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Arc length for the Janowski classes

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Abstract Sharp arc length of the image curve of a given line segment joining the points r^{it} and $-re^{it}$ $(0 < r < 1,\ 0 \leqslant t \leqslant 2\pi)$ under the Janowski starlike and convex functions are derived. Further, length of image curve of the circle $|z| = r < 1$ under the Janowski convex functions is also obtained. Several other results related to the arc length for many classes associated with lemniscate of Bernoulli and exponential functions are also discussed. Relevant connections of our results with the earlier known results are also pointed out.

Keywords arc length · Janowski starlike function · lemniscate of Bernoulli · exponential function

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions defined in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. The subclass of $\mathcal{H}$ of functions normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ is denoted by $\mathcal{A}$. The subclass of $\mathcal{A}$ of univalent functions is denoted by $\mathcal{S}$. An analytic function f is subordinate to another analytic function g, written $f \prec g$, if there is an analytic function w with $|w(z)| \leqslant |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Among several subclasses of $\mathcal{S}$, the classes of starlike functions and convex functions have attracted most because of their nice geometric properties. Analytically, these classes are defined by $\mathcal{S}^* = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1 + z)/(1 - z)\}$ and $\mathcal{K} = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec (1 + z)/(1 - z)\}$, respectively. These classes were generalized by Janowski [5]. For $-1 \leqslant B < A \leqslant 1$, he introduced the classes, known as the classes of the Janowski starlike and convex functions, as follows:

$$\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right) \text{ and } \mathcal{K}[A,B] := \mathcal{K}\left(\frac{1+Az}{1+Bz}\right).$$

Note that $\mathcal{S}^*[1,-1] =: \mathcal{S}^*$ and $\mathcal{K}[1,-1] =: \mathcal{K}$ are the classes of starlike functions and convex functions, respectively. More general form of these classes were considered by Ma and Minda [7] as

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},$$

where φ is an analytic function with $\varphi'(0) > 0, \operatorname{Re}\varphi(z) > 0$ $(z \in \mathbb{D})$ and mapping $\mathbb{D}$ onto a domain starlike with respect to 1, which is symmetrical about the real axis. The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are popularly known as the Ma-Minda starlike and convex functions, respectively. A function f is called close-to-convex if $f \in \mathcal{A}$ satisfies $e^{-i\alpha} zf'(z)/g(z) \prec (1+z)/(1-z)$, for some $g \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$.

Let $C(r,f)$ denotes the image curve of $|z| = r < 1$ under the function $f \in \mathcal{H}$ which bound the area $A(r,f)$. Let $L(r,f)$ be the length of $C(r,f)$ and is given by

$$L(r,f) = \int_0^{2\pi} |zf'(z)|\, dt \quad (z = re^{it}).$$

Let $M(r) := \max\{|f(z)| : |z| = r < 1\}$. In 1959, Keogh [6] proved that if $f \in \mathcal{S}^*$, then

$$L(r,f) = \mathcal{O}\left(M(r)\log(1-r)^{-1}\right) \text{ as } r \to 1,$$

where $\mathcal{O}$ is Landau's symbol. He concluded that if f is bounded, then $L(r,f) = \mathcal{O}\left(\log(1-r)^{-1}\right)$ as $r \to 1$. Further, he raised the question that "whether in $L(r,f) = \mathcal{O}\left(\log(1-r)^{-1}\right)$, $\mathcal{O}$ could be replace by o"? Hayman [4], in 1961, answered this question negatively and established that the result is best possible. Later, for close-to-convex functions, in 1965, Pommerenke [13] has shown that $L(r,f) = \mathcal{O}\left(M(r)\left(\log(3/(1-r))\right)^{5/2}\right)$ as $r \to 1$. For close-to-convex function, Thomas [17], in 1967, proved that $L(r,f) = \mathcal{O}\left(M(r)\log(1-r)^{-1}\right)$ as $r \to 1$. In the same paper he has also shown that if a starlike function maps the disk $|z| \leqslant r$ onto a domain having finite area A, then $L(r,f) \leqslant 8A\left(1 + \log(1-r^2)^{-1}\right)$. Further, in 1968, Thomas [19] proved that if $f \in \mathcal{S}^*$, then

$$2\sqrt{\pi A(r)} \leqslant L(r,f) \leqslant 2\sqrt{\pi A(r)}\left(1 + \log\frac{1+r}{1-r}\right). \tag{1.1}$$

In particular, $L(r,f) \sim 2\sqrt{\pi A(r)}$ as $r \to 0$. Thomas [18], in 1968, has shown that if f is Bazilević function of order β and $|f(z)| < 1$, then $L(r,f) = \mathcal{O}(\log(1-r)^{-1})$ as $r \to 1$. He also proved many other results related to length problem. Nunokawa [10] proved a stronger form of (1.1). He proved that if $zf'(z) \in \mathcal{S}^*$ (i.e. f is convex function), then

$$L(r,f) = \mathcal{O}\left\{A(r,f)\log\frac{1}{1-r}\right\}^{1/2} \quad \text{as} \quad r \to 1.$$

Nunokawa and Sokół [11] obtained a stronger form of the results proved by Pommerenke [17] for Bazilević functions, strongly starlike functions and close-to-convex functions. In 2016, Nunokawa and Sokół [12] discussed arc length problem for functions in the class $\mathcal{S}$ under certain conditions. For some more results in this direction one can refer to [8, 12, 20, 21].

