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Generalized Jordan higher derivable mappings on rings

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Abstract Let $\mathcal{R}$ be a ring and $\mathbb{N}$ be the set of all non-negative integers. Let $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of mappings $d_n : \mathcal{R} \to \mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$, the identity map of $\mathcal{R}$ satisfying $d_n(ab + ba) = \sum\limits_{i+j=n} d_i(a)d_j(b) + d_i(b)d_j(a)$ for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$. In the present paper it is shown that if a family $\Delta = \{G_n\}_{n\in\mathbb{N}}$ of mappings $G_n : \mathcal{R} \to \mathcal{R}$ satisfies $G_n(ab+ba) = \sum\limits_{i+j=n} G_i(a)d_j(b)+G_i(b)d_j(a)$ for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$, then under certain assumptions $G_n(a + b) = G_n(a) + G_n(b)$ for all $a, b \in \mathcal{R}$, and hence $\Delta = \{G_n\}_{n\in\mathbb{N}}$ is a generalized Jordan higher derivation.

Keywords derivation · Jordan derivation · higher derivation · generalized Jordan derivable map · generalized Jordan higher derivable map

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1 Introduction

Let $\mathcal{R}$ be a ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation (resp. Jordan derivation) if $d(ab) = d(a)b + ad(b)$ (resp. $d(a^2) = d(a)a + ad(a)$) holds for all $a, b \in \mathcal{R}$. Every derivation is obviously a Jordan derivation, but the converse need not be true in general. An influential theorem due to Herstein [5] shows that any Jordan derivation on a prime ring of characteristic different from two is a derivation. An additive mapping $G : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists a derivation (resp. Jordan derivation) d such that $G(ab) = G(a)b + ad(b)$ (resp. $G(a^2) = G(a)a + ad(a)$) holds for all $a, b \in \mathcal{R}$. Clearly, every generalized derivation is a generalized Jordan derivation but the converse need not be true in general. Zhu and Xiong [16] proved that every generalized Jordan derivation on a 2-torsion free semiprime ring with identity into itself is a generalized derivation. A map $d : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) is called a derivable (resp. Jordan derivable) if $d(ab) = d(a)b + ad(b)$ (resp.

$d(a^2) = d(a)a + ad(a)$) holds for all $a, b \in \mathcal{R}$. Also a map $G : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) is called a generalized derivable (resp. generalized Jordan derivable) if there exists a derivable (resp. Jordan derivable) map d such that $G(ab) = G(a)b + ad(b)$ (resp. $G(a^2) = G(a)a + ad(a)$) holds for all $a, b \in \mathcal{R}$. Note that not every derivable map is additive. For example, let $\mathcal{R}$ be the ring of all 2×2 strictly upper triangular matrices over $\mathbb{Z}$, the ring of integers. Define $d : \mathcal{R} \to \mathcal{R}$ such that $d\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix}$. Then $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathcal{R}$ but $d(x + y) \neq d(x) + d(y)$.

Let $\mathbb{N}$ be the set of all non-negative integers. A family $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $d_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$, is said to be

(i) a higher derivable map on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $d_n(ab) = \sum\limits_{i+j=n} d_i(a)d_j(b)$, for all $a, b \in \mathcal{R}$.

(ii) a Jordan higher derivable map on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $d_n(a^2) = \sum\limits_{i+j=n} d_i(a)d_j(a)$, for all $a \in \mathcal{R}$.

Note that if $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ is the family of additive mappings $d_n : \mathcal{R} \to \mathcal{R}$ in the above definition then $\mathfrak{D}$ is said to be a higher derivation and a Jordan higher derivation respectively (for reference see Hasse and Schmidt [14]).

Furthermore, motivated by the concept of generalized derivation Cortes and Haetinger [4] introduced the notion of generalized higher derivation. Motivated by the concept of generalized higher higher derivation, we define generalized higher derivable map on $\mathcal{R}$. A family $\Delta = \{G_n\}_{n\in\mathbb{N}}$ of mappings $G_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $G_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$, is said to be

(i) a generalized higher derivable map on $\mathcal{R}$ if there exists a higher derivable mapping $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}$ such that $G_n(ab) = \sum\limits_{i+j=n} G_i(a)d_j(b)$, for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$.

(ii) a generalized Jordan higher derivable map on $\mathcal{R}$ if there exists a Jordan higher derivable mapping $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}$ such that $G_n(a^2) = \sum\limits_{i+j=n} G_i(a)d_j(a)$, for all $a \in \mathcal{R}$ and for each $n \in \mathbb{N}$.

If $\Delta = \{G_n\}_{n\in\mathbb{N}}$ and $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ are families of the additive mappings $G_n : \mathcal{R} \to \mathcal{R}$ and $d_n : \mathcal{R} \to \mathcal{R}$ respectively, in the above definition, then Δ is said to be a generalized higher derivation and generalized Jordan higher derivation respectively.

Now suppose that $\mathcal{R}$ is a ring with a nontrivial idempotent p_1. We write $p_2 = 1 - p_1$. Note that $\mathcal{R}$ need not have the identity element. Put $p_i\mathcal{R}p_j = \mathcal{R}_{ij}$ for any $i, j = 1, 2$. Then, by Peirce decomposition of $\mathcal{R}$, we have $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$. Throughout this paper, the notation a_{ij} will denote an arbitrary element of $\mathcal{R}_{ij}$ and any element $a \in \mathcal{R}$ can be expressed as

$a = a_{11} + a_{12} + a_{21} + a_{22}$. Recall that a ring $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$ and is semiprime if $a\mathcal{R}a = \{0\}$ implies $a = 0$.

In the recent years, characterizing the interrelation between the multiplicative and the additive structures of a ring is an interesting topic and has received attention of many mathematicians. If a ring $\mathcal{R}$ contains a nontrivial idempotent, it is a kind of surprise that every multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring is automatically additive. This result was given by Martindale III in his excellent paper [13]. Inspired by this result, Lu [10] proved that every derivable map on a 2-torsion free unital prime ring containing nontrivial idempotent is a derivation. Motivated by this result, in the year 2011, Jing and Lu [6] characterized Jordan derivable map to a larger class of rings. Further, in the year 2014, the first author together with Parveen [1] extend this result to Jordan higher derivable maps. More precisely, they proved the following:

Theorem 1.1 *[[1], Theorem 2] Let $\mathcal{R}$ be a ring containing a nontrivial idempotent satisfying the following conditions for $i, j, k \in \{1, 2\}$:*

$(S1)$ *If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in \mathcal{R}_{jk}$, then $a_{ij} = 0$;*
$(S2)$ *If $x_{ij}a_{jk} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $a_{jk} = 0$;*
$(S3)$ *If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ for all $x_{ii} \in \mathcal{R}_{ii}$, then $a_{ii} = 0$.*

If the family $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ satisfies

$$d_n(ab + ba) = \sum_{i+j=n} (d_i(a)d_j(b) + d_i(b)d_j(a))$$

for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$, then $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ is additive i.e. $d_n(a + b) = d_n(a) + d_n(b)$ for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$. In addition, if $\mathcal{R}$ is 2-torsion free, then $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ is a Jordan higher derivation.

Over the last decade, a lot of work has been done on the additivity of mappings on rings. We refer the readers to some recent papers (see [2,3, 6,8–13]) where further references can be found. In the present paper we provide a sufficient condition on a ring $\mathcal{R}$ under which a generalized Jordan higher derivable map becomes a generalized Jordan higher derivation on $\mathcal{R}$.

2 Generalized Jordan derivable mapping on rings

The main result of this section states as follows.

