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Generalized para-coKähler structures

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Abstract In this paper, we define and characterize generalized almost para-contact pseudo-metric structures. Then, we define normal generalized para-contact pseudo-metric structures and generalized para-coKähler structures and give some examples of them.

Keywords Generalized almost para-contact structure · Generalized para-Kähler structure

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1 Introduction

The notion of the generalized complex structure, introduced by Hitchin ([4]), is a geometric framework that unifies both complex and symplectic structures. It was developed by his student Gualtieri as the generalized Kähler structure ([3]).

Vaisman introduced the odd dimensional analog of these structures as a generalized almost contact structure ([10]). Poon and Wade studied integrability conditions of generalized almost contact structures. This framework unifies almost contact, contact and cosymplectic structures ([7], [8]).

In the case of para-structures, S. Kaneyuki with his colleague ([5],[6]) defined the almost para-contact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ and then constructed the almost para-complex structure on $M^{(2n+1)} \times \mathbb{R}$. B. Sahin and F. Sahin introduced the generalized almost para-contact manifold and obtained normality conditions in terms of classical tensor fields ([9]), these concepts are defined without any use of the definition of a meter. Buchner and R. Rosca defined the para-coKähler structure ([2]).

Vaisman defined the generalized almost para-Hermitian structure to be a commuting pair $(\mathcal{F}, \mathcal{J})$ of the generalized almost para-complex structure and the generalized almost complex structure with an adequate non-degeneracy condition. If the two structures are integrable the pair is called the generalized para-Kähler structure. He also deduced integrability conditions similar to those of the generalized Kähler structures ([11]).

According to what has been said, it is natural to use some idea from these articles and define an odd dimensional analog of a generalized para-Kähler structures.

In this paper, we define and characterize generalized almost para-contact pseudo-metric structures. Then, we define normal generalized para-contact pseudo-metric structures and generalized para-coKähler structures.

This paper is divided into three sections. In Section 2, we recall the needed background including definitions and theorems about generalized structures. In Section 3, we define normal generalized para-contact pseudo-metric structures and generalized para-coKähler structures and show that generalized almost para-contact pseudo-metric structures $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ are in a one-to-one correspondence with quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, where γ is a pseudo-Riemannian metric of M, ψ is a $2-$form and φ is a $(1,1)-$ tensor field such that $\varphi^2 = -Id + \eta \otimes \xi$, $\gamma(\varphi X, Y) = -\gamma(X, \varphi Y)$, $\gamma^\sharp(\eta) = \xi$ and $\psi^\flat(\xi) = 0$. Then, we give equivalent conditions, by which H and $\bar{H}$ be closed under the Courant bracket.

2 Preliminaries

let M be a smooth manifold and consider the big tangent bundle $\mathbb{T}M = TM \oplus TM^*$. A natural inner product on $\mathbb{T}M = TM \oplus TM^*$ is defined by

$$\langle X+\alpha, Y+\beta\rangle = \mathrm{g}(X+\alpha, Y+\beta) = \frac{1}{2}(\beta(X) + \alpha(Y)),$$

and the Courant bracket by

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \pounds_X\beta - \pounds_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta), \tag{2.1}$$

where $X, Y \in TM$ and $\alpha, \beta \in TM^*$. A subbundle of $TM \oplus TM^*$ is said to be involutive if its sections are closed under the Courant bracket.

A generalized almost complex structure on M is an endomorphism $\mathcal{J}$ of $TM \oplus TM^*$ such that $\mathcal{J} + \mathcal{J}^* = 0$ and $\mathcal{J}^2 = -Id$. Since $\mathcal{J}^2 = -Id$, $\mathcal{J}$ has eigenvalues $\pm\sqrt{-1} = \pm \mathrm{i}$. Let $E \subset \mathbb{T}M \otimes \mathbb{C}$ be the $\mathrm{i}-$ eigenbundle of $\mathcal{J}$, E is maximal isotropic with respect to $\langle , \rangle$ and it satisfies $E \cap \bar{E} = 0$ (the bar denotes complex conjugation). Conversely, any such maximal isotropic subbundle E of $\mathbb{T}M \otimes \mathbb{C}$ defines a generalized almost complex structure on M. $\mathcal{J}$ is called a generalized complex structure (or, $\mathcal{J}$ is integrable) if E is involutive ([3]). The integrability of $\mathcal{J}$ amounts the nullity of the Nijenhuis tensor of $\mathcal{J}$, i.e. for any $X+\alpha, Y+\beta \in \Gamma(E)$, we have

$$\begin{aligned} N_{\mathcal{J}}(X+\alpha, Y+\beta) =& [\![\mathcal{J}(X+\alpha), \mathcal{J}(Y+\beta)]\!] + \mathcal{J}^2[\![X+\alpha, Y+\beta]\!] \\ & - \mathcal{J}([\![X+\alpha, \mathcal{J}(Y+\beta)]\!] - \mathcal{J}[\![\mathcal{J}(X+\alpha), Y+\beta]\!]) = 0. \end{aligned}$$

The analog of generalized almost complex structure for odd dimensional spaces is generalized almost contact structure. We mention here some of the fundamental generalized geometric structures for odd dimensional spaces. But first, it will be worthwhile to recall the formal definition of geometric structures for odd dimensional spaces on a manifold to use them in generalized cases.

First, let review some classical differential geometric structures. Let M^{2n+1} be a smooth manifold with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, then the 1-form η is a contact structure or a contact 1-form. Given a contact 1-form, there is a unique vector field ξ such that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. This vector field is known as the Reeb vector field of the contact form η.

An almost contact metric structure on M is given by tensors (φ, ξ, η, g) where φ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on M, satisfying the following conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.2}$$

and where g is a Riemannian metric compatible with almost contact structure, that means

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for vector fields X and Y on M. We can use the Riemannian metric g and the $(1,1)$ tensor field φ to construct the fundamental 2-form $\Theta(X,Y) = g(X, \varphi Y)$. An almost contact metric structure (φ, ξ, η, g) is called a contact metric structure iff $\Theta = d\eta$. Furthermore, an almost contact metric structure on M is called normal if the Nijenhuis tensor of φ given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [\varphi X, \varphi Y],$$

satisfies $N_\varphi = -2\xi \otimes d\eta$ ([1]). Now, we are ready to return to the definition of generalized structures on odd dimensional spaces.