Motivated by the works cited above, in Section 2, the sharp arc length of the image curve of a given line segment under the Janowski starlike and convex functions are derived. In Section 3, some auxiliary results are established to discuss arc length problem for the class of Janowski convex functions. Section 4 concludes this work with length problems for some geometrically defined classes associated with lemniscate of Bernoulli and exponential function. Also integral means for Ma-Minda starlike and convex functions are discussed. Relevant connections of our results with the existing results are also pointed out.

2 Arc Length: Image of a line segment

The length of the image of a radial segment $z = se^{i\beta}$ $(0 \leqslant s \leqslant r < 1)$ under an analytic function f is defined by (see, [3], p. 203)

$$l(\beta, r, f) = \int_0^r |f'(se^{i\beta})|\, ds.$$

For $f \in \mathcal{S}$, it is easy to see that the following sharp result holds for all $\beta \in [0, 2\pi]$ and $r \in (0,1)$:

$$l(\beta, r, f) \leqslant \int_0^r \frac{1+s}{(1-s)^3}\, ds = \frac{r}{(1-r)^2}.$$

In this case too, the Koebe function acts as an extremal function. Since the Koebe function is also an extremal function for the class of starlike functions, it follows that the above estimate is also true for the class of starlike functions. Similarly for the function $f \in \mathcal{K}$, we have $|f'(z)| \leqslant (1-r)^{-2}$. Therefore, for all $f \in \mathcal{K}$, $\beta \in [0, 2\pi]$ and $r \in (0,1)$, we have

$$l(\beta, r, f) \leqslant \int_0^r \frac{1}{(1-s)^2}\, ds = \frac{r}{1-r}.$$

With equality in case of the function $f(z) = z/(1-z)$ or one of its rotation. In a similar way in case of the Janowski starlike and convex functions, for all $\beta \in [0, 2\pi]$ and $r \in (0,1)$, we have

$$l(\beta, r, f) \leqslant \begin{cases} r(1+Br)^{\frac{A}{B}-1}, & B \neq 0; \\ re^{Ar}, & B = 0 \end{cases}$$

and

$$l(\beta, r, f) \leqslant \begin{cases} \frac{1}{A}[(1+Br)^{A/B}-1], & B \neq 0, A \neq 0; \\ \frac{1}{B}\log(1+Br), & A = 0; \\ \frac{1}{A}\left(e^{Ar}-1\right), & B = 0, \end{cases}$$

respectively. In case of the Janowski starlike functions the result is sharp for the function h_0 defined by

$$h_0(z) = \begin{cases} z(1+Bz)^{\frac{A}{B}-1}, & B \neq 0; \\ ze^{Az}, & B = 0, \end{cases} \tag{2.1}$$

whereas for Janowski convex functions the extremal function is given by

$$k_0(z) = \begin{cases} \frac{1}{A}[(1+Bz)^{\frac{A}{B}}-1], & B \neq 0, A \neq 0; \\ \frac{1}{B}\log(1+Bz), & A = 0; \\ \frac{1}{A}[e^{Az}-1], & B = 0. \end{cases} \tag{2.2}$$

For any analytic function f defined on $\mathbb{D}$, let $l(re^{it}, f)$ be the length of image curve of the line joining the points re^{it} and $-re^{it}$ under the map f. Then, we can write

$$l(re^{it}, f) = \int_{-r}^{r} |f'(\rho e^{it})| \, d\rho.$$

and let

$$l(r, f) = \max_{0 \leqslant t < 2\pi} l(re^{it}, f) = \max\left\{\int_{-r}^{r} |f'(\rho e^{it})| \, d\rho : 0 \leqslant t < 2\pi\right\}.$$

Let $\mathcal{T}$ be a subclass of $\mathcal{H}$ and let $0 < r < 1$. Then we shall write $l(r, \mathcal{T}) \leqslant l(r)$ if and only if for every function $f \in \mathcal{T}$, we have $l(r, f) \leqslant l(r)$ and we say that inequality is best possible if there exists some function in the class $\mathcal{T}$ such that $l(r, \mathcal{T}) = l(r)$, see [16].

Ma and Minda [7] proved the following results:

Lemma 2.1 *[7, Theorem 2, p. 162] Let $f \in \mathcal{S}^*(\varphi)$ and*

$$\varphi(r) = \max\left\{|\varphi(re^{it})| : 0 \leqslant t \leqslant 2\pi\right\}.$$

Let h_φ is a solution of the equation $zh'_\varphi(z)/h_\varphi(z) = \varphi(z)$. Then, for $|z| = r < 1$, we have $h'_\varphi(-r) \leqslant |f'(z)| \leqslant h'_\varphi(r)$. Equality holds for some $z \neq 0$ if and only if f is a rotation of h_φ.

