\right) &= \sum_{n\ge 1}\left\{\prod_{1\le i\le r}(a_i+n-1)\right\}^{-1} B_n\ z^{n-1}\\
&= \sum_{n\ge 0}\varphi_n\ \lambda^{n+1}\ z^n\\
&= \lambda\ {}_rH_s^p(\ell;\lambda z).
\end{aligned}
$$

This proves (2.20). □

3 Special cases

3.1 ℓ-H exponential and ℓ-H Circular functions

In (1.5), if $r = s = 0$, $p = 1$ and $c_1 = 1$ then the particular ℓ-H exponential function is defined as follows (see [3] for more details).

Definition 3.1 *The ℓ-H exponential function is denoted and defined by*

$$
e_H^\ell(z) = H\left[\begin{matrix}-; \\ -; (1:\ell);\end{matrix}\ z\right] = \sum_{n\ge 0}\frac{z^n}{(n!)^{\ell n+1}}, \tag{3.1}
$$

for all $z \in \mathbf{C}$ and $\Re(\ell) \ge 0$.

Remark 3.1 Obviously, $e_H^0(z) = e^z$ *and* $e_H^\ell(0) = 1$.

3.2 ℓ-Analogues of Ramanujan's theorem and Kummer's first formula

It is noteworthy that the generalized ℓ-H-function provides ℓ-extensions to (i) the Ramanujan's theorem [9, Ex.5, p. 106]:

$$ {}_1F_1\begin{bmatrix} a; & x \\ b; & \end{bmatrix} {}_1F_1\begin{bmatrix} a; & -x \\ b; & \end{bmatrix} = {}_2F_3\begin{bmatrix} a, \; b-a; & & & \frac{x^2}{4} \\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}; & \end{bmatrix}, \qquad (3.2)$$

and (ii) the Kummer's first formula [9, p.125]:

$$e^{-z} \; {}_1F_1(a;b;z) = \; {}_1F_1(b-a;b;-z). \qquad (3.3)$$

This is shown below.

Theorem 3.2 *(ℓ-analogue of Ramanujan's Theorem)*
If $\ell \in \mathbf{N} \cup \{0\}$ then

$$\begin{aligned} & {}_1H_1^1\begin{bmatrix} a; & x \\ b; \; (c:\ell); & \end{bmatrix} {}_1H_1^1\begin{bmatrix} a; & -x \\ b; \; (c:\ell); & \end{bmatrix} \\ & = \sum_{n\geq 0} \frac{(a)_n \; x^n}{(b)_n \; (c)_n^{\ell n} \; n!} \\ & \quad \times {}_{3+\ell n}H_2^2\begin{bmatrix} -n, a, 1-b-n, & (1-c-n)^{\ell n}; & (-1)^{\ell(n-1)}(c)_n^\ell \\ b, 1-a-n; & (c, 1-c-n:\ell); & \end{bmatrix}. \end{aligned}$$

Proof. From the definition of the generalized ℓ-H function (1.5) and the formula:

$$(\alpha)_{n-k} = \frac{(-1)^k \; (\alpha)_n}{(1-\alpha-n)_k},$$

$$
\begin{aligned}
&{}_1H_1^1\left[\begin{matrix} a; & x \\ b;\ (c:\ell); & \end{matrix}\right]\ {}_1H_1^1\left[\begin{matrix} a; & -x \\ b;\ (c:\ell); & \end{matrix}\right] \\
&= \sum_{n\geq 0} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{x^n}{n!} \sum_{k\geq 0} \frac{(a)_k}{(b)_k\ (c)_k^{\ell k}} \frac{(-x)^k}{k!} \\
&= \sum_{n\geq 0} \sum_{0\leq k\leq n} \frac{(a)_{n-k}}{(b)_{n-k}\ (c)_{n-k}^{\ell n-\ell k}} \frac{x^n}{(n-k)!} \frac{(a)_k}{(b)_k\ (c)_k^{\ell k}} \frac{(-1)^k}{k!} \\
&= \sum_{n\geq 0} \sum_{0\leq k\leq n} \frac{(-1)^k (a)_n}{(1-a-n)_k} \frac{(1-b-n)_k}{(-1)^k (b)_n} \left[\frac{(1-c-n)_k}{(-1)^k (c)_n}\right]^{\ell n-\ell k} \\
&\quad \times \frac{(-n)_k}{(-1)^k n!} \frac{(a)_k}{(b)_k\ (c)_k^{\ell k}} \frac{(-1)^k}{k!}\ x^n \\
&= \sum_{n\geq 0} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{x^n}{n!} \\
&\quad \times \sum_{0\leq k\leq n} (-1)^{\ell(n-1)k}\ \frac{(-n)_k\ (a)_k\ (1-b-n)_k\ (1-c-n)_k^{\ell n}\ (c)_n^{\ell k}}{(b)_k\ (1-a-n)_k\ (c)_k^{\ell k}\ (1-c-n)_k^{\ell k}\ k!} \\
&= \sum_{n\geq 0} \frac{(a)_n\ x^n}{(b)_n\ (c)_n^{\ell n}\ n!} \\
&\quad \times\ {}_{3+\ell n}H_2^2\left[\begin{matrix} -n, a, 1-b-n, & (1-c-n)^{\ell n}; & (-1)^{\ell(n-1)}(c)_n^{\ell} \\ b, 1-a-n; & (c, 1-c-n:\ell); & \end{matrix}\right].
\end{aligned}
$$

