An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXV, 2019, f. 1

A variational approach for a perturbed second-order impulsive Hamiltonian system

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Abstract In this paper, we study the existence of three solutions for second-order impulsive Hamiltonian systems. By using variational methods, we obtain some new criteria for guaranteeing that the impulsive Hamiltonian systems have three solutions. Some recent results are extended and improved. An example is presented to demonstrate the application of our main results.

Keywords Perturbed Hamiltonian system · Impulsive systems · Three solutions · Variational methods

Mathematics Subject Classification (2010) 34B15 · 47J10

1 Introduction

In this work, we are interested in ensuring the existence of at three classical solutions for the following perturbed Hamiltonian system

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \mu \nabla G(t, u(t)) \\ +\nabla H(u(t)), \qquad a.e.\ t \in [0, T], \\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \qquad i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, p, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \qquad (P_\lambda^F)$$

where $N \geq 1, T > 0$, $\lambda > 0$ and $\mu \geq 0$ are two parameter, $A : [0, T] \to R^{N \times N}$ is a continuous map from the interval $[0, T]$ to the set of $N \times N$ symmetric matrices, t_j, $j = 1, 2, \ldots, p$, $p \geq 2$, are instants in which the impulses occur and $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $\Delta(\dot{u}_i(t_j)) = \dot{u}_i(t_j^+) - \dot{u}_i(t_j^-) = \lim_{t \to t_j^+} \dot{u}_i(t) - \lim_{t \to t_j^-} \dot{u}_i(t)$, $I_{ij} : R \to R$ is a Lipschitz continuous with the Lipschitz constant $L_{ij} > 0$, i.e.,

$$|I_{ij}(s_1) - I_{ij}(s_2)| \leq L_{ij}|s_1 - s_2|$$

for every $s_1, s_2 \in R$, and $I_{ij}(0) = 0$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, p$, and $F, G : [0, T] \times R^N \to R$ are measurable with respect to t, for all $u \in R^N$, continuously differentiable in u, for almost every $t \in [0, T]$, satisfy the following standard summability condition:

$$\sup_{|x| \leq a} \left(\max\{|F(\cdot, x)|,\ |\nabla F(\cdot, x)|, |G(\cdot, x)|,\ |\nabla G(\cdot, x)|\} \right) \in L^1([0, T]) \quad (1.1)$$

for any $a > 0$, $H : R^N \to R$ is continuously differentiable such that ∇H is Lipschitz continuous with the Lipschitz constant $L > 0$, i.e.,

$$|\nabla H(x_1) - \nabla H(x_2)| \leq L|x_1 - x_2|$$

for every $x_1, x_2 \in R^N$ and $\nabla H(0, \ldots, 0) = 0$.

Assume that $\nabla F, \nabla G : [0, T] \times R^N \to R$ are continuous, then the condition (1.1) is satisfied.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics. Due to its significance, in the last few years, the existence and multiplicity of periodic and homoclinic solutions for second-order Hamiltonian systems have been studied extensively, we refer the interested readers to [15, 28, 30, 35]. Inspired by the monographs [29, 31], the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated in many papers (see [8–10, 12–14, 16, 17, 36, 39] and the references therein) via the variational methods.

On the other hand, the theory of impulsive differential equations is emerging as an important area of investigation since it is a lot richer than the corresponding theory of non-impulsive differential equations. Many evolutionary processes in nature are characterized by the fact that at certain moments in time an abrupt change of state is experienced. That is the reason for the rapid development of the theory of impulsive differential equations, for some recent work, we refer the reader to [1, 2, 23, 27, 32] and the references therein. We refer also to the papers [6, 7, 18, 22, 37, 38] in which using critical point theory, the existence and multiplicity of solutions of impulsive problems were discussed.