Lemma 2.2 [7, Theorem 1, p. 159] *Let $f \in \mathcal{K}(\varphi)$. Let k_φ is a solution of the differential equation $1 + zk''_\varphi(z)/k'_\varphi(z) = \varphi(z)$. Then, for $|z| = r < 1$, we have $k'_\varphi(-r) \leqslant |f'(z)| \leqslant k'_\varphi(r)$. Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.*

Note that under the condition of Lemma 2.1, we have

$$\min\left\{h'_\varphi(z) : z = re^{it}, 0 \leqslant t \leqslant 2\pi\right\} = h'_\varphi(-r) \leqslant |f'(z)| \\ \leqslant h'_\varphi(r) = \max\left\{h'_\varphi(z) : z = re^{it}, 0 \leqslant t \leqslant 2\pi\right\}.$$

Integrating both side of the above inequality under the limits from $-r$ to r, for function $f \in \mathcal{S}^*(\varphi)$, we have the following results:

Theorem 2.3 *Let $f \in \mathcal{S}^*(\varphi)$ and satisfies the conditions of Lemma 2.1. Then*

$$\int_{-r}^{r} h'_\varphi(-\rho)\ d\rho \leqslant l\left(r, \mathcal{S}^*(\varphi)\right) \leqslant \int_{-r}^{r} h'_\varphi(\rho)\ d\rho.$$

The result is sharp since equality occurs in case of the function h_φ defined in Lemma 2.1

Theorem 2.4 *Let $f \in \mathcal{K}(\varphi)$ and satisfies the conditions of Lemma 2.2. Then*

$$\int_{-r}^{r} k'_\varphi(-\rho)\ d\rho \leqslant l\left(r, \mathcal{K}(\varphi)\right) \leqslant \int_{-r}^{r} k'_\varphi(\rho)\ d\rho.$$

The result is sharp since equality occurs in case of the function k_φ defined in Lemma 2.2

The following results give applications of Theorems 2.3 and 2.4 to the classes of Janowski starlike and convex functions:

Corollary 2.5 *For function $f \in \mathcal{S}^*[A, B]$, we have*

$$l\left(r, \mathcal{S}^*[A, B]\right) \leqslant \begin{cases} r\left[(1 + Br)^{\frac{A}{B}-1} + (1 - Br)^{\frac{A}{B}-1}\right], & B \neq 0; \\ r\left[e^{Ar} + e^{-Ar}\right], & B = 0. \end{cases}$$

The result is best possible, as equality holds in case of the function defined in (2.1).

Remark 2.1 In particular, if we take $A = 1$ and $B = -1$, then for function $f \in \mathcal{S}^*$, we have

$$l\left(r, f\right) \leqslant \frac{r}{(1 - r)^2} + \frac{1}{(1 + r)^2} = \frac{2r\left(1 + r^2\right)}{\left(1 - r^2\right)^2}.$$

Therefore,

$$l\left(r, \mathcal{S}^*\right) \leqslant \frac{2r\left(1 + r^2\right)}{\left(1 - r^2\right)^2}.$$

Equality in the result holds in case of the function $f(z) = z/(1-z)^2$, since $f'(z) = (z+1)/(1-z)^3$ and

$$\int_{-r}^{r} f'(z)dz = \frac{2r\left(1+r^2\right)}{\left(1-r^2\right)^2}.$$

Since the extremal functions for the classes $\mathcal{S}^*$ and $\mathcal{S}$ are the same, it follows that arc length estimate for the classes $\mathcal{S}^*$ and $\mathcal{S}$ are same.

Theorem 2.6 *For function* $f \in \mathcal{K}[A,B]$, *we have*

$$l\left(r, \mathcal{K}[A,B]\right) \leqslant \begin{cases} \frac{1}{A}\left[(1+Br)^{\frac{A}{B}} - (1-Br)^{\frac{A}{B}}\right], & B \neq 0, A \neq 0; \\ \frac{1}{B}\log\left(\frac{1+Br}{1-Br}\right), & A = 0; \\ \frac{1}{A}\left(e^{Ar} - e^{-Ar}\right), & B = 0. \end{cases}$$

The result is best possible, as equality holds in case of the function defined in (2.2).

Remark 2.2 In particular, if we take $A = 1$ and $B = -1$, then, for function $f \in \mathcal{K}$, we have

$$l\left(r, f\right) \leqslant \frac{1}{1-r} - \frac{1}{1+r} = \frac{2r}{1-r^2}.$$

Therefore

$$l\left(r, \mathcal{K}\right) \leqslant \frac{2r}{1-r^2}.$$

The result is best possible as equality holds in case of the function $f(z) = 1/(1-z)^2$, since for this function, we see that $f'(z) = 1/(1-z)$ and $\int_{-r}^{r} f'(z)dz = 2r/(1-r^2)$.