□

It may be seen that for $\ell = 0$, Theorem 3.2 gets reduced to (3.2) by replacing first n by $2n$ in the last series of the proof and then using the summation formula [9, Ex.5, p. 106]:

$$
{}_3F_2\left[\begin{matrix} -2n, & a, 1-b-2n; 1 \\ 1-a-2n, & b; \end{matrix}\right] = \frac{(2n)!\ (a)_n\ (b-a)_n}{n!\ (a)_{2n}\ (b)_n}.
$$

Theorem 3.3 *(ℓ-Analogue of Kummer's first formula)*
If $\ell \in \mathbf{N} \cup \{0\}$ and $z \in \mathbf{C}$ then

$$
\begin{aligned}
&e_H^{\ell}(-z)\ {}_1H_1^1\left[\begin{matrix} a; & z \\ b;\ (c:\ell); & \end{matrix}\right] \\
&= \sum_{n\geq 0} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}} \\
&\quad \times\ {}_{2+\ell n}H_1^2\left[\begin{matrix} -n, -n, \ldots, -n, & a; & (1)_n^{\ell}(-1)^{\ell(n-1)} \\ b; & (c, -n:\ell); & \end{matrix}\right]. \qquad (3.4)
\end{aligned}
$$

Proof. We begin with

$$
\begin{aligned}
& e_H^\ell(-z)\ {}_1H_1^1 \left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] \\
&= \sum_{n\geq 0} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}} \sum_{k\geq 0} \frac{(a)_k}{(b)_k\ (c)_k^{\ell k}} \frac{z^k}{k!} \\
&= \sum_{n\geq 0} \sum_{k\geq 0} (-1)^{n-k} \frac{(a)_k}{[(n-k)!]^{\ell n-\ell k+1}\ (b)_k\ (c)_k^{\ell k}} \frac{z^n}{k!} \\
&= \sum_{n\geq 0} \sum_{0\leq k\leq n} (-1)^{n-k} \left[\frac{(-1)^k\ (-n)_k}{n!}\right]^{\ell n-\ell k+1} \frac{(a)_k\ \ z^n}{(b)_k\ (c)_k^{\ell k}\ k!} \\
&= \sum_{n\geq 0} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}} \sum_{0\leq k\leq n} \frac{(-n)_k^{\ell n+1}\ n!^{\ell k}\ (a)_k}{(-n)_k^{\ell k}\ (b)_k\ (c)_k^{\ell k}\ k!}\ (-1)^{\ell nk-\ell k} \\
&= \sum_{n\geq 0} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}} \sum_{0\leq k\leq n} \frac{(-n)_k^{\ell n+1}\ (a)_k\ \left[(n!)^\ell(-1)^{\ell(n-1)}\right]^k}{(-n)_k^{\ell k}\ (b)_k\ (c)_k^{\ell k}\ k!} \\
&= \sum_{n\geq 0} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}}\ {}_{2+\ell n}H_1^2 \left[\begin{matrix} -n,-n,\ldots,-n,a; & & (n!)^\ell(-1)^{\ell(n-1)} \\ b; & (c,-n:\ell); & \end{matrix}\right].
\end{aligned}
$$

□

Remark 3.2 When $\ell = 0$, (3.4) reduces to (3.3).

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