Theorem 2.1 *Let $\mathcal{R}$ be a ring containing a nontrivial idempotent satisfying the following conditions for $i, j, k \in \{1, 2\}$:*

$(S1)$ *If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in \mathcal{R}_{jk}$, then $a_{ij} = 0$;*
$(S2)$ *If $x_{ij}a_{jk} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $a_{jk} = 0$;*
$(S3)$ *If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ for all $x_{ii} \in \mathcal{R}_{ii}$, then $a_{ii} = 0$.*

If a mapping $G : \mathcal{R} \to \mathcal{R}$ satisfies

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in \mathcal{R}$, then G is additive. In addition, if $\mathcal{R}$ is 2-torsion free, then G is a generalized Jordan derivation.

Throughout the remaining part of this section, we always assume that $\mathcal{R}$ is a ring with a nontrivial idempotent p_1 and satisfies conditions $(S1), (S2)$ and $(S3)$. We facilitate our discussion with the following lemmas.

Lemma 2.2

(i) $G(a_{11} + b_{12}) = G(a_{11}) + G(b_{12})$,
(ii) $G(a_{11} + b_{21}) = G(a_{11}) + G(b_{21})$,
(iii) $G(a_{22} + b_{12}) = G(a_{22}) + G(b_{12})$,
(iv) $G(a_{22} + b_{21}) = G(a_{22}) + G(b_{21})$.

Proof. We only prove (i). The rest of the proof follow similarly.
For any $x_{22} \in \mathcal{R}_{22}$, we compute

$$\begin{aligned}&G[(a_{11} + b_{12})x_{22} + x_{22}(a_{11} + b_{12})]\\ &= G(a_{11}+b_{12})x_{22}+(a_{11}+b_{12})d(x_{22})+G(x_{22})(a_{11}+b_{12})+x_{22}d(a_{11}+b_{12}).\end{aligned}$$

On the other way

$$\begin{aligned}G[(a_{11}+b_{12})x_{22}+x_{22}(a_{11}+b_{12})] &= G(b_{12}x_{22})\\ &= G(a_{11}x_{22}+x_{22}a_{11})+G(b_{12}x_{22}+x_{22}b_{12})\\ &= G(a_{11})x_{22}+a_{11}d(x_{22})+G(x_{22})a_{11}+x_{22}d(a_{11})\\ &\quad +G(b_{12})x_{22}+b_{12}d(x_{22})+G(x_{22})b_{12}+x_{22}d(b_{12}).\end{aligned}$$

Comparing the above two equalities, we obtain

$$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]x_{22}+x_{22}[d(a_{11}+b_{12})-d(a_{11})-d(b_{12})]=0.$$

In view of Theorem 1.1, d is additive and hence

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]x_{22} = 0.$$

By condition $(S1)$, we have

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{12} = 0,$$

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{22} = 0.$$

We now show that $[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{11} = 0 = [G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{21}$.
For any $x_{12} \in \mathcal{R}_{12}$, we have

$$\begin{aligned}G[(a_{11}+b_{12})x_{12}+x_{12}(a_{11}+b_{12})] &= G(a_{11}x_{12})\\ &= G(a_{11}x_{12}+x_{12}a_{11})+G(b_{12}x_{12}+x_{12}b_{12}).\end{aligned}$$

Applying the definition to both sides of this identity, one can find that

$$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]x_{12}+x_{12}[d(a_{11}+b_{12})-d(a_{11})-d(b_{12})]=0.$$

Again by applying Theorem 1.1, we arrive at

$$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]x_{12}=0.$$

Consequently, by condition $(S1)$, we have

$$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]_{11}=0,$$

$$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]_{21}=0.$$

This completes the proof. □

Lemma 2.3

(i) $G(a_{12}+b_{12}c_{22})=G(a_{12})+G(b_{12}c_{22})$,
(ii) $G(a_{21}+b_{22}c_{21})=G(a_{21})+G(b_{22}c_{21})$.

Proof. (i) Using Lemma 2.2, we obtain

$$\begin{aligned} G(a_{12}+b_{12}c_{22}) &= G[(p_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(p_1+b_{12})] \\ &= G(p_1+b_{12})(a_{12}+c_{22})+(p_1+b_{12})d(a_{12}+c_{22}) \\ &\quad +G(a_{12}+c_{22})(p_1+b_{12})+(a_{12}+c_{22})d(p_1+b_{12}) \\ &= [G(p_1)+G(b_{12})](a_{12}+c_{22})+(p_1+b_{12})[d(a_{12})+d(c_{22})] \\ &\quad +[G(a_{12})+G(c_{22})](p_1+b_{12})+(a_{12}+c_{22})[d(p_1)+d(b_{12})] \\ &= G(a_{12})+G(b_{12}c_{22}). \end{aligned}$$

(ii) Note that

$$a_{21}+b_{22}c_{21}=(p_1+c_{21})(a_{21}+b_{22})+(a_{21}+b_{22})(p_1+c_{21}).$$

Now the proof is similarly as that of (i). □

Lemma 2.4

(i) $G(a_{12}+b_{12})=G(a_{12})+G(b_{12})$,
(ii) $G(a_{21}+b_{21})=G(a_{21})+G(b_{21})$.

Proof. We only prove (i). The proof of (ii) is similar.

For any $x_{22}\in\mathcal{R}_{22}$, we calculate $G[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})]$ in two different ways.

On one hand,

$$\begin{aligned} &G[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})] \\ &=G(a_{12}+b_{12})x_{22}+(a_{12}+b_{12})d(x_{22})+G(x_{22})(a_{12}+b_{12})+x_{22}d(a_{12}+b_{12}). \end{aligned}$$

On the other hand, by Lemma 2.3

$$\begin{aligned} G[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})] &= G(a_{12}x_{22}+b_{12}x_{22}) \\ &= G(a_{12}x_{22})+G(b_{12}x_{22}) \\ &= G(a_{12}x_{22}+x_{22}a_{12})+G(b_{12}x_{22}+x_{22}b_{12}) \\ &= G(a_{12})x_{22}+a_{12}d(x_{22})+G(x_{22})a_{12}+x_{22}d(a_{12}) \\ &\quad +G(b_{12})x_{22}+b_{12}d(x_{22})+G(x_{22})b_{12}+x_{22}d(b_{12}). \end{aligned}$$

Comparing the above two equalities, we find that

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{22}+x_{22}[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]=0$$

for all $x_{22}\in\mathcal{R}_{22}$.
This implies that

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{22}=0.$$

By condition $(S1)$, we can find that

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{12}=0,$$

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{22}=0.$$

We now show that

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{11}=0=[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{21}.$$

Now for any $x_{12}\in\mathcal{R}_{12}$, we have

$$\begin{aligned} &G[(a_{12}+b_{12})x_{12}+x_{12}(a_{12}+b_{12})] \\ &=G((a_{12}+b_{12}))x_{12}+(a_{12}+b_{12})d(x_{12})+G(x_{12})(a_{12}+b_{12})+x_{12}d(a_{12}+b_{12}). \end{aligned}$$

On the other hand

$$\begin{aligned} 0 &= G(a_{12}x_{12}+x_{12}a_{12})+G(b_{12}x_{12}+x_{12}b_{12}) \\ &= G(a_{12})x_{12}+a_{12}d(x_{12})+G(x_{12})a_{12}+x_{12}d(a_{12}) \\ &\quad +G(b_{12})x_{12}+b_{12}d(x_{12})+G(x_{12})b_{12}+x_{12}d(b_{12}). \end{aligned}$$

From the last two expressions, we have

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{12}+x_{12}[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]=0.$$

In view of Theorem 1.1, d is additive and hence

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{12}=0.$$

By condition $(S1)$ we can infer that

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{11}=0,$$

$$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{21}=0.$$

Hence, $[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]=0.$ □

Lemma 2.5

(*i*) $G(a_{11}+b_{11})=G(a_{11})+G(b_{11})$,
(*ii*) $G(a_{22}+b_{22})=G(a_{22})+G(b_{22})$.