In generalized geometry case, using the definition given in [8], a pair $(\Phi, Z + \eta)$ is called a generalized almost contact structure iff

$$\begin{aligned} &\Phi + \Phi^* = 0, \ \ \Phi^2 = -I + Z \odot \eta, \\ &\eta(Z) = 1, \ \ \Phi(Z) = 0 \ \ \textit{and}\ \Phi(\eta) = 0, \end{aligned} \tag{2.3}$$

where Φ is an endomorphism of $TM \oplus TM^*$, and $Z + \eta$ is a section of $TM \oplus TM^*$ and $Z \odot \eta(X + \alpha) := \eta(X)Z + \alpha(Z)\eta$, for any $X + \alpha \in \Gamma(\mathbb{TM})$. Given a generalized almost contact pair $(\Phi, Z + \eta)$, we define

$$\begin{aligned} E^{(1,0)} &= \{X + \alpha - \mathrm{i}\Phi(X + \alpha) | X + \alpha \in \ker \eta \oplus \ker Z\}, \\ E^{(0,1)} &= \{X + \alpha + \mathrm{i}\Phi(X + \alpha) | X + \alpha \in \ker \eta \oplus \ker Z\}. \end{aligned} \tag{2.4}$$

The endomorphism Φ is linearly extended to the complexified bundle $\mathbb{T}M \otimes \mathbb{C}$. It has three eigenvalues, namely, $\lambda = 0$ and i and $-$i. The corresponding eigenbundles are

$$L_Z \oplus L_\eta, \ E^{(1,0)} \ \ \textit{and} \ \ E^{(0,1)}, \tag{2.5}$$

respectively, where L_Z and L_η are the complex vector bundles of rank 1 generated by Z and η. Define

$$L := L_Z \oplus E^{(1,0)}, \quad L^* := L_\eta \oplus E^{(0,1)},$$
$$\bar{L} := L_Z \oplus E^{(0,1)}, \quad \bar{L}^* := L_\eta \oplus E^{(1,0)}. \tag{2.6}$$

The generalized almost contact pair $(\Phi, Z+\eta)$ is a generalized contact structure or $(\Phi, Z+\eta)$ is integrable, if L is involutive. Morever, if both L and L^* be involutive, the pair $(\Phi, Z+\eta)$ is called a strong generalized contact structure. The strong generalized contact structure $(\Phi, Z+\eta)$ is called a normal generalized contact structure if $\pounds_Z \eta = 0$ ([8]).

A generalized almost para-complex structure is an endomorphism $\mathcal{F}$ of $TM \oplus TM^*$ such that it is skew-symmetric with respect to the metric $\langle , \rangle$ and $\mathcal{F}^2 = Id$ ([11]), and a generalized almost para-Hermitian structure is a commuting pair $(\mathcal{F}, \mathcal{J})$, where $\mathcal{F}$ is a generalized almost para-complex structure and $\mathcal{J}$ is a generalized almost complex structure, such that the symmetric bivector $\gamma(\alpha, \beta) = -2\langle \mathcal{F}(0,\alpha), \mathcal{J}(0,\beta)\rangle$ is non-degenerate. If $\mathcal{F}$ is integrable, the structure is generalized para-Hermitian. If $\mathcal{J}$ is integrable, the structure is generalized almost para-Kähler. If both $\mathcal{F}$ and $\mathcal{J}$ are integrable, the structure is generalized para-Kähler. Vaisman shows ([11]) that a generalized almost para-Hermitian structure is equivalent to a triple (γ, ψ, P), where γ is a (pseudo) Riemannian metric on M, ψ is a $2-$form and $P \in T^c M$ (the index c denotes complexification) satisfying $P^2 = Id$ and $\gamma(PX, Y) + \gamma(X, PY) = 0$. The generalized para-Kähler structure associated to (γ, ψ, P) is characterized by the involutivity of the eigenbundles of the endomorphism P together with

$$d\psi(X, Y, Z) = \mathrm{i}\; d\omega(PX, PY, PZ), \tag{2.7}$$

where $\omega(X, Y) = \gamma(X, PY)$.

The analog of generalized almost para-complex structure for odd dimensional spaces is generalized almost para-contact structure. We mention here some of the fundamental generalized geometric para-structures for odd dimensional spaces.

A pair $(\mathcal{A}, Z+\eta)$, for an odd dimensional manifold M, is called a generalized almost para-contact structure ([9]) iff

$$\mathcal{A} + \mathcal{A}^* = 0, \;\; \mathcal{A}^2 = Id - Z \odot \eta,$$
$$\eta(Z) = 1, \;\; \mathcal{A}(Z) = 0 \;\; \textit{and}\; \mathcal{A}(\eta) = 0, \tag{2.8}$$

where $\mathcal{A}$ is an endomorphism of $TM \oplus TM^*$, and $Z + \eta$ is a section of $TM \oplus TM^*$ and $Z \odot \eta(X+\alpha) := \eta(X)Z + \alpha(Z)\eta$, for any $X + \alpha \in \Gamma(\mathbb{T}M)$. Given a generalized almost para-contact pair $(\mathcal{A}, Z+\eta)$, we define

$$E^{(1,0)}_{\mathcal{A}} = \{X + \alpha - \mathcal{A}(X+\alpha) | X + \alpha \in \ker\eta \oplus \ker Z\},$$
$$E^{(0,1)}_{\mathcal{A}} = \{X + \alpha + \mathcal{A}(X+\alpha) | X + \alpha \in \ker\eta \oplus \ker Z\}.$$

The endomorphism $\mathcal{A}$ is linearly extended to the complexified bundle $\mathbb{T}M \otimes \mathbb{C}$. It has three eigenvalues, namely, 0, 1 and -1. The corresponding eigenbundles are $L_Z \oplus L_\eta$, $E_{\mathcal{A}}^{(1,0)}$ and $E_{\mathcal{A}}^{(0,1)}$, where L_Z and L_η are vector bundles of rank 1 generated by Z and η, respectively. Define

$$L_{\mathcal{A}} := L_Z \oplus E_{\mathcal{A}}^{(1,0)}, \quad L_{\mathcal{A}}^* := L_\eta \oplus E_{\mathcal{A}}^{(0,1)},$$
$$\bar{L}_{\mathcal{A}} := L_Z \oplus E_{\mathcal{A}}^{(0,1)}, \quad \bar{L}_{\mathcal{A}}^* := L_\eta \oplus E_{\mathcal{A}}^{(1,0)}.$$

The generalized almost para-contact pair $(\mathcal{A}, Z + \eta)$ is called a generalized para-contact structure, or $(\mathcal{A}, Z + \eta)$ is integrable, if $L_{\mathcal{A}}$ is involutive. Morever, if both $L_{\mathcal{A}}$ and $L_{\mathcal{A}}^*$ be involutive, the pair $(\mathcal{A}, Z + \eta)$ is called a strong generalized para-contact structure. The strong generalized para-contact structure $(\mathcal{A}, Z + \eta)$ is called a normal generalized para-contact structure if $\pounds_Z \eta = 0$.