Recently, problems of second-order impulsive Hamiltonian systems have been studied by a number of authors. For the background, theory and applications of impulsive differential equations, we refer the interest readers to [11, 33, 34, 40] and the references therein. For example, Zhou and Li in [40] by means of some critical point theorems, established some sufficient conditions for the existence of solutions for the second-order Hamiltonian systems with impulsive effects. Sun et al. in [33] based on variational methods, studied the existence of infinitely many solutions for a class of second-order impulsive Hamiltonian systems. Chen and He in [11] by using a variational method and some critical points theorems of Ricceri, studied the existence of three solutions for second-order impulsive Hamiltonian systems. In [19], based on variational methods and critical point theory, we studied the existence of infinitely many periodic classical solutions for the problem (P_λ^F), while in [21], using variational methods and critical point theory, we investigated the existence of nontrivial periodic solutions for the problem (P_λ^F), in the case $\mu = 0$. Also in [20], based on variational methods, the existence of three distinct periodic solutions for the system (P_λ^F) was discussed.

In the present paper, motivated by the above facts, using a three critical points theorem due to Bonanno and Candito [4] which we recall in Section 2 (Theorem 2.1) we obtain the existence of at least three classical solutions for the system (P_λ^F) (see Theorem 3.1). In particular, we require that there is a growth of F which is greater than quadratic in a suitable interval (see, for instance, condition (A_2) of Theorem 3.2), and which is less than quadratic in a following suitable interval (see, for instance, condition (A_2) of Theorem 3.2). We present Example 3.1 in which the hypotheses of Theorem 3.2 are fulfilled. Finally, as a special case of Theorem 3.2, we obtain Theorem 3.3 when $\mu = 0$.

This paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our abstract results.

2 Preliminaries

Our main tool to ensure the existence of three solutions for the system (P_λ^F) is the following three critical point theorem due to Bonanno and Candito. Let X be a nonempty set and $\varPhi, \varPsi : X \to R$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \varPhi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \varPhi^{-1}(]-\infty, r[)} \frac{(\sup_{u \in \varPhi^{-1}(]-\infty, r[)} \varPsi(u)) - \varPsi(u)}{r - \varPhi(u)},$$

$$\beta(r_1, r_2) := \inf_{u \in \varPhi^{-1}(]-\infty, r_1[)} \sup_{v \in \varPhi^{-1}([r_1, r_2[)} \frac{\varPsi(v) - \varPsi(u)}{\varPhi(v) - \varPhi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u \in \varPhi^{-1}(]-\infty, r_2 + r_3[)} \varPsi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 2.1 *[4, Theorem 3.3] Let X be a reflexive real Banach space, $\varPhi : X \to R$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\varPsi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(a_1) $\inf_X \varPhi = \varPhi(0) = \varPsi(0) = 0$;
(a_2) *for every $u_1, u_2 \in X$ such that $\varPsi(u_1) \geq 0$ and $\varPsi(u_2) \geq 0$, one has*

$$\inf_{s \in [0,1]} \varPsi(su_1 + (1 - s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;
(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;
(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that

$$u_1 \in \Phi^{-1}(]-\infty, r_1[),\ \ u_2 \in \Phi^{-1}([r_1, r_2[) \text{ and } u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[).$$

We refer the interested reader to the papers [5,24–26] in which Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for some boundary value problems.

We assume that A satisfies the following conditions:

(A1) $A(t) = (a_{kl}(t))$, $k = 1, \ldots, N$, $l = 1, \ldots, N$ is a symmetric matrix with $a_{kl} \in L^\infty[0, T]$ for any $t \in [0, T]$;

(A2) There exists $\kappa > 0$ such that $(A(t)x, x) \geq \kappa|x|^2$ for any $x \in R^N$ and a.e. $t \in [0, T]$.

Let us recall some basic concepts. Denote

$$E = \Big\{u : [0, T] \to R^N |\ u \text{ is absolutely continuous},$$

$$u(0) = u(T),\ \ \dot{u} \in L^2([0, T], R^N)\Big\}$$

with the inner product

$$\prec u, v \succ_E = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]dt$$

where $(\cdot, \cdot)$ denotes the inner product in R^N. The corresponding norm is defined by

$$\|u\|_E = \int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt,\ \forall\ u \in E.$$

For every $u, v \in E$, we define

$$\prec u, v \succ = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]dt,$$

and we observe that, by assumptions (A1) and (A2), it defines an inner product in E. Then E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}}\ ,\quad \forall u \in E.$$

Obviously, E is a uniformly convex Banach space.