Proof. We only prove (*i*). For any $x_{22}\in\mathcal{R}_{22}$, we have

$$\begin{aligned}G(a_{11}+b_{11})x_{22}&+(a_{11}+b_{11})d(x_{22})+G(x_{22})(a_{11}+b_{11})+x_{22}d(a_{11}+b_{11})\\&=G[(a_{11}+b_{11})x_{22}+x_{22}(a_{11}+b_{11})]\\&=0\\&=G(a_{11}x_{22}+x_{22}a_{11})+G(b_{11}x_{22}+x_{22}b_{11})\\&=G(a_{11})x_{22}+a_{11}d(x_{22})+G(x_{22})a_{11}+x_{22}d(a_{11})\\&\quad+G(b_{11})x_{22}+b_{11}d(x_{22})+G(x_{22})b_{11}+x_{22}d(b_{11}).\end{aligned}$$

This gives us

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]x_{22}+x_{22}[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]=0.$$

Again in view of Theorem 1.1, we get

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]x_{22}=0.$$

By condition $(S1)$, we have

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{12}=0,$$

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{22}=0.$$

Similarly, by considering $(a_{11}+b_{11})x_{12}+x_{12}(a_{11}+b_{11})$ and using Lemma 2.4, one can find that

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{11}=0,$$

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{21}=0.$$

Hence, $[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]=0$. □

Lemma 2.6 $G(a_{12}+b_{21})=G(a_{12})+G(b_{21})$.

Proof. From $a_{12}+b_{21}=(a_{12}+b_{21})p_1+p_1(a_{12}+b_{21})$, we have

$$G(a_{12}+b_{21})=G(a_{12}+b_{21})p_1+(a_{12}+b_{21})d(p_1)+G(p_1)(a_{12}+b_{21})+p_1d(a_{12}+b_{21}).$$

Multiplying this equality from the left by p_2, we get

$$p_2G(a_{12}+b_{21})=p_2G(a_{12}+b_{21})p_1+b_{21}d(p_1)+p_2G(p_1)(a_{12}+b_{21}).$$

Similarly, we have

$$p_2G(b_{21})=p_2G(b_{21})p_1+b_{21}d(p_1)+p_2G(p_1)b_{21},$$

$$p_2G(a_{12})=p_2G(a_{12})p_1+p_2G(p_1)a_{12}.$$

From above three expressions, we have

$p_2[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})] = p_2[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]p_1.$

This implies that

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{22} = 0.$$

Now for any $x_{12} \in \mathcal{R}_{12}$, we have

$$\begin{aligned} G(a_{12}+b_{21})x_{12} + (a_{12}+b_{21})d(x_{12}) + G(x_{12})(a_{12}+b_{21}) + x_{12}d(a_{12}+b_{21}) \\ = G[(a_{12}+b_{21})x_{12} + x_{12}(a_{12}+b_{21})] \\ = G(b_{21}x_{12} + x_{12}b_{21}) \\ = G(b_{21}x_{12} + x_{12}b_{21}) + G(a_{12}x_{12} + x_{12}a_{12}) \\ = G(b_{21})x_{12} + b_{21}d(x_{12}) + G(x_{12})b_{21} + x_{12}d(b_{21}) \\ +G(a_{12})x_{12} + a_{12}d(x_{12}) + G(x_{12})a_{12} + x_{12}d(a_{12}), \end{aligned}$$

which leads to

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]x_{12} = 0.$$

It follows from condition $(S1)$ that

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{11} = 0,$$

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{21} = 0.$$

Similarly, by considering $G[(a_{12}+b_{21})x_{21}+x_{21}(a_{12}+b_{21})]$ for all $x_{21} \in \mathcal{R}_{21}$, we can find

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]x_{21} = 0.$$

Consequently, $[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{12} = 0$.
This completes the proof. □

Lemma 2.7

(*i*) $G(a_{11}+b_{12}+c_{21}) = G(a_{11}) + G(b_{12}) + G(c_{21})$,
(*ii*) $G(a_{12}+b_{21}+c_{22}) = G(a_{12}) + G(b_{21}) + G(c_{22})$.

Proof. We only prove (i). For any $x_{22} \in \mathcal{R}_{22}$, we have

$$\begin{aligned} &G[(a_{11}+b_{12}+c_{21})x_{22} + x_{22}(a_{11}+b_{12}+c_{21})] \\ &= G(a_{11}+b_{12}+c_{21})x_{22} + (a_{11}+b_{12}+c_{21})d(x_{22}) \\ &\quad +G(x_{22})(a_{11}+b_{12}+c_{21}) + x_{22}d(a_{11}+b_{12}+c_{21}). \end{aligned}$$

On the other hand, by Lemma 2.6, we also have

$$\begin{aligned} &G[(a_{11}+b_{12}+c_{21})x_{22} + x_{22}(a_{11}+b_{12}+c_{21})] \\ &= G(b_{12}x_{22} + x_{22}c_{21}) \\ &= G(b_{12}x_{22}) + G(x_{22}c_{21}) \\ &= G(a_{11}x_{22} + x_{22}a_{11}) + G(b_{12}x_{22} + x_{22}b_{12}) + G(c_{21}x_{22} + x_{22}c_{21}) \\ &= G(a_{11})x_{22} + a_{11}d(x_{22}) + G(x_{22})a_{11} + x_{22}d(a_{11}) + G(b_{12})x_{22} + b_{12}d(x_{22}) \\ &\quad +G(x_{22})b_{12} + x_{22}d(b_{12}) + G(c_{21})x_{22} + c_{21}d(x_{22}) + G(x_{22})c_{21} + x_{22}d(c_{21}). \end{aligned}$$

It follows that

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]x_{22}=0.$$

Then we can obtain that

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{12}=0,$$

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{22}=0.$$

Now for any $x_{12}\in\mathcal{R}_{12}$, from

$$\begin{aligned}&G[(a_{11}+b_{12}+c_{21})x_{12}+x_{12}(a_{11}+b_{12}+c_{21})]\\&=G[(a_{11}+c_{21})x_{12}+x_{12}(a_{11}+c_{21})]+G[b_{12}x_{12}+x_{12}b_{12}]\end{aligned}$$

and using Lemma 2.2, one can easily get

$$\begin{aligned}&[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]x_{12}\\&+x_{12}[d(a_{11}+b_{12}+c_{21})-d(a_{11})-d(b_{12})-d(c_{21})]=0.\end{aligned}$$

It follows that

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]x_{12}=0,$$

and hence we get

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{11}=0,$$

$$[G(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{21}=0,$$

which completes the proof. □

Lemma 2.8 $G(a_{11}+b_{12}+c_{21}+w_{22})=G(a_{11})+G(b_{12})+G(c_{21})+G(w_{22})$.

Proof. For any $x_{11}\in\mathcal{R}_{11}$, by Lemma 2.7, we have

$$\begin{aligned}&G(a_{11}+b_{12}+c_{21}+w_{22})x_{11}+(a_{11}+b_{12}+c_{21}+w_{22})d(x_{11})\\&\quad+G(x_{11})(a_{11}+b_{12}+c_{21}+w_{22})+x_{11}d(a_{11}+b_{12}+c_{21}+w_{22})\\&=G[(a_{11}+b_{12}+c_{21}+w_{22})x_{11}+x_{11}(a_{11}+b_{12}+c_{21}+w_{22})]\\&=G(a_{11}x_{11}+c_{21}x_{11}+x_{11}a_{11}+x_{11}b_{12})\\&=G(a_{11}x_{11}+x_{11}a_{11})+G(x_{11}b_{12})+G(c_{21}x_{11})\\&=G(a_{11}x_{11}+x_{11}a_{11})+G(b_{12}x_{11}+x_{11}b_{12})+G(c_{21}x_{11}+x_{11}c_{21})+G(w_{22}x_{11}+x_{11}w_{22}).\end{aligned}$$

This gives us

$$\begin{aligned}&[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]x_{11}\\&+x_{11}[d(a_{11}+b_{12}+c_{21}+w_{22})-d(a_{11})-d(b_{12})-d(c_{21})-d(w_{22})]=0.\end{aligned}$$

Again using additivity of d we find that

$$[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]x_{11}=0.$$

We can infer that

$$[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]_{11}=0,$$

$$[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]_{21}=0.$$

Similarly, one can get

$$[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]_{12}=0,$$

$$[G(a_{11}+b_{12}+c_{21}+w_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(w_{22})]_{22}=0.$$

This completes the proof. □

Now we are ready to prove our main result.