3 Genaralized para-coKähler manifolds

In classical differential geometry, an almost para-contact structure (F, ξ, η) on a $(2n+1)-$ dimensional manifold M^{2n+1} consistes of a tensor field F of type $(1,1)$, a vector field ξ and a $1-$form η satisfying the following compatibility conditions

(i) $F(\xi) = 0, \;\; \eta \circ F = 0$,
(ii) $\eta(\xi) = 1, \;\; F^2 = Id - \eta \otimes \xi$,
(iii) The tensor field F induces an almost para-complex structure on each fibre of $\mathbb{D} = \ker \eta$.

If an almost para-contact manifold (M^{2n+1}, F, ξ, η) admits a pseudo-Riemannian metric γ such that

$$\gamma(FX, FY) = -\gamma(X, Y) + \eta(X)\eta(Y),$$

then we say that M^{2n+1} has an almost para-contact pseudo-metric structure and γ is called a compatible metric. Any compatible metric γ with a given almost para-contact structure is necessarily of signature $(n, n+1)$. The fundamental $2-$form $\Omega(X, Y) = \gamma(X, FY)$ is non-degenerate on the distribution $\mathbb{D}$ and $\eta \wedge \Omega^n \neq 0$ ([12]). If $\gamma(X, FY) = d\eta(X, Y)$, then η is a para-contact form and the almost para-contact pseudo-metric manifold $(M, \varphi, \xi, \eta, \gamma)$ is said to be para-contact pseudo-metric manifold.

Furthermore, if the $2-$form $\Omega(X, Y) = \gamma(X, FY)$ and $1-$form η are closed in an almost para-contact pseudo-metric manifold, the structure is almost para-coKähler and if, in addition, $N_F = 0$ where N_F is the Nijenhuis tensor of F given by

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y],$$

the structure is para-coKähler ([2]).

In an almost para-contact pseudo-metric manifold $(M, F, \xi, \eta, \gamma)$, with the musical isomorphisms $\gamma^\flat, \Omega^\flat, \gamma^\sharp = (\gamma^\flat)^{-1}$ *and* $\Omega^\sharp = -(\Omega^\flat)^{-1}$, we have

$$\begin{aligned} &\gamma^\flat \circ F = -F^* \circ \gamma^\flat, \quad F \circ \gamma^\sharp = -\gamma^\sharp \circ F^*, \\ &\Omega^\flat = F^* \circ \gamma^\flat = -\gamma^\flat \circ F, \quad \Omega^\sharp = -\gamma^\sharp \circ F^* = F \circ \gamma^\sharp, \\ &\Omega^\flat \circ \gamma^\sharp = -\gamma^\flat \circ \Omega^\sharp = F^*, \quad \Omega^\sharp \circ \gamma^\flat = -\gamma^\sharp \circ \Omega^\flat = F. \end{aligned} \tag{3.1}$$

These objects may be encoded in the endomorphisms $\Phi, \mathcal{H}, \mathcal{A} \in End(\mathbb{T}M)$, given by the action of the matrices

$$\Phi = \begin{pmatrix} 0 & -F \circ \gamma^\sharp \\ \gamma^\flat \circ F & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} F & 0 \\ 0 & -F^* \end{pmatrix}, \tag{3.2}$$

on columns $\begin{pmatrix} X \\ \alpha \end{pmatrix}$, $X \in TM$ and $\alpha \in T^*M$. Then, Φ and $\mathcal{A}$ are skew-symmetric with respect to the pairing metric $\langle , \rangle$ and $\Phi^2 = -Id + \eta \odot \xi$, $\Phi(\xi) = 0 = \Phi(\eta)$ and $\mathcal{A}^2 = Id - \eta \odot \xi$, $\mathcal{A}(\xi) = 0 = \mathcal{A}(\eta)$, which means that $(\Phi, \xi + \eta)$ is a generalized almost contact structure and $(\mathcal{A}, \xi + \eta)$ is a generalized almost para-contact structure. The endomorphism $\mathcal{H}$ satisfies the properties

$$\mathcal{H}^2 = -Id, \tag{3.3}$$

$$\langle \mathcal{H}X, Y \rangle = \langle X, \mathcal{H}Y \rangle, \quad (\text{g-}\textit{symmetric}) \tag{3.4}$$

$$\mathcal{H} \circ \Phi = \Phi \circ \mathcal{H} = \mathcal{A}, \tag{3.5}$$

$$\mathcal{H}(\xi) = -\eta, \quad \mathcal{H}(\eta) = \xi. \tag{3.6}$$

These observations suggest the following definition:

Definition 3.1 *A generalized almost para-contact pseudo-metric structure is a quadruple $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$, where $(\Phi, \xi+\eta)$ is a generalized almost contact structure, $\mathcal{H}$ is a* g*−symmetric matrix with $\mathcal{H}^2 = -Id$, $\mathcal{H}(\eta) = \xi$, $\Phi \circ \mathcal{H} = \mathcal{H} \circ \Phi = \mathcal{A}$ and $(\mathcal{A}, \xi + \eta)$ is a generalized almost para-contact structure and*

$$\gamma(\alpha, \beta) := 2\langle \mathcal{H}(0, \alpha), (0, \beta) \rangle, \tag{3.7}$$

$$\nu(X, Y) := 2\langle \mathcal{H}(X, 0), (Y, 0) \rangle \tag{3.8}$$

are non-degenerate.