A simple computation shows that

$$(A(t)x, x) = \sum_{k,l=1}^{N} a_{kl}(t)x_k x_l \leq \sum_{k,l=1}^{N} \|a_{kl}\|_\infty |x|^2$$

for every $t \in [0, T]$ and $x \in R^N$, and this along with Assumption (A2) yields

$$\sqrt{m}\|u\|_E \leq \|u\| \leq \sqrt{M}\|u\|_E \tag{2.1}$$

where $m = \min\{1, \kappa\}$ and $M = \max\{1, \sum_{k,l=1}^{N} \|a_{kl}\|_\infty\}$, which means the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$.

Since $(E, \|\cdot\|)$ is compactly embedded in $C([0, T], R^N)$ (see [29]), there exists a positive constant c such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{2.2}$$

where $\|u\|_\infty = \max_{t\in[0,T]} \mid u(t) \mid$ and $c = \sqrt{\frac{2}{m}} \max\{\frac{1}{\sqrt{T}}, \sqrt{T}\}$ (see [11]).

If $u \in E$, then u is absolutely continuous and $\dot{u} \in L^2([0, T], R^N)$. In this case, $\Delta\dot{u}(t) = \dot{u}(t^+) - \dot{u}(t^-) = 0$ is not necessarily valid for every $t \in (0, T)$, and the derivative $\dot{u}$ may possess some discontinuities that lead to the impulsive effects.

A function $u \in \{u \in E : \ \dot{u} \in (W^{1,2}(t_j, t_{j+1}))^N, \ j = 0, 1, 2, \ldots, p\}$ is said to be a classical solution of the system (P_λ^F) if u satisfies (P_λ^F).

We mean by a (weak) solution of the system (P_λ^F), any $u \in E$ such that

$$\int_0^T \Big[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\Big]dt$$
$$+\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \lambda \int_0^T (\nabla F(t, u(t)), v(t))dt$$
$$-\mu \int_0^T (\nabla G(t, u(t)), v(t))dt = 0$$

for every $v \in E$.

Lemma 2.2 *[19, Lemma 2.2] If $u \in E$ is a weak solution of (P_λ^F), then u is a classical solution of (P_λ^F).*

We suppose that

$$K := c^2(2LT + \sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}) < 1.$$

Moreover, set $G^\theta := \displaystyle\int_{[0,T]} \max_{|x|\leq\theta} G(t, x)dt$, for every $\theta > 0$ and $G_\eta := \inf_{[0,T]\times[0,\eta]^N} G$, for every $\eta > 0$ where $[0, \eta]^N = [0, \eta] \times \cdots \times [0, \eta]$. If G is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

3 Main results

Put

$$D = \frac{(T-t_p)^2}{t_1 t_p^2} + \frac{t_1}{3t_p^2}(t_p^2 + t_p T + T^2) + (t_p - t_1) + \frac{T-t_p}{t_p^2} + \frac{1}{3t_p^2}(T^3 - t_p^3) > 0.$$

We fix four positive constants θ_1, θ_2, θ_3 and η, put

$$\delta_1 := \min\Bigg\{ \frac{1}{2c^2} \min\Big\{ \frac{(1-K)\theta_1^2 - 2\lambda c^2 \int_0^T \sup_{|t|\le\theta_1} F(t,x)dt}{G^{\theta_1}}, \tag{3.1}$$
$$\frac{(1-K)\theta_2^2 - 2\lambda c^2 \int_0^T \sup_{|t|\le\theta_2} F(t,x)dt}{G^{\theta_2}},$$
$$\frac{(1-K)(\theta_3^2-\theta_2^2) - 2\lambda c^2 \int_0^T \sup_{|t|\le\theta_3} F(t,x)dt}{G^{\theta_3}}\Big\},$$
$$\frac{\frac{1}{2}(1+K)DM\eta^2 - \lambda\big(\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \sup_{|t|\le\theta_1} F(t,x)dt\big)}{G_\eta - G^{\theta_1}}\Bigg\}$$

where $\varepsilon = (1,0,\cdots,0) \in R^N$.