Proof of Theorem 2.1. For any $a, b \in \mathcal{R}$, we write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. Applying Lemmas 2.3-2.7, we have

$$\begin{aligned}
G(a+b) &= G(a_{11}+a_{12}+a_{21}+a_{22}+b_{11}+b_{12}+b_{21}+b_{22})\\
&= G((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{21}+b_{21})+(a_{22}+b_{22}))\\
&= G(a_{11}+b_{11})+G(a_{12}+b_{12})+G(a_{21}+b_{21})+G(a_{22}+b_{22})\\
&= G(a_{11})+G(b_{11})+G(a_{12})+G(b_{12})+G(a_{21})+G(b_{21})+G(a_{22})+G(b_{22})\\
&= G(a_{11}+a_{12}+a_{21}+a_{22})+G(b_{11}+b_{12}+b_{21}+b_{22})\\
&= G(a)+G(b).
\end{aligned}$$

Hence G is additive.
In addition, if $\mathcal{R}$ is 2-torsion free, then for any $a \in \mathcal{R}$, we have

$$2G(a^2) = G(2a^2) = G(aa + aa) = 2(G(a)a + ad(a)).$$

Therefore, G is a generalized Jordan derivation. □

If the underlying ring is semiprime, applying Lemma 1.5 of [6] and the fact that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation (see [15], Theorem 2.7), we have the following corollary.

Corollary 2.9 *Let $\mathcal{R}$ be a 2-torsion free semiprime ring containing a non-trivial idempotent satisfying the following conditions for $i, j, k \in \{1, 2\}$:*

$(S1)$ *If $a_{ii}x_{ij} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $a_{ii} = 0$;*
$(S2)$ *If $x_{ji}a_{ii} = 0$ for all $x_{ji} \in \mathcal{R}_{ji}$, then $a_{ii} = 0$.*

If a mapping $G : \mathcal{R} \to \mathcal{R}$ satisfies

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in \mathcal{R}$; then G is additive. Moreover, G is a generalized derivation.

If the underlying ring is prime, applying Lemma 1.6 of [6] and the fact that every generalized Jordan derivation on a prime ring of characteristic not two is a generalized derivation (see [7]), we can easily find the following result.

Corollary 2.10 *Let $\mathcal{R}$ be a prime ring with characteristic different from two containing a nontrivial idempotent, and let $G : \mathcal{R} \to \mathcal{R}$ be a mapping satisfying*

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in \mathcal{R}$, then G is additive. Moreover, G is a generalized derivation.

Since every standard operator algebra is prime, we can easily find the following corollary.

Corollary 2.11 *Let $\mathcal{A}$ be a standard operator algebra in a Banach space X whose dimension is greater than 1. Suppose that $G : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying*

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in \mathcal{A}$, then G is additive. Moreover, G is a generalized derivation.

3 Generalized Jordan higher derivable mappings on rings

The main result of this section states as follows.

Theorem 3.1 *Let $\mathcal{R}$ be a ring containing a nontrivial idempotent satisfying the following conditions for $i, j, k \in \{1, 2\}$:*

$(S1)$ *If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in \mathcal{R}_{jk}$, then $a_{ij} = 0$;*
$(S2)$ *If $x_{ij}a_{jk} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $a_{jk} = 0$;*
$(S3)$ *If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ for all $x_{ii} \in \mathcal{R}_{ii}$, then $a_{ii} = 0$.*

If the family $\Delta = \{G_n\}_{n\in\mathbb{N}}$ of mappings $G_n : \mathcal{R} \to \mathcal{R}$ such that $G_0 = I_{\mathcal{R}}$, the identity map of $\mathcal{R}$, satisfies

$$G_n(ab + ba) = \sum_{i+j=n} G_i(a)d_j(b) + G_i(b)d_j(a)$$

for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$, then $\Delta = \{G_n\}_{n\in\mathbb{N}}$ is additive. In addition, if $\mathcal{R}$ is 2-torsion free, then $\Delta = \{G_n\}_{n\in\mathbb{N}}$ is a generalized Jordan higher derivation.

We shall prove our result by using induction on n. We may see that, for $n = 1$, $G_n(ab + ba) = \sum_{i+j=n} G_i(a)d_j(b) + G_i(b)d_j(a)$ reduces to $G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$ for all $a, b \in \mathcal{R}$. In view of Theorem 2.1, it is clear that G is additive. We will use this result in the following lemmas whenever needed without specific mention. Throughout this section we assume that $\mathcal{R}$ satisfies the hypothesis of our theorem. In order to prove our main result of this section, we begin with the following sequence of lemmas:

Lemma 3.2

(*i*) $G_n(a_{11}+b_{12})=G_n(a_{11})+G_n(b_{12})$,
(*ii*) $G_n(a_{11}+b_{21})=G_n(a_{11})+G_n(b_{21})$,
(*iii*) $G_n(a_{22}+b_{12})=G_n(a_{22})+G_n(b_{12})$,
(*iv*) $G_n(a_{22}+b_{21})=G_n(a_{22})+G_n(b_{21})$.

Proof. We only prove (i). The rest of the proof follow similarly. Since G is additive, we have $G(a_{11}+b_{12})=G(a_{11})+G(b_{12})$. Now by induction hypothesis let the result hold for all positive integer $m<n$. For any $x_{22}\in\mathcal{R}_{22}$, we compute

$$\begin{aligned}&G_n[(a_{11}+b_{12})x_{22}+x_{22}(a_{11}+b_{12})]\\&=\sum_{i+j=n}G_i(a_{11}+b_{12})d_j(x_{22})+\sum_{i+j=n}G_i(x_{22})d_j(a_{11}+b_{12})\\&=G_n(a_{11}+b_{12})x_{22}+(a_{11}+b_{12})d_n(x_{22})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(a_{11}+b_{12})d_j(x_{22})\\&\quad+G_n(x_{22})(a_{11}+b_{12})+x_{22}d_n(a_{11}+b_{12})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(x_{22})d_j(a_{11}+b_{12}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&G_n[(a_{11}+b_{12})x_{22}+x_{22}(a_{11}+b_{12})]\\&=G_n(b_{12}x_{22})\\&=G_n(a_{11}x_{22}+x_{22}a_{11})+G_n(b_{12}x_{22}+x_{22}b_{12})\end{aligned}$$

$$\begin{aligned}&=\sum_{i+j=n}G_i(a_{11})d_j(x_{22})+\sum_{i+j=n}G_i(x_{22})d_j(a_{11})\\&\quad+\sum_{i+j=n}G_i(b_{12})d_j(x_{22})+\sum_{i+j=n}G_i(x_{22})d_j(b_{12})\\&=G_n(a_{11})x_{22}+a_{11}d_n(x_{22})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(a_{11})d_j(x_{22})+G_n(x_{22})a_{11}+x_{22}d_n(a_{11})\\&\quad+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(x_{22})d_j(a_{11})+G_n(b_{12})x_{22}+b_{12}d_n(x_{22})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(b_{12})d_j(x_{22})\\&\quad+G_n(x_{22})b_{12}+x_{22}d_n(b_{12})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(x_{22})d_j(b_{12}).\end{aligned}$$