Definition 3.2 *In the generalized almost para-contact pseudo-metric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$, if $(\Phi, \xi+\eta)$ is normal generalized contact structure, the structure is generalized almost para-coKähler. If $(\mathcal{A}, \xi+\eta)$ is normal generalized para-contact structure, the structure is normal generalized para-contact pseudo-metric. If both $(\Phi, \xi + \eta)$ and $(\mathcal{A}, \xi + \eta)$ are normal , the generalized almost para-contact pseudo-metric structure is generalized para-coKähler.*

Let $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ be a generalized almost para-contact pseudo-metric structure. We write down the following matrix representations of $\Phi, \mathcal{H}, \mathcal{A}$:

$$\Phi = \begin{pmatrix} P & \phi^\sharp \\ \theta^\flat & -P^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} Q & \gamma^\sharp \\ \nu^\flat & Q^* \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & \pi^\sharp \\ \sigma^\flat & -A^* \end{pmatrix}, \tag{3.9}$$

where $\phi, \theta, \pi, \sigma$ are skew-symmetric, $\gamma \in End(TM)$ and $\nu \in End(T^*M)$ are symmetric $2-$vector field and symmetric $2-$covector fields, rspectively. Also, we get

$$\begin{aligned}
&P^2 = -Id - \phi^\sharp \circ \theta^\flat + \eta \otimes \xi, \;\; \phi^\sharp \circ P^* = P \circ \phi^\sharp, \;\; \theta^\flat \circ P = P^* \circ \theta^\flat, \\
&A^2 = Id - \pi^\sharp \circ \sigma^\flat - \eta \otimes \xi, \;\; \pi^\sharp \circ A^* = A \circ \pi^\sharp, \;\; \sigma^\flat \circ A = A^* \circ \sigma^\flat, \\
&Q^2 = -Id - \gamma^\sharp \circ \nu^\flat, \;\; \gamma^\sharp \circ Q^* = -Q \circ \gamma^\sharp, \;\; \nu^\flat \circ Q = -Q^* \circ \nu^\flat, \\
&A = P \circ Q + \phi^\sharp \circ \nu^\flat = Q \circ P + \gamma^\sharp \circ \theta^\flat, \\
&\pi^\sharp = P \circ \gamma^\sharp + \phi^\sharp \circ Q^* = Q \circ \phi^\sharp - \gamma^\sharp \circ P^*, \\
&\sigma^\flat = \nu^\flat \circ P + Q^* \circ \theta^\flat = \theta^\flat \circ Q - P^* \circ \nu^\flat,
\end{aligned} \tag{3.10}$$

these relations ensure that $\Phi, \mathcal{H}, \mathcal{A}$ are structures of the required type and $\Phi \circ \mathcal{H} = \mathcal{A}$.

Define $\tau : T^cM \to T^cM \oplus T^cM^*$ by

$$\tau(X' + \mathrm{i}X'') := (X' + \mathrm{i}X'', -\gamma^\flat(X'' + QX') + \mathrm{i}\gamma^\flat(X' - QX'')). \tag{3.11}$$

Using formulas (3.10), we have

$$\begin{aligned}
\mathcal{H}(\tau(X' + \mathrm{i}X'')) =& \mathcal{H}(X', -\gamma^\flat(X'' + QX')) + \mathrm{i}\mathcal{H}(X'', \gamma^\flat(X' - QX'')) \\
=& (-X'', \nu^\flat X' - Q^*\gamma^\flat(X'' + QX')) + \mathrm{i}(X', \nu^\flat X'' \\
&+ Q^*\gamma^\flat(X' - QX'')) \\
=& \mathrm{i}(X' + \mathrm{i}X'', -\gamma^\flat(X'' + QX') + \mathrm{i}\gamma^\flat(X' - QX'')) \\
=& \mathrm{i}\tau(X' + \mathrm{i}X'').
\end{aligned}$$

Thus, $im\tau \subset H$ in which $H = im(Id - \mathrm{i}\mathcal{H})$ is the $\mathrm{i}-$eigenspace of $\mathcal{H}$. If we put $\tau(X' + \mathrm{i}X'') = (X' + \mathrm{i}X'', \alpha' + \mathrm{i}\alpha'')$, then we have

$$QX' + \gamma^\sharp\alpha' = -X'', \;\; QX'' + \gamma^\sharp\alpha'' = X', \tag{3.12}$$

$$\nu^\flat X' + Q^*\alpha' = -\alpha'', \;\; \nu^\flat X'' + Q^*\alpha'' = \alpha'. \tag{3.13}$$

Now, by non-degeneracy of $\nu^\flat$ and (3.13), it is evident that $\tau : T^cM \to H$ is an isomorphism.

By defining $\psi^\flat = -\gamma^\flat \circ Q$, we get ψ as a $2-$form, and using (3.11), we have

$$\tau(X) = (X, (\psi^\flat + \mathrm{i}\gamma^\flat)X). \tag{3.14}$$

An interesting consequence of formula (3.14) is

$$\gamma(X,Y) = -\mathrm{i}\,\langle \tau X, \tau Y\rangle, \tag{3.15}$$

where $X, Y \in T^cM$ (we should have written γ^{-1}, but, we follow the custom of Riemannian geometry).

If we define $\bar{\tau} : T^cM \to T^cM \oplus T^cM^*$ by

$$\bar{\tau}(X' + \mathrm{i}X'') := (X' + \mathrm{i}X'', -\gamma^\flat(QX' - X'') - \mathrm{i}\gamma^\flat(X' + QX'')), \tag{3.16}$$

similarly, one can see $im\bar{\tau} \subset \bar{H}$ in which $\bar{H} = im(Id + \mathrm{i}\mathcal{H})$ is the $(-\mathrm{i})-$eigen -space of $\mathcal{H}$ and

$$i)\ \bar{\tau}(X) = (X, (\psi^\flat - \mathrm{i}\gamma^\flat)X), \quad ii)\ \gamma(X,Y) = \mathrm{i}\,\langle \bar{\tau}X, \bar{\tau}Y\rangle. \tag{3.17}$$

It is interesting to emphasize that there may be generalized almost para-contact pseudo-metric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ such that $(\mathcal{A}, \xi + \eta)$ is normal, but $(\Phi, \xi + \eta)$ is not normal as the following example shows.