Theorem 3.1 *Let $F : [0,T] \times R^N \to R$ be a non-negative function. Assume that there exist positive constants θ_1, θ_2, θ_3 and η with*

$$\frac{\theta_1}{c\sqrt{Dm}} < \eta < \frac{\theta_2\sqrt{1-K}}{c\sqrt{DM(1+K)}}$$

and $\theta_2 < \theta_3$ such that

(A_1)

$$\max\left\{ \frac{\int_0^T \max_{|\xi|\le\theta_1} F(t,\xi)dt}{\theta_1^2}, \frac{\int_0^T \max_{|\xi|\le\theta_2} F(t,\xi)dt}{\theta_2^2}, \frac{\int_0^T \max_{|\xi|\le\theta_3} F(t,\xi)dt}{\theta_3^2-\theta_2^2} \right\}$$
$$< \frac{1-K}{c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \max_{|\xi|\le\theta_1} F(t,\xi)dt}{\eta^2}.$$

Then, for every

$$\lambda \in \Big(\frac{\frac{(1+K)DM}{2}\eta^2}{\displaystyle\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \max_{|\xi|\le\theta_1} F(t,\xi)dt}, \frac{1-K}{2c^2} \min\Big\{ \frac{\theta_1^2}{\displaystyle\int_0^T \max_{|\xi|\le\theta_1} F(t,\xi)dt}, \frac{\theta_2^2}{\displaystyle\int_0^T \max_{|\xi|\le\theta_2} F(t,\xi)dt}, \frac{\theta_3^2-\theta_2^2}{\displaystyle\int_0^T \max_{|\xi|\le\theta_3} F(t,\xi)dt} \Big\} \Big)$$

and for every non-negative function $G : [0,T]\times R^N \to R$ which is measurable with respect to t, for all $u \in R^N$, continuously differentiable in u, for almost every $t \in [0,T]$, satisfying (1.1), there exists $\delta_1 > 0$ given by (3.1) such that, for each $\mu \in [0,\delta_1)$, the system (P_λ^F) possesses at least three classical solutions $u^1 = (u_1^1,\dots,u_N^1)$, $u^2 = (u_1^2,\dots,u_N^2)$, and $u^3 = (u_1^3,\dots,u_N^3)$ such that

$$\max_{t\in[0,T]} |u^1(t)| < \theta_1, \ \max_{t\in[0,T]} |u^2(t)| < \theta_2 \ \textit{and} \ \max_{t\in[0,T]} |u^3(t)| < \theta_3.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. Fix λ, g and μ as in the conclusion. Take $X = E$ and introduce the functionals Φ, $\Psi : X \to R$ defined by

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N} \int_0^{u_i(t_j)} I_{ij}(s)ds - \int_0^T H(u(t))dt$$

and

$$\Psi(u) = \int_0^T [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt$$

for every $u \in X$. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, it is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = \int_0^T \Big(\nabla F(t,u(t)) + \frac{\mu}{\lambda}\nabla G(t,u(t)), v(t)\Big)dt$$

for every $v \in X$, and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_0^T \Big[(\dot u(t),\dot v(t)) + (A(t)u(t),v(t)) - (\nabla H(u(t)),v(t))\Big]dt$$

$$+\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j)$$

for every $v \in X$, while [20, Proposition 2.4] gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, let $u_n \in X$ with $u_n \to u$ weakly in X, we have $\liminf_{n\to+\infty} \|u_n\| \geq \|u\|$ and $u_n \to u$ uniformly on $[0,T]$. Hence, since H is continuous, one has

$$\liminf_{n\to+\infty}\Big(\frac{1}{2}\|u_n\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_{ni}(t_j)} I_{ij}(s)ds - \int_0^T H(u_n(t))dt\Big)$$

$$\geq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(s)ds - \int_0^T H(u(t))dt,$$