Since the result holds for all $m<n$, on comparing the above two equalities we obtain

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]x_{22}+x_{22}[d_n(a_{11}+b_{12})-d_n(a_{11})-d_n(b_{12})]=0.$$

By Theorem 1.1, we get

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]x_{22}=0.$$

On using condition $(S1)$, we get

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{12}=0,$$

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{22}=0.$$

We now show that

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{11}=0=[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{21}.$$

For any $x_{12}\in\mathcal{R}_{12}$, we have

$$\begin{aligned}
&G_n[(a_{11}+b_{12})x_{12}+x_{12}(a_{11}+b_{12})]\\
&=G_n(a_{11}x_{12})\\
&=G_n(a_{11}x_{12}+x_{12}a_{11})+G_n(b_{12}x_{12}+x_{12}b_{12})\\
&=G_n(a_{11})x_{12}+a_{11}d_n(x_{12})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(a_{11})d_j(x_{12})+G_n(x_{12})a_{11}+x_{12}d_n(a_{11})\\
&\quad+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(x_{12})d_j(a_{11})+G_n(b_{12})x_{12}+b_{12}d_n(x_{12})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(b_{12})d_j(x_{12})\\
&\quad+G_n(x_{12})b_{12}+x_{12}d_n(b_{12})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(x_{12})d_j(b_{12}).
\end{aligned}$$

The above yields that

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]x_{12}+x_{12}[d_n(a_{11}+b_{12})-d_n(a_{11})-d_n(b_{12})]=0.$$

By Theorem 1.1, we find

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]x_{12}=0.$$

Consequently, by condition $(S1)$, we have

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{11}=0,$$

$$[G_n(a_{11}+b_{12})-G_n(a_{11})-G_n(b_{12})]_{21}=0,$$

which completes the proof. □

Lemma 3.3

(*i*) $G_n(a_{12}+b_{12}c_{22})=G_n(a_{12})+G_n(b_{12}c_{22})$,
(*ii*) $G_n(a_{21}+b_{22}c_{21})=G_n(a_{21})+G_n(b_{22}c_{21})$.

Proof. (i) Using Lemma 3.2, we find

$$
\begin{aligned}
G_n(a_{12}+b_{12}c_{22}) &= G_n[(p_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(p_1+b_{12})] \\
&= \sum_{i+j=n} G_i(p_1+b_{12})d_j(a_{12}+c_{22}) + \sum_{i+j=n} G_i(a_{12}+c_{22})d_j(p_1+b_{12}) \\
&= \sum_{i+j=n} (G_i(p_1)+G_i(b_{12}))(d_j(a_{12})+d_j(c_{22})) \\
&\quad + \sum_{i+j=n} (G_i(a_{12})+G_i(c_{22}))(d_j(p_1)+d_j(b_{12})) \\
&= (G_n(p_1)+G_n(b_{12}))(a_{12}+c_{22})+(p_1+b_{12})(d_n(a_{12})+d_n(c_{22})) \\
&\quad + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} (G_i(p_1)+G_i(b_{12}))(d_j(a_{12})+d_j(c_{22})) \\
&\quad + (G_n(a_{12})+G_n(c_{22}))(p_1+b_{22})+(a_{12}+c_{22})(d_n(p_1)+d_n(b_{12})) \\
&\quad + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} (G_i(a_{12})+G_i(c_{22}))(d_j(p_1)+d_j(b_{12})) \\
&= G_n(a_{12})+G_n(b_{12}c_{22}).
\end{aligned}
$$

(ii) On using the same technique as used in (i), we obtain the result. □

Lemma 3.4

(i) $G_n(a_{12}+b_{12}) = G_n(a_{12})+G_n(b_{12})$,
(ii) $G_n(a_{21}+b_{21}) = G_n(a_{21})+G_n(b_{21})$.

Proof. (i) We only prove (i), proof of (ii) follows similarly. Since G is additive, we have $G(a_{12}+b_{12}) = G(a_{12})+G(b_{12})$. Now by induction hypothesis let the result be true for all positive integer $m < n$. For any $x_{22} \in \mathcal{R}_{22}$, we have

$$
\begin{aligned}
&G_n[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})] \\
&= \sum_{i+j=n} G_i(a_{12}+b_{12})d_j(x_{22}) + \sum_{i+j=n} G_i(x_{22})d_j(a_{11}+b_{12}) \\
&= G_n(a_{12}+b_{12})x_{22}+(a_{12}+b_{12})d_n(x_{22}) + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(a_{12}+b_{12})d_j(x_{22}) \\
&\quad + G_n(x_{22})(a_{12}+x_{22})d_n(a_{12}+b_{12}) + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(x_{22})d_j(a_{12}+b_{12}).
\end{aligned}
$$

On the other hand, by Lemma 3.3, we have

$$
\begin{aligned}
&G_n[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})]\\
&=G_n(a_{12}x_{22}+b_{12}x_{22})\\
&=G_n(a_{12}x_{22})+G_n(b_{12}x_{22})\\
&=G_n(a_{12}x_{22}+x_{22}a_{12})+G_n(b_{12}x_{22}+x_{22}b_{12})\\
&=\sum_{i+j=n}G_n(a_{12})d_j(x_{22})+\sum_{i+j=n}G_n(x_{22})d_j(a_{12})\\
&\quad+\sum_{i+j=n}G_n(b_{12})d_j(x_{22})+\sum_{i+j=n}G_n(x_{22})d_j(b_{12})\\
&=G_n(a_{12})x_{22}+a_{12}d_n(x_{22})+\sum_{\substack{i+j=n}}^{0<i,j\leq n-1}G_i(a_{12})d_j(x_{22})+G_n(x_{22})a_{12}+x_{22}d_n(a_{12})\\
&\quad+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(x_{22})d_j(a_{12})+G_n(b_{12})x_{22}+b_{12}d_n(x_{22})+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(b_{12})d_j(x_{22})\\
&\quad+G_n(x_{22})b_{12}+x_{22}d_n(b_{12})+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(x_{22})d_j(b_{12}).
\end{aligned}
$$

On comparing the above two equalities and using the hypothesis that the result holds for all $m<n$, we get

$$[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]x_{22}+x_{22}[d_n(a_{12}+b_{12})-d_n(a_{12})-d_n(b_{12})]=0$$

for all $x_{22}\in\mathcal{R}_{22}$.
Using Theorem 1.1, we get

$$[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]x_{22}=0.$$

By condition $(S1)$, we can find that

$$[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]_{12}=0,$$

$$[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]_{22}=0.$$

We now show that

$$[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]_{11}=0=[G_n(a_{12}+b_{12})-G_n(a_{12})-G_n(b_{12})]_{21}.$$

Now for any $x_{12} \in \mathcal{R}_{12}$, we have

$$
\begin{aligned}
0 &= G_n[(a_{12}+b_{12})x_{12} + x_{12}(a_{12}+b_{12})] \\
&= \sum_{i+j=n} G_i(a_{12}+b_{12})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(a_{12}+b_{12}) \\
&= G_n(a_{12}+b_{12})x_{12} + (a_{12}+b_{12})d_n(x_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(a_{12}+b_{12})d_j(x_{12}) \\
&\quad + G_n(x_{12})(a_{12}+b_{12}) + x_{12}d_n(a_{12}+b_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{12})d_j(a_{12}+b_{12}).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
0 &= G_n(a_{12}x_{12} + x_{12}a_{12}) + G_n(b_{12}x_{12} + x_{12}b_{12}) \\
&= \sum_{i+j=n} G_i(a_{12})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(a_{12}) \\
&\quad + \sum_{i+j=n} G_i(b_{12})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(b_{12}) \\
&= G_n(a_{12})x_{12} + a_{12}d_n(x_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(a_{12})d_j(x_{12}) + G_n(x_{12})a_{12} + x_{12}d_n(a_{12}) \\
&\quad + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{12})d_j(a_{12}) + G_n(b_{12})x_{12} + b_{12}d_n(x_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(b_{12})d_j(x_{12}) \\
&\quad + G_n(x_{12})b_{12} + x_{12}d_n(b_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{12})d_j(b_{12}).
\end{aligned}
$$