Example 3.1 let $M = \mathbb{R}^3$ and choose a local frame $\{X_1, X_2, X_3\}$ and a dual basis $\{\sigma^1, \sigma^2, \sigma^3\}$ such that

$$[X_3, X_1] = -X_2,\ [X_3, X_2] = X_1 \text{ and } [X_1, X_2] = 0,$$

thus $d\sigma^2 = \sigma^1 \wedge \sigma^3$, $d\sigma^1 = -\sigma^2 \wedge \sigma^3$ and $d\sigma^3 = 0$. One can construct a generalized contact structure associated to an almost contact structure $(\varphi, \xi, \eta = \sigma^3)$ given by

$$\xi = X_3, \quad \varphi = \sigma^1 \otimes X_2 - \sigma^2 \otimes X_1,$$

by setting $\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}$, then we set

$$L = span\{X_3, X_1 - \mathrm{i}X_2, \sigma^1 - \mathrm{i}\sigma^2\}, \quad L^* = span\{\sigma^3, X_1 + \mathrm{i}X_2, \sigma^1 + \mathrm{i}\sigma^2\}.$$

By an elementary computation we can verify that the structure equations for subbundles L and L^* respectively are given by

$$[\![X_3, X_1 - \mathrm{i}X_2]\!] = -\mathrm{i}(X_1 - \mathrm{i}X_2), \quad [\![X_3, \sigma^1 - \mathrm{i}\sigma^2]\!] = \mathrm{i}(\sigma^1 - \mathrm{i}\sigma^2),$$
$$[\![X_1 - \mathrm{i}X_2, \sigma^1 - \mathrm{i}\sigma^2]\!] = 0 = [\![X_1 + \mathrm{i}X_2, \sigma^1 + \mathrm{i}\sigma^2]\!],$$

and $\mathcal{L}_\xi \eta = 0$. Thus $(\Phi, \xi + \eta)$ is a normal generalized contact structure.

Now, define a g$-$symmetric matrix $\mathcal{H}$ on $TM \oplus TM^*$ by $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ in which $\gamma(X_i, X_j) = \delta_{ij}$ for $i = 1, 2, 3$. Then we have $\mathcal{H}(X_i) = -\sigma^i$ and $\mathcal{H}(\sigma^i) = X_i$, for $i = 1, 2, 3$. So that $(\Phi \circ \mathcal{H} = \mathcal{A}, \xi + \eta)$ defines a generalized almost para-contact structure and $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ defines a generalized almost para-contact pseudo-metric structure which is normal. Now it is evident that

$$L_{\mathcal{A}} = span\{X_3, X_1 + \sigma^2, \sigma^1 - X_2\}.$$

The Courant brackets for $L_{\mathcal{A}}$ are

$$[\![X_3, X_1 + \sigma^2]\!] = -(\sigma^1 - X_2), \; [\![X_3, \sigma^1 - X_2]\!] = -(X_1 - \sigma^2),$$
$$[\![\sigma^1 - X_2, X_1 + \sigma^2]\!] = 0.$$

The phrase $[\![X_3, \sigma^1 - X_2]\!] = -(X_1 - \sigma^2) \notin L_{\mathcal{A}}$ showes that $(\mathcal{A}, \xi + \eta)$ is not integrable, thus the generalized almost para-coKähler structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is not normal generalized para-contact pseudo-metric structure.

In the following, we give an example of a generalized para-coKähler structure.

Example 3.2 On the three-dimensional Heisenberg group H_3, choose a basis $\{X_1, X_2, X_3\}$ for its algebra $\mathfrak{h}_3$ so that $[X_1, X_2] = -X_3$. Let $\{\sigma^1, \sigma^2, \sigma^3\}$ be a dual frame. Then $d\sigma^3 = \sigma^1 \wedge \sigma^2$. Now, we define a generalized almost contac structure assosiated to the following cosymplectic structure

$$\eta = \sigma^1 \quad and \quad \theta = \sigma^2 \wedge \sigma^3.$$

by setting

$$\Phi = \begin{pmatrix} 0 & X_3 \wedge X_2 \\ \sigma^2 \wedge \sigma^3 & 0 \end{pmatrix}$$

and the Reeb field $\xi = X_1$. Then

$$L = span\{X_1, X_2 - \mathrm{i}\sigma^3, X_3 + \mathrm{i}\sigma^2\}, \quad L^* = span\{\sigma^1, X_2 + \mathrm{i}\sigma^3, X_3 - \mathrm{i}\sigma^2\}.$$

It is now an elementary computation to verify that the structure equations for subbundles L are given by

$$[\![X_1, X_2 - \mathrm{i}\sigma^3]\!] = -(X_3 + \mathrm{i}\sigma^2),$$

and the rest of the brackets are equal to zero. Similarly, for L^* we compute the Courant brackets and we see that all of them are equal to zero and $\mathcal{L}_\xi \eta = 0$. Thus $(\Phi, \xi + \eta)$ is a normal generalized contact structure.

Now, define a symmetric matrix $\mathcal{H}$ on $TM \oplus TM^*$ by $\mathcal{H} = \begin{pmatrix} 0 & \gamma^\sharp \\ -\gamma^\flat & 0 \end{pmatrix}$ in which $\gamma(X_i, X_j) = \delta_{ij}$. Then we have $\mathcal{H}(X_i) = -\sigma^i$ and $\mathcal{H}(\sigma^i) = X_i$, for $i = 1, 2, 3$. So that $(\Phi \circ \mathcal{H} = \mathcal{A}, \xi + \eta)$ defines a generalized almost para-contact structure and $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ defines a generalized almost para-contact pseudo-metric structure.

Now it is evident that

$$L_{\mathcal{A}} = span\{X_1, X_2 - X_3, X_3 + X_2\}, \quad L^*_{\mathcal{A}} = span\{\sigma^1, X_2 + X_3, X_3 - X_2\}.$$

Similarly, for $L_{\mathcal{A}}$, we compute the Courant brackets and we obtain

$$[\![X_1, X_2 - X_3]\!] = -X_3, \; [\![X_1, X_3 + X_2]\!] = -X_3,$$
$$[\![X_2 - X_3, X_3 + X_2]\!] = 0,$$

thus $L_{\mathcal{A}}$ is involutive. Finally, for $L^*_{\mathcal{A}}$, we compute the Courant brackets and we see that all of them are equal to zero. Therefore, $(\mathcal{A}, \xi+\eta)$ is a strong structure and $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ is generalized para-coKähler structure.