For a function φ with positive real part and some function g which is starlike of order half, let $\mathcal{K}_s(\varphi)$ denotes the class of normalized analytic functions f such that $z^2 f'(z)/(g(z)g(-z)) \prec \varphi(z)$. This class was introduced by Wang *et al.* [22]. In 2011, Cho *et al.* [2], proved that if $f \in \mathcal{K}_s(\varphi)$ and $\varphi(-r) = \min\{|\varphi(z)| : |z| = r < 1\}$, and $\varphi(r) = \max\{|\varphi(z)| : |z| = r < 1\}$, then $\varphi(-r)/(1+r^2) \leqslant |f'(z)| \leqslant \varphi(r)/(1-r^2)$. Thus, for this class we have

$$\int_{-r}^{r} \frac{\varphi(-\rho)}{1+\rho^2}\, d\rho \leqslant l\left(r, f\right) \leqslant \int_{-r}^{r} \frac{\varphi(\rho)}{1-\rho^2}\, d\rho.$$

Let $\varphi(z) = (1+Az)/(1+Bz)$, $-1 \leqslant B < A \leqslant 1$. Then $\varphi(-r) = (1 - Ar)/(1-Br)$ and $\varphi(r) = (1+Ar)/(1+Br)$. Therefore, we have

$$\int_{-r}^{r} \frac{1-A\rho}{(1-B\rho)(1+\rho^2)}\, d\rho \leqslant l\left(r, f\right) \leqslant \int_{-r}^{r} \frac{1+A\rho}{(1+B\rho)(1-\rho^2)}\, d\rho.$$

In particular, if $A = 1$ and $B = -1$, then

$$2\tanh^{-1}(r) \leqslant l\left(r, f\right) \leqslant \frac{2r}{1-r^2}.$$

3 Arc Length: Janowski convex functions

In this section, we first establish certain results related to the the class $\mathcal{P}[A,B]$. Then using the association of the functions in between the classes $\mathcal{P}[A,B]$ and $\mathcal{K}[A,B]$, we derive arc length for functions in the class $\mathcal{K}[A,B]$. For further discussion, we recall the following result:

Lemma 3.1 (Parseval-Gutzmer Formula) [1] *Let $g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ is an analytic function in $\mathbb{D}$, then, for $z = re^{it}$ $(0 \leqslant r < 1, 0 \leqslant t \leqslant 2\pi)$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |g(z)|^2 \; dt = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

We owe the following result to Rogosiniki:

Lemma 3.2 [15] *If $f \prec g$ in $\mathbb{D}$, then for all $\mu > 0$*

$$\int_0^{2\pi} |f(z)|^{\mu} dt \leqslant \int_0^{2\pi} |g(z)|^{\mu} dt \quad (z \in \mathbb{D}).$$

Let $p \in \mathcal{P}[A,B]$. Then by definition of the class $\mathcal{P}[A,B]$ and properties of subordination, we have

$$p(z) \prec \frac{1+Az}{1+Bz} \;\text{ and }\; |p(z)| \leqslant \max_{|z|\leqslant r}\left|\frac{1+Az}{1+Bz}\right|. \tag{3.1}$$

Further, we can write

$$\frac{1+Az}{1+Bz} = \sum_{k=0}^{\infty} \phi_k z^k,$$

where

$$\phi_k = \begin{cases} 1, & k = 0; \\ (-1)^k B^{k-1}(B-A), & k = 1,2,3,\ldots. \end{cases} \tag{3.2}$$

In the view of Lemmas 3.1, 3.2 and Eqn. (3.1), for $B \neq 0$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2 \; dt &\leqslant \sum_{k=0}^{\infty} \phi_k^2 r^{2k} \\ &= 1 + \frac{(B-A)^2}{B^2}\sum_{k=1}^{\infty}(Br)^{2k} \\ &= 1 + \frac{(B-A)^2}{B^2}\left(\frac{1}{1-(Br)^2} - 1\right) \\ &= 1 + \frac{(B-A)^2}{B^2}\left(\frac{B^2r^2}{1-B^2r^2}\right). \end{aligned} \tag{3.3}$$

Now using (3.3), we have

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,dtd\rho \leqslant \int_0^r\sum_{k=0}^{\infty}\phi_k^2\rho^{2k}d\rho$$
$$= r+\frac{(B-A)^2}{2B^3}\left(\log\left(\frac{1+Br}{1-Br}\right)-2Br\right). \quad (3.4)$$

Further, if $B=0$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\,dt \leqslant 1+A^2r^2$$

and

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,dtd\rho \leqslant \int_0^r(1+A^2\rho^2)d\rho$$
$$= r+\frac{A^2r^3}{3}.$$

In particular, if $A=1$ and $B=-1$, then from (3.4), we have

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,dtd\rho \leqslant 2\log\left(\frac{1+r}{1-r}\right)-3r.$$

In the view of the above discussion and Lemma 3.1, we have the following result:

Theorem 3.3 *Let $p\in\mathcal{P}[A,B]$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\,dt \leqslant \begin{cases} 1+\frac{(B-A)^2}{B^2}\left(\frac{B^2r^2}{1-B^2r^2}\right), & B\neq 0;\\ 1+A^2r^2, & B=0.\end{cases}$$

and

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,dtd\rho \leqslant \begin{cases} r+\frac{(B-A)^2}{2B^3}\left(\log\left(\frac{1+Br}{1-Br}\right)-2Br\right), & B\neq 0;\\ r+\frac{A^2r^3}{3}, & B=0.\end{cases}$$

In particular, for $p\in\mathcal{P}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\,dt \leqslant 1+4\sum_{k=1}^{\infty}r^{2k}=\frac{1+3r^2}{1-r^2} \quad (3.5)$$

and

$$\frac{1}{2\pi}\int_0^r\int_0^{2\pi}|p(z)|^2\,dtd\rho \leqslant 2\log\left(\frac{1+r}{1-r}\right)-3r. \quad (3.6)$$

Remark 3.1 Inequality (3.5) was proved by Nunokawa and Sokół [Lemma 2.1, [11]], whereas Inequality (3.6) improves the result proved by Thomas (see the proof of Theorem 2 in [17]). In fact, he proved that

$$\int_0^r \int_0^{2\pi} |p(z)|^2 \, dtd\rho \leqslant 4\pi \log\left(\frac{1+r}{1-r}\right).$$

The following theorem gives arc length for the function $f \in \mathcal{K}[A, B]$.

Theorem 3.4 *Let $A(r, f)$ be the area bounded by the image curve of the circle $|z| = r < 1, r \neq 0$ under the function $f \in \mathcal{K}[A, B]$. Then*

$$L(r,f) \leqslant \begin{cases} 2\left(\frac{\pi A(r,f)}{r}\left(r + \frac{(B-A)^2}{2B^3}\left(\log\left(\frac{1+Br}{1-Br}\right) - 2Br\right)\right)\right)^{1/2}, & B \neq 0; \\ 2\left(\frac{\pi A(r,f)}{r}\left(r + \frac{A^2 r^3}{3}\right)\right)^{1/2}, & B = 0. \end{cases} \tag{3.7}$$

Proof. Let $f \in \mathcal{K}[A, B]$. Then by definition of arc length

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)| dt \\ &= \int_0^r \int_0^{2\pi} |f'(z) + zf''(z)| dtd\rho \\ &= \int_0^r \int_0^{2\pi} \left| f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| dtd\rho \\ &\leqslant \left(\int_0^r \int_0^{2\pi} \left|f'(z)\right|^2 dtd\rho\right)^{1/2} \left(\int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 dtd\rho\right)^{1/2} \end{aligned} \tag{3.8}$$

We note that, for $f(z) = \sum_{k=1}^{\infty} a_k z^k$ $(a_1 = 1)$, we have

$$A(r,f) = \int_0^r \int_0^{2\pi} \rho \left|f'(z)\right|^2 dtd\rho = \pi \sum_{k=1}^{\infty} k|a_k|^2 r^{2k}. \tag{3.9}$$

Now

$$\begin{aligned} \int_0^r \int_0^{2\pi} \left|f'(z)\right|^2 dtd\rho &\leqslant 2\pi \int_0^r \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2(k-1)} d\rho \\ &= 2\pi \sum_{k=1}^{\infty} \frac{k^2}{2k-1} |a_k|^2 r^{2k-1} \\ &\leqslant 2\pi \sum_{k=1}^{\infty} k|a_k|^2 r^{2k-1}. \end{aligned}$$

Thus, in the view of (3.9), we have

$$\int_0^r \int_0^{2\pi} \left|f'(z)\right|^2 dt d\rho \leqslant \frac{2A(r,f)}{r}. \tag{3.10}$$

Since $f \in \mathcal{K}[A,B]$, setting $p(z) = 1 + zf''(z)/f'(z)$ in Theorem 3.3, we have

$$\int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 dt d\rho \leqslant \begin{cases} 2\pi\left(r + \frac{(B-A)^2}{2B^3}\left(\log\left(\frac{1+Br}{1-Br}\right) - 2Br\right)\right), & B \neq 0; \\ 2\pi\left(r + \frac{A^2r^3}{3}\right), & B = 0. \end{cases} \tag{3.11}$$

From (3.8), (3.10) and (3.11), we have the desired Inequality (3.7). □

Remark 3.2 As a consequence of above theorem, we derive that, for function $f \in \mathcal{K}$

$$L(r,f) \leqslant 2\left(\frac{\pi A(r,f)}{r}\left(2\log\left(\frac{1+r}{1-r}\right) - 3r\right)\right)^{1/2}$$

or we can write

$$L(r,f) = \mathcal{O}\left\{\left(A(r,f)\log\left(\frac{1}{1-r}\right)\right)^{1/2}\right\} \quad \text{as} \quad r \to 1,$$

which is the same as proved by Nunokawa [9] (see also [10]).