that is $\liminf_{n\to+\infty} \Phi(u_n) \geq \Phi(u)$, which means Φ is sequentially weakly lower semicontinuous. From $|\nabla H(x_1) - \nabla H(x_2)| \leq L|x_1 - x_2|$ for every $x_1, x_2 \in R^N$ and the fact $H(0,\dots,0) = 0$, we have $|H(\xi)| \leq L|\xi|^2$ for all $\xi \in R^N$. This in conjunction with the fact $-L_{ij}|s|^2 \leq I_{ij}(s)s \leq L_{ij}|s|^2$ for every $s \in R$ for all $i = 1,2,\dots,N,\ j = 1,2,\dots,p$, taking (2.2) into account, for $u \in X$ yields

$$\frac{1}{2}(1-K)\|u\|^2 \leq \Phi(u) \leq \frac{1}{2}(1+K)\|u\|^2. \tag{3.2}$$

Put $r_1 = \frac{1}{2}(1-K)(\frac{\theta_1}{c})^2$,$r_2 = \frac{1}{2}(1-K)(\frac{\theta_2}{c})^2$, $r_3 = \frac{1}{2}(1-K)(\frac{\theta_3^2-\theta_2^2}{c^2})$ and

$$w(t) = \begin{cases} (T + \frac{t_p - T}{t_1}t)\frac{\eta\varepsilon}{t_p}, & t \in [0,t_1) \\ \eta\varepsilon, & t \in [t_1,t_p], \\ \frac{\eta\varepsilon}{t_p}t, & t \in (t_p,T]. \end{cases}$$

It is easy to see that $w \in X$ and, it is clear that $w \in E$, and $\|w\|_E^2 = D\eta^2$. Hence, taking (2.1) into account, one has

$$Dm\eta^2 \leq \|w\|^2 \leq DM\eta^2.$$

Form (3.2), we have

$$\frac{1}{2}(1-K)Dm\eta^2 \leq \Phi(w) \leq \frac{1}{2}(1+K)DM\eta^2$$

and this together with the conditions $\theta_2 < \theta_3$ and $\frac{\theta_1}{c\sqrt{Dm}} < \eta < \frac{\theta_2\sqrt{1-K}}{c\sqrt{DM(1+K)}}$, ensure that $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. Recalling (2.2), from the inequality (3.2) we see that for each $u \in X$,

$$\begin{aligned}\Phi^{-1}(-\infty, r_1) &= \{u \in X;\ \Phi(u) < r_1\} \\ &\subseteq \left\{u \in X;\ \frac{1}{2}(1-K)\|u\|^2 < r_1\right\} \\ &\subseteq \{u \in X;\ |u(t)| < \theta_1 \text{ for each } t \in [0,T]\},\end{aligned}$$

and it follows that

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)=\sup_{u\in\Phi^{-1}(-\infty,r_1)}\int_0^T[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt$$
$$\leq\int_0^T\sup_{|\xi|\leq\theta_1}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_1}.$$

In a similar way, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)\leq\int_0^T\sup_{|\xi|\leq\theta_2}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_2}$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_3+r_2)}\Psi(u)\leq\int_0^T\sup_{|\xi|\leq\theta_3}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_3}.$$

Therefore, since $0\in\Phi^{-1}(-\infty,r_1)$ and $\Phi(0)=\Psi(0)=0$, one has

$$\varphi(r_1)=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u))-\Psi(u)}{r_1-\Phi(u)}$$
$$\leq\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}$$
$$\leq\frac{\int_0^T\sup_{|\xi|\leq\theta_1}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{1}{2}(1-K)(\frac{\theta_1}{c})^2},$$

$$\varphi(r_2)\leq\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\leq\frac{\int_0^T\sup_{|\xi|\leq\theta_2}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_2}}{\frac{1}{2}(1-K)(\frac{\theta_2}{c})^2}$$

and

$$\gamma(r_2,r_3)\leq\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}\leq\frac{\int_0^T\sup_{|\xi|\leq\theta_3}F(t,\xi)dx+\frac{\mu}{\lambda}G^{\theta_3}}{\frac{1}{2}(1-K)(\frac{\theta_3^2-\theta_2^2}{c^2})}.$$