Proposition 3.3 *Any para-coKähler manifold is a generalized para-coKäh -ler manifold.*

Proof. Let (F, ξ, η, γ) be a para-coKähler structure on M. Define $\Phi, \mathcal{H}$ and $\mathcal{A}$ as in (3.2). The para-coKähler structure (F, ξ, η, γ) on M is normal and thus $\pounds_\xi \eta = 0$. Moreover, we know that the generalized almost contact structure (Φ, ξ, η) is strong ([8]). Thus (M, Φ, ξ, η) is a normal generalized contact manifold. Similar to what is shown in [8], the generalized almost para-contact structure $(\mathcal{A}, \xi+\eta)$ is strong. Therefore, $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ is a generalized para-coKähler structure. $\square$

Theorem 3.4 *The generalized almost para-contact pseudo-metric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi+\eta)$ introduced by matrices in (3.9) are in a one-to-one correspondence with quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, where γ is a pseudo-Riemannian metric of M, ψ is a $2-$form and φ is a $(1,1)-$ tensor field such that*

$$\begin{aligned} &i)\ \varphi^2 = -Id + \eta \otimes \xi, \ \ \gamma(\varphi X, Y) = -\gamma(X, \varphi Y), \\ &ii)\ \gamma^\sharp(\eta) = \xi, \ \ \psi^\flat(\xi) = 0. \end{aligned} \tag{3.18}$$

Proof. From $\mathcal{H}(\eta) = \xi$, we have $\gamma^\sharp(\eta) = \xi$ and $\psi^\flat(\xi) = 0$ in which $\psi^\flat := -\gamma^\flat \circ Q$. Since $\Phi \circ \mathcal{H} = \mathcal{H} \circ \Phi$, then Φ preserves $\mathcal{H}$ and leads to a tensor $\varphi \in End(T^cM)$ given by

$$\varphi|_H := \tau^{-1}\Phi\tau \quad and \quad \varphi|_{\bar{H}} := \bar{\tau}^{-1}\Phi\bar{\tau}, \tag{3.19}$$

then by (3.14), we get

$$\begin{aligned} \varphi^2 X =& \tau^{-1}\Phi^2(X, (\psi^\flat + \mathrm{i}\gamma^\flat)X) \\ =& \tau^{-1}(-X + \eta(X)\xi, -(\psi^\flat + \mathrm{i}\gamma^\flat)X + (\psi + \mathrm{i}\gamma)(X, \xi)\eta) \\ =& \tau^{-1}(-X + \eta(X)\xi, (\psi^\flat + \mathrm{i}\gamma^\flat)(-X + \eta(X)\xi)) \\ =& -X + \eta(X)\xi. \end{aligned}$$

for $X \in H$. Similarly we get $\varphi^2 = -Id + \eta \otimes \xi$ on $\bar{H}$.

Using (3.15) for $X \in H$, we get

$$\begin{aligned} \gamma(\varphi X, Y) =& \gamma(\tau^{-1}\Phi(\tau X), Y) = -\mathrm{i}\langle \Phi(\tau X), \tau Y \rangle = \mathrm{i}\langle \tau X, \Phi(\tau Y) \rangle \\ =& -\gamma(X, \tau^{-1}\Phi(\tau Y)) = -\gamma(X, \varphi Y). \end{aligned} \tag{3.20}$$

Similarly, we can get (3.20), for $X \in \bar{H}$ by using (3.17).

Conversely, the pair (γ, ψ) allows us to reconstruct $\mathcal{H} = \begin{pmatrix} Q & \gamma^\sharp \\ \nu^\flat & Q^* \end{pmatrix}$ in which $Q := -\gamma^\sharp \circ \psi^\flat$ and $\nu^\flat := -\gamma^\flat + \psi^\flat \circ Q$, then $\mathcal{H}^2 = -I$ and $\mathcal{H}$ is g−symmetric. Also, from (3.18 ii), we get $\mathcal{H}(\eta) = \xi$. As a result of $\mathcal{H}|_H = iH$ and $\mathcal{H}|_{\bar{H}} = -i\bar{H}$, one can show that

$$
\begin{aligned}
i)\ \alpha &= (\psi^\flat + i\gamma^\flat)X \quad for \quad X + \alpha \in H,\\
ii)\ \alpha &= (\psi^\flat - i\gamma^\flat)X \quad for \quad X + \alpha \in \bar{H}.
\end{aligned} \tag{3.21}
$$

Now, we are able to reconstruct Φ on H and $\bar{H}$ by

$$
\Phi|_H := \tau\varphi\tau^{-1} \quad and \quad \Phi|_{\bar{H}} := \bar{\tau}\varphi\bar{\tau}^{-1}, \tag{3.22}
$$

respectively and $\Phi(\xi + \eta) = 0$. The resulting Φ commutes with $\mathcal{H}$. By using (3.18) and (3.21 i), we have

$$
\begin{aligned}
\Phi|_H^2(X, (\psi^\flat + \mathrm{i}\gamma^\flat)X) =& \tau\varphi\tau^{-1}\tau\varphi\tau^{-1}(X, (\psi^\flat + \mathrm{i}\gamma^\flat)X) = \tau\varphi^2 X\\
=& (-X + \eta(X)\xi, (\psi^\flat + \mathrm{i}\gamma^\flat)(-X + \eta(X)\xi))\\
=& -(X, (\psi^\flat + \mathrm{i}\gamma^\flat)X) + \eta \odot \xi(X, (\psi^\flat + \mathrm{i}\gamma^\flat)X),
\end{aligned}
$$

and similarly from (3.21 ii), we get $\Phi|_{\bar{H}}^2 = -I + \eta \odot \xi$. Also, from (3.15), (3.18 i) and (3.21), it is obvious that

$$
\langle \Phi(X + \alpha), (Y + \beta)\rangle = -\langle (X + \alpha), \Phi(Y + \beta)\rangle, \tag{3.23}
$$

for any $X + \alpha, Y + \beta \in H$ (or $\bar{H}$).

Finally, we take $\mathcal{A} = \Phi \circ \mathcal{H}$. Therefore, one can get easily $\mathcal{A}^2 = -\Phi^2 = I - \eta \odot \xi$ and $\mathcal{A}(\xi + \eta) = 0$. Since we have (3.4) and (3.23), then

$$
\langle \mathcal{A}(X + \alpha), (Y + \beta)\rangle = -\langle (X + \alpha), \mathcal{A}(Y + \beta)\rangle,
$$

and it completes the proof. □

Consider $D_{\pm}$, the $\pm i-$eigenbundles, and S, the $0-$eigenbundle of $\varphi \in End(T^cM)$. In the following theorem, we give neccesary and sufficient condition for H and $\bar{H}$ to be closed under the Courant bracket, using the quintuple associated to the generalized almost para-contact pseudo-metric structure.