From the last two expressions, we have

$$[G_n(a_{12}+b_{12}) - G_n(a_{12}) - G_n(b_{12})]x_{12} + x_{22}[d_n(a_{12}+b_{12}) - d_n(a_{12}) - d_n(b_{12})] = 0.$$

By Theorem 1.1, we find that

$$[G_n(a_{12}+b_{12}) - G_n(a_{12}) - G_n(b_{12})]x_{12} = 0.$$

By condition $(S1)$ we can infer that

$$[G_n(a_{12}+b_{12}) - G_n(a_{12}) - G_n(b_{12})]_{11} = 0,$$

$$[G_n(a_{12}+b_{12}) - G_n(a_{12}) - G_n(b_{12})]_{21} = 0.$$

Hence, $[G_n(a_{12}+b_{12}) - G_n(a_{12}) - G_n(b_{12})] = 0.$ $\square$

Lemma 3.5

(i) $G_n(a_{11}+b_{11}) = G_n(a_{11}) + G_n(b_{11})$,
(ii) $G_n(a_{22}+b_{22}) = G_n(a_{22}) + G_n(b_{22})$.

Proof. (i) We only prove (i), proof of (ii) follows similarly. Since G is additive, we have $G(a_{11}+b_{11}) = G(a_{11}) + G(b_{11})$. Now by induction hypothesis let the result true for all positive integer $m < n$. For any $x_{22} \in \mathcal{R}_{22}$, we have

$$\begin{aligned}
0 &= G_n[(a_{11}+b_{11})x_{22} + x_{22}(a_{11}+b_{11})] \\
&= \sum_{i+j=n} G_i(a_{11}+b_{11})d_j(x_{22}) + \sum_{i+j=n} G_i(x_{22})d_j(a_{11}+b_{11}) \\
&= G_n(a_{11}+b_{11})x_{22} + (a_{11}+b_{11})d_n(x_{22}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(a_{11}+b_{11})d_j(x_{22}) \\
&\quad + G_n(x_{22})(a_{11}+b_{11}) + x_{22}d_n(a_{11}+b_{11}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{22})d_j(a_{11}+b_{11}).
\end{aligned}$$

Also,

$$\begin{aligned}
0 &= G_n(a_{11}x_{22} + x_{22}a_{11}) + G_n(b_{11}x_{22} + x_{22}b_{11}) \\
&= \sum_{i+j=n} G_i(a_{11})d_j(x_{22}) + \sum_{i+j=n} G_i(x_{22})d_j(a_{11}) \\
&\quad + \sum_{i+j=n} G_i(b_{11})d_j(x_{22}) + \sum_{i+j=n} G_i(x_{22})d_j(b_{11}) \\
&= G_n(a_{11})x_{22} + a_{11}d_n(x_{22}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(a_{11})d_j(x_{22}) + G_n(x_{22})a_{11} + x_{22}d_n(a_{11}) \\
&\quad + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{22})d_j(a_{11}) + G_n(b_{11})x_{22} + b_{11}d_n(x_{22}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(b_{11})d_j(x_{22}) \\
&\quad + G_n(x_{22})b_{11} + x_{22}d_n(b_{11}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{22})d_j(b_{11}).
\end{aligned}$$

On comparing the above two equalities and using the hypothesis that the result holds for all $m < n$, we get

$$[G_n(a_{11}+b_{11}) - G_n(a_{11}) - G_n(b_{11})]x_{22} + x_{22}[d_n(a_{11}+b_{11}) - d_n(a_{11}) - d_n(b_{11})] = 0.$$

Using Theorem 1.1, we get

$$[G_n(a_{11}+b_{11}) - G_n(a_{11}) - G_n(b_{11})]x_{22} = 0.$$

By condition $(S1)$, we have

$$[G_n(a_{11}+b_{11}) - G_n(a_{11}) - G_n(b_{11})]_{12} = 0,$$

$$[G_n(a_{11}+b_{11})-G_n(a_{11})-G_n(b_{11})]_{22}=0.$$

Similarly, by considering $(a_{11}+b_{11})x_{12}+x_{12}(a_{11}+b_{11})$ and using Lemma 3.4, one can find that

$$[G_n(a_{11}+b_{11})-G_n(a_{11})-G_n(b_{11})]_{11}=0,$$
$$[G_n(a_{11}+b_{11})-G_n(a_{11})-G_n(b_{11})]_{21}=0.$$

This completes the proof. $\square$

Lemma 3.6 $G_n(a_{12}+b_{21})=G_n(a_{12})+G_n(b_{21})$.

Proof. Since G is additive, we have $G(a_{12}+b_{21})=G(a_{12})+G(b_{21})$. Now by induction hypothesis, let the result hold for all positive integer $m<n$. Consider $a_{12}+b_{21}=(a_{12}+b_{21})p_1+p_1(a_{12}+b_{21})$. Then we find that

$$\begin{aligned}G_n(a_{12}+b_{21}) &= G_n((a_{12}+b_{21})p_1+p_1(a_{12}+b_{21}))\\ &= \sum_{i+j=n} G_i(a_{12}+b_{21})d_j(p_1)+\sum_{i+j=n} G_i(p_1)d_j(a_{12}+b_{21})\\ &= G_n(a_{12}+b_{21})p_1+(a_{12}+b_{21})d_n(p_1)+\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(a_{12}+b_{21})d_j(p_1)\\ &\quad +G_n(p_1)(a_{12}+b_{21})+p_1d_n(a_{12}+b_{21})+\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(p_1)d_j(a_{12}+b_{21}).\end{aligned}$$

On multiplying left hand side by p_2 and using the fact that result holds good for all $m<n$ in the above relation, we obtain

$$\begin{aligned}&p_2G_n(a_{12}+b_{21})\\ &= p_2G_n(a_{12}+b_{21})p_1+p_2(a_{12}+b_{21})d_n(p_1)+\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} p_2(G_i(a_{12})+G_i(b_{21}))d_j(p_1)\\ &\quad +p_2G_n(p_1)a_{12}+p_2G_n(p_1)b_{21}+\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} p_2G_i(p_1)(d_j(a_{12})+d_j(b_{21})).\end{aligned}$$

Also on using the same technique, we compute $G_n(a_{12})=G_n(a_{12}p_1+p_1a_{12})$ and $G_n(b_{21})=G_n(b_{21}p_1+p_1b_{21})$, to obtain

$$\begin{aligned}&G_n(a_{12})=G_n(a_{12}p_1+p_1a_{12})\\ &= \sum_{i+j=n} G_i(a_{12})d_j(p_1)+\sum_{i+j=n} G_i(p_1)d_j(a_{12})\\ &= G_n(a_{12})p_1+a_{12}d_n(p_1)+\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(a_{12})d_j(p_1)+G_n(p_1)a_{12}+p_1d_n(a_{12})\\ &\quad +\sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(p_1)d_j(a_{12}).\end{aligned}$$

On multiplying left hand side by p_2, we get

$$p_2G_n(a_{12}) = p_2G_n(a_{12})p_1 + \sum_{i+j=n}^{0<i,j\leq n-1} p_2G_i(a_{12})d_j(p_1) + p_2G_n(p_1)a_{12} + \sum_{i+j=n}^{0<i,j\leq n-1} p_2G_i(p_1)d_j(a_{12}).$$

Similarly, we can find

$$p_2G_n(b_{21}) = p_2G_n(b_{21})p_1 + \sum_{i+j=n}^{0<i,j\leq n-1} p_2G_i(b_{21})d_j(p_1) + p_2G_n(p_1)b_{21} + \sum_{i+j=n}^{0<i,j\leq n-1} p_2G_i(p_1)d_j(b_{21}).$$