4 Some more length problems

We now define the following classes:

$$\mathcal{K}_c := \mathcal{K}\left(1 + \sum_{k=1}^{\infty}\frac{z^k}{k}\right) = \left\{f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sum_{k=1}^{\infty}\frac{z^k}{k}\right\}.$$

$$\mathcal{K}_l := \mathcal{K}\left(\sqrt{1+z}\right) = \left\{f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \sqrt{1+z}\right\}$$

and

$$\mathcal{K}_e := \mathcal{K}\left(e^z\right) = \left\{f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec e^z\right\}.$$

We note that

$$\sqrt{1+z} = \sum_{k=0}^{\infty}\binom{1/2}{k}z^k \quad \text{and} \quad e^z = \sum_{k=0}^{\infty}\frac{1}{k!}z^k.$$

Let us denote the corresponding classes related with the quantity zf'/f by:

$$\mathcal{S}_c^* := \mathcal{S}^*\left(1 + \sum_{k=1}^{\infty}\frac{z^k}{k}\right), \ \mathcal{S}_l^* := \mathcal{S}^*\left(\sqrt{1+z}\right) \text{ and } \mathcal{S}_e^* := \mathcal{S}^*\left(e^z\right).$$

In the following theorems we let ${}_mF_n$, $\mathrm{Li}_n\,(x) = \sum_{k=1}^{\infty}(x^k/k^n)$, $J_n(x)$ and $\mathrm{Li}_n\,(x) = \sum_{k=1}^{\infty}(x^k/k^n)$ denote the hypergeometric, the polylogarithm, and the modified Bessel functions of the first kind, respectively.

Theorem 4.1 *Let $A_c(r,f)$, $A_l(r,f)$, and $A_e(r,f)$ be the area bounded by the image curve of the circle $|z| = r < 1$, $r \neq 0$ under the function f in the classes $\mathcal{K}_c$, $\mathcal{K}_l$ and $\mathcal{K}_e$, respectively.*

1. *If $f \in \mathcal{K}_c$, then*

$$L(r,f) \leqslant 2\left(\frac{\pi A_c(r,f)}{r}\left(rLi_2\left(r^2\right) + 4\tanh^{-1}(r) + 2r\log\left(1-r^2\right) - 3r\right)\right)^{1/2}.$$

2. *If $f \in \mathcal{K}_l$, then*

$$L(r,f) \leqslant 2\left(\pi A_l(r,f)\,{}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; r^2\right)\right)^{1/2}.$$

3. *If $f \in \mathcal{K}_e$, then*

$$L(r,f) \leqslant 2\left(\pi A_e(r,f)\,{}_1F_2\left(\frac{1}{2}; 1, \frac{3}{2}; r^2\right)\right)^{1/2}.$$

Proof. (1) Let $f \in \mathcal{K}_c$. Proceeding as in the proof of Theorem 3.4, we have

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)|dt \\ &= \int_0^r \int_0^{2\pi} \left|f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| dtd\rho \\ &\leqslant \left(\int_0^r \int_0^{2\pi} \left|f'(z)\right|^2 dtd\rho\right)^{1/2} \left(\int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 dtd\rho\right)^{1/2}. \end{aligned} \tag{4.1}$$

Also note that

$$A(r) \geqslant \int_0^r \int_0^{2\pi} \left|f'(z)\right|^2 dtd\rho. \tag{4.2}$$

Using Lemma 3.1 and Lemma 3.2, for function $g(z) = 1 + zf''(z)/f'(z)$, we have

$$\int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 dtd\rho \leqslant 2\pi \sum_{k=0}^{\infty} \delta_k^2 \left(\frac{r^{2k+1}}{2k+1}\right), \tag{4.3}$$

where $\delta_0 = 1$ and $\delta_k = 1/k, k = 1, 2, 3, \cdots$. Thus, from (4.3), we have

$$\begin{aligned} \int_0^r \int_0^{2\pi} \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 dtd\rho &\leqslant 2\pi\left(r + \sum_{k=1}^{\infty} \frac{1}{k^2}\left(\frac{r^{2k+1}}{2k+1}\right)\right) \\ &= 2\pi\left(r + r\mathrm{Li}_2\left(r^2\right) + 2r\left(\log\left(1-r^2\right) - 2\right) + 4\tanh^{-1}(r)\right) \\ &= 2\pi\left(r\mathrm{Li}_2\left(r^2\right) + 2r\log\left(1-r^2\right) + 4\tanh^{-1}(r) - 3r\right), \end{aligned} \tag{4.4}$$

where $\mathrm{Li}_2\left(r^2\right) = \sum_{k=1}^{\infty}(r^{2k}/k^2)$. Now the results follows from (4.1), (4.2), (4.3) and (4.4). Other parts of this theorem have similar proofs and therefore we skip them here. □

Theorem 4.2 *Let us assume that*

$$M_c(r,f) = \max\{|f(z)| : f \in \mathcal{S}_c^*,\ |z| \leqslant r < 1\},$$
$$M_l(r,f) = \max\{|f(z)| : f \in \mathcal{S}_l^*,\ |z| \leqslant r < 1\}$$

and

$$M_e(r,f) = \max\{|f(z)| : f \in \mathcal{S}_e^*,\ |z| \leqslant r < 1\}.$$

1. *If $f \in \mathcal{S}_c^*$, then*

$$L(r,f) \leqslant 2\pi M_c(r,f)\left(1 + Li_2\left(r^2\right)\right)^{1/2}.$$

2. *If $f \in \mathcal{S}_l^*$, then*

$$L(r,f) \leqslant 2\pi M_l(r,f)\left({}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; r^2\right)\right)^{1/2}.$$