Theorem 3.5 *If a generalized almost para-contact pseudo-metric structure $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is associated to a quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, then H and $\bar{H}$ are closed under the Courant bracket iff*

$$
\gamma((\nabla^\gamma_Z\varphi)Y, \varphi X) = \frac{1}{2}\{d\psi(\varphi X, Y, Z) - d\psi(\varphi^2 X, \varphi Y, Z)\}, \tag{3.24}
$$

where ∇^γ is the Levi-Civita connection of the metric γ and $X, Y, Z \in T^cM$.

Proof. By the isomorphisms $\tau : T^cM \to H$ and $\bar{\tau} : T^cM \to \bar{H}$, the restriction of the Courant bracket on H and $\bar{H}$ is given by the following formula (see also [3], [10])

$$\begin{aligned}&[\![X + (\psi^\flat \underline{+} \mathrm{i}\gamma^\flat)X, Y + (\psi^\flat \underline{+} \mathrm{i}\gamma^\flat)Y]\!] \\ &\quad = [X, Y] + (\psi^\flat \underline{+} \mathrm{i}\gamma^\flat)([X, Y]) + i_X i_Y d\psi \underline{+} \mathrm{i}(\pounds_X i_Y \gamma - i_X \pounds_Y \gamma), \end{aligned} \tag{3.25}$$

in which $X, Y \in T^cM$.

Thus by (3.25), H and $\bar{H}$ are closed under the Courant bracket iff

$$i_X i_Y d\psi = \mp\, \mathrm{i}(\pounds_X i_Y \gamma - i_X \pounds_Y \gamma). \tag{3.26}$$

Let ∇^γ be the Levi-Civita connection of the metric γ, and ∇ be a metric connection such that $\nabla\varphi = 0$ and $\nabla\gamma = 0$. Let $\Theta(X, Y) = \nabla_X Y - \nabla^\gamma_X Y$, thus from $\nabla\gamma = 0$, we have

$$\gamma(\Theta(X, Y), Z) + \gamma(Y, \Theta(X, Z)) = 0, \tag{3.27}$$

and by $\nabla\varphi = 0$, we have

$$\Theta(X, \varphi Y) - \varphi\Theta(X, Y) = -(\nabla^\gamma_X \varphi)(Y). \tag{3.28}$$

By a simple computation, we get

$$\begin{aligned}(\pounds_X \gamma)(Y, Z) =& \gamma(T(X, Y), Z) + \gamma(T(X, Z), Y) + \gamma(\nabla_Y X, Z) \\ &+ \gamma(\nabla_Z X, Y),\end{aligned}$$

where T is torsion of ∇. Using the above formula, we have

$$\begin{aligned}(\pounds_X i_Y \gamma - i_X \pounds_Y \gamma)Z =& \gamma(\nabla_Z X, Y) - \gamma(\nabla_Z Y, X) + \gamma(T(X, Z), Y) \\ &- \gamma(T(Y, X), Z) - \gamma(T(Y, Z), X),\end{aligned} \tag{3.29}$$

for $X, Y, Z \in T^cM$. Now, let $X, Y \in D_{\underline{+}}$ or $X \in D_{\underline{+}}, Y \in S$, then the first two terms of the right-hand side of (3.29) vanishes and we get

$$(\pounds_X i_Y \gamma - i_X \pounds_Y \gamma)Z = \gamma(T(X, Z), Y) - \gamma(T(Y, X), Z) - \gamma(T(Y, Z), X). \tag{3.30}$$

If we insert $T(X, Y) = \Theta(X, Y) - \Theta(Y, X)$ in (3.30) and using (3.26) gives

$$d\psi(X, Y, Z) = \mp\, 2\mathrm{i}\gamma(\Theta(Z, Y), X), \tag{3.31}$$

for $X, Y \in D_{\underline{+}}$ or $X \in D_{\underline{+}}, Y \in S$.

Replacing once, $X, Y \in D_{\underline{+}}$ by $\varphi^2 X \underline{+} \mathrm{i}\varphi X, \varphi^2 Y \underline{+} \mathrm{i}\varphi Y \in D_{\underline{+}}$, and once $X \in D_{\underline{+}}, Y \in S$ by $\varphi^2 X \underline{+} \mathrm{i}\varphi X \in D_{\underline{+}}$ and $Y + \varphi^2 Y \in S$ in (3.31) we get the following equalities

$$d\psi(\varphi^2 X \underline{+} \mathrm{i}\varphi X, \varphi^2 Y \underline{+} \mathrm{i}\varphi Y, Z) = \mp\, 2\mathrm{i}\gamma(\Theta(Z, \varphi^2 Y \underline{+} \mathrm{i}\varphi Y), \varphi^2 X \underline{+} \mathrm{i}\varphi X),$$
$$d\psi(\varphi^2 X \underline{+} \mathrm{i}\varphi X, Y + \varphi^2 Y, Z) = \mp\, 2\mathrm{i}\gamma(\Theta(Z, Y + \varphi^2 Y), \varphi^2 X \underline{+} \mathrm{i}\varphi X),$$

in which $X, Y, Z \in T^cM$. By simplifying the above equalities and comparing the real and imaginary part of them, we see that some of them are equivalent by the change $X \to \varphi X$. Finally, we get the following two equalities.

$$\begin{aligned}\gamma(\Theta(Z,\varphi Y),\varphi X) - \gamma(\Theta(Z,\varphi^2 Y),\varphi^2 X) &= \frac{1}{2}\{d\psi(\varphi X,\varphi^2 Y,Z) \\ &\qquad + d\psi(\varphi^2 X,\varphi Y,Z)\}, \\ \gamma(\Theta(Z,Y),\varphi^2 X) + \gamma(\Theta(Z,\varphi^2 Y),\varphi^2 X) &= -\frac{1}{2}d\psi(\varphi X, Y+\varphi^2 Y, Z). \end{aligned} \tag{3.32}$$