From the above three expressions, we have

$$p_2[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]=p_2[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]p_1.$$

This implies that

$$[G_n(a_{12}+b_{21}) - G_n(a_{12}) - G_n(b_{21})]_{22} = 0.$$

Now for any $x_{12} \in \mathcal{R}_{12}$, we have

$$G_n(a_{12}+b_{21})x_{12} + (a_{12}+b_{21})d_n(x_{12}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(a_{12}+b_{21})d_j(x_{12}) + G_n(x_{12})(a_{12}+b_{21}) + x_{12}d_n(a_{12}+b_{21}) + \sum_{i+j=n}^{0<i,j\leq n-1} G_i(x_{12})d_j(a_{12}+b_{21})$$
$$= \sum_{i+j=n} G_i(a_{12}+b_{21})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(a_{12}+b_{21})$$

$$
\begin{aligned}
&= G_n((a_{12}+b_{21})x_{12}+x_{12}(a_{12}+b_{21}))\\
&= G_n(b_{21}x_{12}+x_{12}b_{21})\\
&= G_n(b_{21}x_{12}+x_{12}b_{21})+G_n(a_{12}x_{12}+x_{12}a_{12})\\
&= \sum_{i+j=n} G_i(b_{21})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(b_{21})\\
&\quad + \sum_{i+j=n} G_i(a_{12})d_j(x_{12}) + \sum_{i+j=n} G_i(x_{12})d_j(a_{12})\\
&= G_n(b_{21})x_{12}+b_{21}d_n(x_{12}) + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(b_{12})d_j(x_{12}) + G_n(x_{12})b_{21}+x_{12}d_n(b_{21})\\
&\quad + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(x_{12})d_j(b_{21}) + G_n(a_{12})x_{12}+a_{12}d_n(x_{12}) + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(a_{12})d_j(x_{12})\\
&\quad + G_n(x_{12})a_{12}+x_{12}d_n(a_{12}) + \sum_{\substack{i+j=n}}^{0<i,j\leq n-1} G_i(x_{12})d_j(a_{12}).
\end{aligned}
$$

This further leads to

$$[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]x_{12}+x_{12}[d_n(a_{12}+b_{21})-d_n(a_{12})-d_n(b_{21})]=0.$$

By Theorem 1.1, we get

$$[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]x_{12}=0.$$

It follows from condition $(S1)$ that

$$[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]_{11}=0,$$

$$[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]_{21}=0.$$

Similarly, by considering $G_n[(a_{12}+b_{21})x_{21}+x_{21}(a_{12}+b_{21})]$ for all $x_{21}\in\mathcal{R}_{21}$, we can find

$$[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]x_{21}=0.$$

Consequently, $[G_n(a_{12}+b_{21})-G_n(a_{12})-G_n(b_{21})]_{12}=0$, which completes the proof. □

Lemma 3.7

(*i*) $G_n(a_{11}+b_{12}+c_{21})=G_n(a_{11})+G_n(b_{12})+G_n(c_{21})$,
(*ii*) $G_n(a_{12}+b_{21}+c_{22})=G_n(a_{12})+G_n(b_{21})+G_n(c_{22})$.

Proof. (i) We only prove (i), the proof of (ii) follows similarly. Since G is additive, we have $G(a_{11}+b_{12}+c_{21}) = G(a_{11})+G(b_{12})+G(c_{21})$. Now by induction hypothesis let the result hold for all positive integer $m<n$. For any $x_{22}\in\mathcal{R}_{22}$, we have

$$\begin{aligned}
&G_n[(a_{11}+b_{12}+c_{21})x_{22}+x_{22}(a_{11}+b_{12}+c_{21})]\\
&=\sum_{i+j=n} G_i(a_{11}+b_{12}+c_{21})d_j(x_{22})+\sum_{i+j=n} G_i(x_{22})d_j(a_{11}+b_{12}+c_{21})\\
&=G_n(a_{11}+b_{12}+c_{21})x_{22}+(a_{11}+b_{12}+c_{21})d_n(x_{22})\\
&\quad+\sum_{\substack{i+j=n}}^{0<i,j\le n-1} G_i(a_{11}+b_{12}+c_{21})d_j(x_{22})\\
&\quad+G_n(x_{22})(a_{11}+b_{12}+c_{21})+x_{22}d_n(a_{11}+b_{12}+c_{21})\\
&\quad+\sum_{\substack{i+j=n}}^{0<i,j\le n-1} G_i(x_{22})d_j(a_{11}+b_{12}+c_{21}).
\end{aligned}$$

On the other hand, by Lemma 3.6, we also have

$$\begin{aligned}
&G_n[(a_{11}+b_{12}+c_{21})x_{22}+x_{22}(a_{11}+b_{12}+c_{21})]\\
&=G_n(b_{12}x_{22}+x_{22}c_{21})\\
&=G_n(b_{12}x_{22})+G_n(x_{22}c_{21})\\
&=G_n(a_{11}x_{22}+x_{22}a_{11})+G_n(b_{12}x_{22}+x_{22}b_{12})+G_n(c_{21}x_{22}+x_{22}c_{21})\\
&=\sum_{i+j=n} G_i(a_{11})d_j(x_{22})+\sum_{i+j=n} G_i(x_{22})d_j(a_{11})+\sum_{i+j=n} G_i(b_{12})d_j(x_{22})\\
&\quad+\sum_{i+j=n} G_i(x_{22})d_j(b_{12})+\sum_{i+j=n} G_i(c_{21})d_j(x_{22})+\sum_{i+j=n} G_i(x_{22})d_j(c_{21})\\
&=G_n(a_{11})x_{22}+a_{11}d_n(x_{22})+\sum_{i+j=n}^{0<i,j\le n-1} G_i(a_{11})d_j(x_{22})+G_n(x_{22})a_{11}+x_{22}d_n(a_{11})\\
&\quad+\sum_{i+j=n}^{0<i,j\le n-1} G_i(x_{22})d_j(a_{11})+G_n(b_{12})x_{22}+b_{12}d_n(x_{22})+\sum_{i+j=n}^{0<i,j\le n-1} G_i(b_{12})d_j(x_{22})\\
&\quad+G_n(x_{22})b_{12}+x_{22}d_n(b_{12})+\sum_{i+j=n}^{0<i,j\le n-1} G_i(x_{22})d_j(b_{12})+G_n(c_{21})x_{22}+c_{21}d_n(x_{22})\\
&\quad+\sum_{i+j=n}^{0<i,j\le n-1} G_i(c_{21})d_j(x_{22})+G_n(x_{22})c_{21}+x_{22}d_n(c_{21})+\sum_{i+j=n}^{0<i,j\le n-1} G_i(x_{22})d_j(c_{21}).
\end{aligned}$$