3. *If $f \in \mathcal{S}_e^*$, then*

$$L(r,f) \leqslant 2\pi M_e(r,f)\left(J_0(2x)\right)^{1/2}.$$

Proof. (1) Let $f \in \mathcal{S}_c^*$. Then

$$p(z) = \frac{zf'(z)}{f(z)} \prec 1 + \sum_{k=1}^{\infty}\frac{z^k}{k}.$$

Now, we have

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)|dt \\ &= \int_0^{2\pi} |p(z)f(z)|dt \\ &= \left(\int_0^{2\pi} |f(z)|^2 dt\right)^2 \left(\int_0^{2\pi} |p(z)|^2 dt\right)^2. \end{aligned} \tag{4.5}$$

Also note that, if we denote $M_c(r,f) = \max_{|z|\leqslant r<1}|f(z)|$, then

$$\int_0^{2\pi} |f(z)|^2 dt \leqslant 2\pi M_c(r,f)^2. \tag{4.6}$$

Setting $g(z) = p(z)$ and using Lemma 3.1 and Lemma 3.2, we have

$$\int_0^{2\pi} |p(z)|^2\ dt \leqslant 2\pi\left(1 + \sum_{k=1}^{\infty}\frac{1}{k^2}r^{2k}\right). \tag{4.7}$$

Now the result follows from (4.5), (4.6) and (4.7). The rest parts of this theorem have similar proofs and therefore we skip them here. □

Now at the end of this manuscript, for function f either in the class $\mathcal{K}(\varphi)$ or $\mathcal{S}^*(\varphi)$, we shall find the estimate for integral means

$$I(r) = \int_0^{2\pi} |f'(z)|^\mu dt \text{ and } I(r) = \int_0^{2\pi} |f(z)|^\mu dt, \quad \mu > 0,$$

respectively. These results will be applied to the classes $\mathcal{K}[A,B]$ or $\mathcal{S}^*[A,B]$. To discuss these results we need the following results due to Ma and Minda:

Lemma 4.3 [7, Theorem 1'] *Let h_φ be a solution of the differential equation $zh'_\varphi(z)/h_\varphi(z) = \varphi(z)$. If $f \in \mathcal{S}^*(\varphi)$, then $zf'(z)/f(z) \prec zh'_\varphi(z)/h_\varphi(z)$ and $f(z)/z \prec h_\varphi(z)/z$.*

Lemma 4.4 [7, Theorem 1] *Let k_φ be a solution of the differential equation $zk'_\varphi(z)/k_\varphi(z) = \varphi(z)$. If $f \in \mathcal{K}(\varphi)$, then $1 + zf''(z)/f'(z) \prec 1 + zk''_\varphi(z)/k'_\varphi(z)$ and $f'(z) \prec k'_\varphi(z)$.*

As a straight forward applications of Lemmas 3.2, 4.3 and 4.4, we have

Theorem 4.5 *If $f \in \mathcal{S}^*(\varphi)$ and h_φ is a solution of the differential equation $zh'_\varphi(z)/h_\varphi(z) = \varphi(z)$, then, for $\mu > 0$, we have*

$$\int_0^{2\pi} |f(z)|^\mu dt \leqslant \int_0^{2\pi} |h_\varphi(z)|^\mu dt.$$

Theorem 4.6 *If $f \in \mathcal{K}(\varphi)$ and k_φ is a solution of the differential equation $1 + zk''_\varphi(z)/k'_\varphi(z) = \varphi(z)$, then, for $\mu > 0$, we have*

$$\int_0^{2\pi} |f'(z)|^\mu dt \leqslant \int_0^{2\pi} |k'_\varphi(z)|^\mu dt.$$

Let $\varphi(z) = (1 + Az)/(1 + Bz), -1 \leqslant B < A \leqslant 1$. Then $zh'_\varphi(z)/h_\varphi(z) = (1+Az)/(1+Bz)$, gives h_φ as defined in (2.1), and solution k_φ of the equation $1 + zk''_\varphi(z)(z)/k'_\varphi(z) = (1 + Az)/(1 + Bz)$ is given by (2.2). Thus in view of Theorem 4.5, if $f \in \mathcal{S}^*[A,B]$, then for $z = re^{it}$ $(0 < r < 1,\ 0 \leqslant t \leqslant 2\pi)$, we have

$$\int_0^{2\pi} |f(z)| dt \leqslant \begin{cases} \int_0^{2\pi} r^\mu \left|(1 + Bre^{it})\right|^{\left(\frac{A}{B}-1\right)\mu} dt, & B \neq 0; \\ \int_0^{2\pi} r^\mu e^{\mu Ar\cos t} dt, & B = 0 \end{cases} \tag{4.8}$$

and for $f \in \mathcal{K}[A,B]$, Theorem 4.6 gives

$$\int_0^{2\pi} |f'(z)| dt \leqslant \begin{cases} \int_0^{2\pi} \left|1 + Bre^{it}\right|^{\left(\frac{A}{B}-1\right)\mu} dt, & B \neq 0; \\ \int_0^{2\pi} e^{\mu Ar\cos t} dt, & B = 0. \end{cases} \tag{4.9}$$