Using (3.27) and (3.28) we get the following system, which is equivalent to (3.32), and does not contain Θ anymore

$$\begin{aligned}\gamma((\nabla^\gamma_Z\varphi)\varphi Y,\varphi^2 X) &= \frac{1}{2}\{d\psi(\varphi X,\varphi^2 Y,Z) + d\psi(\varphi^2 X,\varphi Y,Z)\}, \\ \gamma((\nabla^\gamma_Z\varphi)\varphi Y,\varphi^2 X) + \gamma((\nabla^\gamma_Z\varphi)Y,\varphi X) &= \frac{1}{2}d\psi(\varphi X, Y+\varphi^2 Y, Z).\end{aligned}$$

Subtracting the above two equalities, we get

$$\gamma((\nabla^\gamma_Z\varphi)Y,\varphi X) = \frac{1}{2}\{d\psi(\varphi X, Y, Z) - d\psi(\varphi^2 X,\varphi Y,Z)\}, \tag{3.33}$$

that is equivalent with the required condition (3.24) and it completes the proof. □

Corollary 3.6 *If a generalized almost para-contact pseudo-metric structure* $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ *is associated to a quintuple* $(\varphi, \xi, \eta, \gamma, \psi)$ *such that* $d\psi = 0$*, then* H *and* $\bar{H}$ *are closed under Courant bracket iff* $\nabla^\gamma\varphi \in S$.

Furthermore, consider $E^{(1,0)}$, $E^{(0,1)}$ and $L_\xi \oplus L_\eta$ difined by (2.5). The commutation properties $\mathcal{H} \circ \Phi = \Phi \circ \mathcal{H}$ ensures that the projections of vectors of $E^{(1,0)}$, $E^{(0,1)}$ and $L_\xi \oplus L_\eta$ by H and $\bar{H}$ belong to $E^{(1,0)}$, $E^{(0,1)}$ and $L_\xi \oplus L_\eta$ respectively and the projections of vectors of H and $\bar{H}$ by $E^{(1,0)}$, $E^{(0,1)}$ and $L_\xi \oplus L_\eta$ belong to H and $\bar{H}$ respectively. This leads to the existence of the following decompositions

$$\begin{aligned} E^{(1,0)} &= (E^{(1,0)} \cap H) \oplus (E^{(1,0)} \cap \bar{H}), \\ E^{(0,1)} &= (E^{(0,1)} \cap H) \oplus (E^{(0,1)} \cap \bar{H}), \\ L_\xi \oplus L_\eta &= ((L_\xi \oplus L_\eta) \cap H) \oplus ((L_\xi \oplus L_\eta) \cap \bar{H}), \\ H &= (E^{(1,0)} \cap H) \oplus (E^{(0,1)} \cap H) \oplus ((L_\xi \oplus L_\eta) \cap H), \\ \bar{H} &= (E^{(1,0)} \cap \bar{H}) \oplus (E^{(0,1)} \cap \bar{H}) \oplus ((L_\xi \oplus L_\eta) \cap \bar{H}). \end{aligned} \tag{3.34}$$

We define $H_+ = E^{(1,0)} \cap H$, $H_- = E^{(0,1)} \cap H$, $Z = (L_\xi \oplus L_\eta) \cap H$ and their conjugation by $\bar{H}_+ = E^{(1,0)} \cap \bar{H}$, $\bar{H}_- = E^{(0,1)} \cap \bar{H}$, $\bar{Z} = (L_\xi \oplus L_\eta) \cap \bar{H}$.

Lemma 3.7 *In the generalized almost para-contact pseudo-metric structure associated to a quintuple $(\varphi, \xi, \eta, \gamma, \psi)$, we have $H_+ = \tau(D_+)$, $H_- = \bar{\tau}(D_-)$, $Z = \tau(S)$, $\bar{Z} = \bar{\tau}(S)$, $H_+ \oplus Z = \tau(D_+ \oplus S)$ and $H_- \oplus \bar{Z} = \bar{\tau}(D_- \oplus S)$.*

Proof. One can easily prove the first four equalities. For the fifth one, let $X \in D_+ \oplus S$, then we have $X = X_0 + \epsilon\xi$ in which $X_0 \in D_+$ and $\epsilon\xi \in S$. It is obvious that $\tau(X) \in H$. Also, using (3.22), we have $\Phi(\tau X_0) = i\tau X_0$ and $\Phi(\tau(\epsilon\xi)) = 0$. Therefore, $\tau(X) \in H_+ \oplus Z$.

Conversely, let $X \in H_+ \oplus Z$, then $X = X_1 + X_2$ in which $X_1 \in E^{(1,0)} \cap H$ and $X_2 \in (L_\xi \oplus L_\eta) \cap H$. Thus $X = X_1 + X_2 \in H$, and we can write $X = \tau(Y)$ in which $Y \in T^c M$, then

$$Y = \tau^{-1}(X) = \tau^{-1}(X_1) + \tau^{-1}(X_2).$$

It suffices to show $\tau^{-1}(X_1) \in D_+$ and $\tau^{-1}(X_2) \in S$. By (3.19) we have

$$\varphi\tau^{-1}(X_1) = \tau^{-1}\Phi(X_1) = \tau^{-1}(iX_1) = i\tau^{-1}(X_1),$$

thus $\tau^{-1}(X_1) \in D_+$. Also, since $X_2 \in L_\xi \oplus L_\eta$, then $X_2 = (\alpha\xi, \beta\eta)$ and we have

$$\varphi\tau^{-1}(X_2) = \tau^{-1}\Phi(\alpha\xi, \beta\eta) = 0,$$

thus $\tau^{-1}(X_2) \in S$, and the fifth equality is proved. The assertion for the sixth one is the same as the previous one. □

Proposition 3.8 *If a generalized almost para-contact pseudo-metric struc -ture $(\Phi, \mathcal{H}, \mathcal{A}, \xi + \eta)$ is associated to a quintuple $(\varphi, \xi, \eta, \gamma, \psi)$ and (3.24) is satisfied, then $H_+, H_-, H_+ \oplus Z$ and $H_- \oplus \bar{Z}$ are closed under the Courant bracket iff $D_+, D_-, D_+ \oplus S$ and $D_- \oplus S$ are closed under the Courant bracket.*

Proof. Using Theorem 3.5, we know that equation (3.24) is satisfied iff H and $\bar{H}$ are closed under the Courant bracket. Thus, by using Lemma 3.7, it is obvious that $H_+, H_-, H_+ \oplus Z$ and $H_- \oplus \bar{Z}$ are closed under the Courant bracket iff $D_+, D_-, D_+ \oplus S$ and $D_- \oplus S$ are closed under the Courant bracket. □

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