On comparing the above two equalities and using Theorem 1.1, we get

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]x_{22}=0.$$

Hence, by condition $(S1)$ we obtain that

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]_{12}=0,$$

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]_{22}=0.$$

Now for any $x_{12}\in\mathcal{R}_{12}$, we have

$$\begin{aligned}
&G_n(a_{11}+b_{12}+c_{21})x_{12}+(a_{11}+b_{12}+c_{21})d_n(x_{12})\\
&\quad+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(a_{11}+b_{12}+c_{21})d_j(x_{12})\\
&\quad+G_n(x_{12})(a_{11}+b_{12}+c_{21})+x_{12}d_n(a_{11}+b_{12}+c_{21})\\
&\quad+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(x_{12})d_j(a_{11}+b_{12}+c_{21})\\
&=G_n[(a_{11}+b_{12}+c_{21})x_{12}+x_{12}(a_{11}+b_{12}+c_{21})]\\
&=G_n(a_{11}x_{12}+c_{21}x_{12}+x_{12}c_{21})\\
&=G_n[(a_{11}+c_{21})x_{12}+x_{12}(a_{11}+c_{21})]+G_n(b_{12}x_{12}+x_{12}b_{12})\\
&=G_n(a_{11}+c_{21})x_{12}+(a_{11}+c_{21})d_n(x_{12})+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(a_{11}+c_{21})d_j(x_{12})\\
&\quad+G_n(x_{12})(a_{11}+c_{21})+x_{12}d_n(a_{11}+c_{21})+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(x_{12})d_j(a_{11}+c_{21})\\
&\quad+G_n(b_{12})x_{12}+b_{12}d_n(x_{12})+\sum_{i+j=n}^{0<i,j\leq n-1}G_i(b_{12})d_j(x_{12})\\
&\quad+G_n(x_{12})b_{12}+x_{12}d_n(b_{12})\sum_{i+j=n}^{0<i,j\leq n-1}G_i(x_{12})d_j(b_{12}).
\end{aligned}$$

It follows that

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]x_{12}=0,$$

consequently, we have

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]_{11}=0,$$

$$[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]_{21}=0.$$

Hence, $[G_n(a_{11}+b_{12}+c_{21})-G_n(a_{11})-G_n(b_{12})-G_n(c_{21})]=0.$ □

Lemma 3.8 $G_n(a_{11}+b_{12}+c_{21}+w_{22})=G_n(a_{11})+G_n(b_{12})+G_n(c_{21})+G_n(w_{22})$.

Proof. Since G is additive, we have $G(a_{11}+b_{12}+c_{21}+w_{22}) = G(a_{11}) + G(b_{12}) + G(c_{21}) + G(w_{22})$. Now by induction hypothesis let the result hold for all positive integer $m < n$. For any $x_{11} \in \mathcal{R}_{11}$, we have

$$\begin{aligned}
&G_n(a_{11}+b_{12}+c_{21}+w_{22})x_{11} + (a_{11}+b_{12}+c_{21}+w_{22})d_n(x_{11})\\
&+ \sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(a_{11}+b_{12}+c_{21}+w_{22})d_j(x_{11})\\
&+G_n(x_{11})(a_{11}+b_{12}+c_{21}+w_{22}) + x_{11}d_n(a_{11}+b_{12}+c_{21}+w_{22})\\
&+ \sum_{\substack{0<i,j\leq n-1\\ i+j=n}} G_i(x_{11})d_j(a_{11}+b_{12}+c_{21}+w_{22})\\
=&\sum_{i+j=n} G_i(a_{11}+b_{12}+c_{21}+w_{22})d_j(x_{11}) + \sum_{i+j=n} G_i(x_{11})d_j(a_{11}+b_{12}+c_{21}+w_{22})\\
=&\ G_n[(a_{11}+b_{12}+c_{21}+w_{22})x_{11} + x_{11}(a_{11}+b_{12}+c_{21}+w_{22})]\\
=&\ G_n(a_{11}x_{11} + c_{21}x_{11} + x_{11}a_{11} + x_{11}b_{12})\\
=&\ G_n(a_{11}x_{11} + x_{11}a_{11}) + G_n(x_{11}b_{12}) + G_n(c_{21}x_{11})\\
=&\ G_n(a_{11}x_{11} + x_{11}a_{11}) + G_n(b_{12}x_{11} + x_{11}b_{12})\\
&+G_n(c_{21}x_{11} + x_{11}c_{21}) + G_n(w_{22}x_{11} + x_{11}w_{22})\\
=&\sum_{i+j=n} G_i(a_{11})d_j(x_{11}) + \sum_{i+j=n} G_i(x_{11})d_j(a_{11}) + \sum_{i+j=n} G_i(b_{12})d_j(x_{11})\\
&+ \sum_{i+j=n} G_i(x_{11})d_j(b_{12}) + \sum_{i+j=n} G_i(c_{21})d_j(x_{11}) + \sum_{i+j=n} G_i(x_{11})d_j(c_{11})\\
&+ \sum_{i+j=n} G_i(w_{22})d_j(x_{11}) + \sum_{i+j=n} G_i(x_{11})d_j(w_{22}).
\end{aligned}$$

This gives us

$$\begin{aligned}
&[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]x_{11}\\
&+x_{11}[d_n(a_{11}+b_{12}+c_{21}+w_{22}) - d_n(a_{11}) - d_n(b_{12}) - d_n(c_{21}) - d_n(w_{22})] = 0.
\end{aligned}$$

From this we can infer that

$$[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]x_{11} = 0.$$

Hence, we get

$$[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]_{11} = 0,$$

$$[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]_{21} = 0.$$

Similarly, one can get

$$[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]_{12} = 0,$$

$$[G_n(a_{11}+b_{12}+c_{21}+w_{22}) - G_n(a_{11}) - G_n(b_{12}) - G_n(c_{21}) - G_n(w_{22})]_{22} = 0.$$

This completes the proof. □

Now we are ready to prove our main result of this section.

Proof of Theorem 3.1. For any $a, b \in \mathcal{R}$, we write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. Applying Lemmas 3.2-3.8, we have

$$\begin{aligned} G_n(a+b) &= G_n(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\ &= G_n[(a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})] \\ &= G_n(a_{11} + b_{11}) + G_n(a_{12} + b_{12}) + G_n(a_{21} + b_{21}) + G_n(a_{22} + b_{22}) \\ &= G_n(a_{11}) + G_n(b_{11}) + G_n(a_{12}) + G_n(b_{12}) \\ &\quad + G_n(a_{21}) + G_n(b_{21}) + G_n(a_{22}) + G_n(b_{22}) \\ &= G_n(a_{11} + a_{12} + a_{21} + a_{22}) + G_n(b_{11} + b_{12} + b_{21} + b_{22}) \\ &= G_n(a) + G_n(b). \end{aligned}$$

Hence we conclude that $\Delta = \{G_i\}_{i\in\mathbb{N}}$ is the family of additive mappings $G_i : \mathcal{R} \to \mathcal{R}$.
In addition, if $\mathcal{R}$ is 2-torsion free, then for any $a \in \mathcal{R}$, we have

$$2G_n(a^2) = G_n(2a^2) = G_n(aa + aa) = 2 \sum_{i+j=n} G_i(a)d_j(a).$$

Therefore, $\Delta = \{G_i\}_{i\in\mathbb{N}}$ is a generalized Jordan higher derivation. □

If the underlying ring is semiprime, applying Lemma 1.5 of [6] and using the fact that every generalized Jordan higher derivation on a 2-torsion free semiprime ring is a generalized higher derivation (see [15]), we have the following corollary.

Corollary 3.9 *Let $\mathcal{R}$ be a 2-torsion free semiprime ring containing a non-trivial idempotent satisfying the following conditions for $i, j, k \in \{1, 2\}$:*

(S1) *If $a_{ii}x_{ij} = 0$ for all $x_{ij} \in \mathcal{R}_{ij}$, then $a_{ii} = 0$;*
(S2) *If $x_{ji}a_{ii} = 0$ for all $x_{ji} \in \mathcal{R}_{ji}$, then $a_{ii} = 0$.*

If the family $\Delta = \{G_n\}_{n\in\mathbb{N}}$ of mappings $G_n : \mathcal{R} \to \mathcal{R}$ such that $G_0 = I_{\mathcal{R}}$ satisfies

$$G_n(ab + ba) = \sum_{i+j=n} G_i(a)d_j(b) + G_i(b)d_j(a)$$

for all $a, b \in \mathcal{R}$ and for each $n \in \mathbb{N}$, then $\Delta = \{G_n\}_{n\in\mathbb{N}}$ is additive. Moreover, $\Delta = \{G_n\}_{n\in\mathbb{N}}$ is a generalized higher derivation.

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