Both results are sharp. Further, since the functions h_φ and k_φ are univalent and non-vanishing in $0 < |z| \leqslant r < 1$, it follows that if $f \in \mathcal{S}^*[A,B]$, then $z/f(z) \prec z/h_\varphi(z)$, and if $f \in \mathcal{K}[A,B]$, then $1/f'(z) \prec 1/k'_\varphi(z)$. Thus, we conclude that Inequalities (4.8) and (4.9) are true for all real μ. In particular, if $f \in \mathcal{S}^*$, then, for all real μ,

$$\begin{aligned}\int_0^{2\pi} |f(z)|dt &\leqslant \int_0^{2\pi} r^{\mu}|1-re^{it}|^{-2\mu}dt \\ &\leqslant \int_0^{2\pi} |1-re^{it}|^{-2\mu}dt \\ &= \int_0^{2\pi} |1+re^{it}|^{-2\mu}dt. \end{aligned} \tag{4.10}$$

Note that Inequality (4.10) was proved by Robertson [Theorem 4.4, [14]] in 1936.

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References

1. Ahlfors, L.V. – *Complex analysis. An introduction to the theory of analytic functions of one complex variable. Third edition.*, International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York (1978), 331 pp.
2. Cho, N.E.; Kwon, O.S.; Ravichandran, V. – *Coefficient, distortion and growth inequalities for certain close-to-convex functions*, J. Inequal. Appl. (2011), Art. ID 100, 7 pp.
3. Goodman, A.W. – *Univalent functions. Vol. II*, Mariner Publishing Co., Inc., Tampa, FL (1983), 311 pp.
4. Hayman, W.K. – *On functions with positive real part,* J. London Math. Soc. **36** (1961), 35-48.
5. Janowski, W. – *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math. **23** (1970/1971), 159-177.
6. Keogh, F.R. – *Some theorems on conformal mapping of bounded star-shaped domains*, Proc. London Math. Soc. **9** (1959), no. 3, 481-491.
7. Ma, W.C.; Minda, D. – *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis (Tianjin), 1992 (Z. Li, F.Y. Ren, L. Yang, S.Y. Zhang, eds.), Conf. Proc. Lecture Notes Anal., Vol. 1, Int. Press, Massachusetts, 1994, 157-169.
8. Meng, F.; Najva, A.; Yuan, W. – *On some length problems for univalent functions*, Pac. J. Appl. Math. **8** (2016), no. 1, 59-63.
9. Nunokawa, M. – *On Bazilevič and convex functions*, Trans. Amer. Math. Soc. **143** (1969), 337-341.
10. Nunokawa, M. – *A note on convex and Bazilevič functions*, Proc. Amer. Math. Soc. **24** (1970), 332-335.
11. Nunokawa M.; Sokół, J. – *On some length problems for analytic functions*, Osaka J. Math. **51** (2014), no. 3, 695-707.
12. Nunokawa, M.; Sokół, J. – *On some length problems for univalent functions,* Math. Methods Appl. Sci. **39** (2016), no. 7, 1662-1666.
13. Pommerenke, C. – *Über nahezu konvexe analytische funktionen* (German), Arch. Math. (Basel) **16** (1965), 344-347.
14. Robertson, M.I.S. – *On the theory of univalent functions*, Ann. of Math. **37** (1936), no. 2, 374-408.

15. Rogosinski, W. – *On the coefficients of subordinate functions*, Proc. London Math. Soc. **48** (1943), no. 2, 48-82.
16. Sokół J.; Nunokawa, M. – *On some new length problems for analytic functions*, Hacet. J. Math. Stat. **46** (2017), no. 3, 427-435.
17. Thomas, D.K. – *On starlike and close-to-convex univalent functions*, J. London Math. Soc. **42** (1967), 427-435.
18. Thomas, D.K. – *On Bazilevič functions,* Trans. Amer. Math. Soc. **132** (1968), 353-361.
19. Thomas, D.K. – *A note on starlike functions*, J. London Math. Soc. **43** (1968), 703-706.
20. Ul-Haq, W.; Noor, K.I. – *A certain class of analytic functions and the growth rate of Hankel determinant*, J. Inequal. Appl. **2012**, Art. ID 309, 11 pp.
21. Ul-Haq, W.; Arif, M.; Khan, A. – *Arc length inequality for a certain class of analytic functions related to conic regions*, J. Complex Anal. **2013**, Art. ID 407596, 4 pp.
22. Wang, Z.; Gao, C.; Yuan, S. – *On certain subclass of close-to-convex functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **22** (2006), no. 2, 171-177.

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