For each $u\in\Phi^{-1}(-\infty,r_1)$ one has

$$\beta(r_1,r_2)\geq\frac{\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt-\int_0^T\max_{|\xi|\leq\theta_1}F(t,\xi)dt+\frac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\Phi(w)-\Phi(u)}$$
$$\geq\frac{\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt-\int_0^T\max_{|\xi|\leq\theta_1}F(t,\xi)dt+\frac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\frac{1}{2}(1+K)DM\eta^2}.$$

Due to (A_1) we get

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption (a_2) of Theorem 2.1. Let $u^\star = (u_1^\star, \ldots, u_N^\star)$ and $u^{\star\star} = (u_1^{\star\star}, \ldots, u_N^{\star\star})$ be two local minima for $\Phi - \lambda\Psi$. Then $u^\star$ and $u^{\star\star}$ are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the system (P_λ^F). Since F and G are non-negative, $F(t, su^\star + (1-s)u^{\star\star}) \geq 0$ and $G(t, su^\star + (1-s)u^{\star\star}) \geq 0$, and consequently, $\Psi(su^\star + (1-s)u^{\star\star}) \geq 0$ for all $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \Big(\frac{\frac{(1+K)DM}{2}\eta^2}{\displaystyle\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \max_{|\xi|\leq\theta_1} F(t,\xi)dt}, \frac{1-K}{2c^2} \min\Big\{ \frac{\theta_1^2}{\displaystyle\int_0^T \max_{|\xi|\leq\theta_1} F(t,\xi)dt}, \frac{\theta_2^2}{\displaystyle\int_0^T \max_{|\xi|\leq\theta_2} F(t,\xi)dt}, \frac{\theta_3^2 - \theta_2^2}{\displaystyle\int_0^T \max_{|\xi|\leq\theta_3} F(t,\xi)dt} \Big\} \Big)$$

and $\mu \in [0, \delta_1)$, the functional I_λ has three critical points $u^1 = (u_1^1, \ldots, u_N^1)$, $u^2 = (u_1^2, \ldots, u_N^2)$, and $u^3 = (u_1^3, \ldots, u_N^3)$ such that $\Phi(u^1) < r_1$, $\Phi(u^2) < r_2$ and $\Phi(u^3) < r_2 + r_3$, that is,

$$\max_{t\in[0,T]} |u^1(t)| < \theta_1, \ \max_{t\in[0,T]} |u^2(t)| < \theta_2 \text{ and } \max_{t\in[0,T]} |u^3(t)| < \theta_3.$$

Then, taking into account the fact that the solutions of the system (P_λ^F) are exactly critical points of the functional I_λ we have the desired conclusion. □

For positive constants θ_1, θ_4 and η, set

$$\delta_2 := \min\Bigg\{ \frac{1}{2c^2} \min\Big\{ \frac{(1-K)\theta_1^2 - 2\lambda c^2 \displaystyle\int_0^T \sup_{|t|\leq\theta_1} F(t,x)dt}{G^{\theta_1}}, \frac{(1-K)\theta_4^2 - 4\lambda c^2 \displaystyle\int_0^T \sup_{|t|\leq\frac{\theta_4}{\sqrt{2}}} F(t,x)dt}{2G^{\frac{\theta_4}{\sqrt{2}}}}, \frac{(1-K)\theta_4^2 - 4\lambda c^2 \displaystyle\int_0^T \sup_{|t|\leq\theta_4} F(t,x)dt}{2G^{\theta_4}} \Big\}, \frac{\frac{1}{2}(1+K)DM\eta^2 - \lambda\Big(\displaystyle\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \sup_{|t|\leq\theta_1} F(t,x)dt\Big)}{G_\eta - G^{\theta_1}} \Bigg\}, \tag{3.3}$$

where ε is as in (3.1). Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.2 *Let $F:[0,T]\times R^N\to R$ be a non-negative function. Assume that there exist positive constants θ_1, θ_4 and η with $\theta_1<\min\{\eta,\sqrt{Dm}c\eta\}$ and $\sqrt{\frac{2DM(1+K)}{1-K}}c\eta<\theta_4$ such that*

(A_2)

$$\max\left\{\frac{\int_0^T \max_{|\xi|\le\theta_1}F(t,\xi)dt}{\theta_1^2},\ \frac{2\int_0^T \max_{|\xi|\le\theta_4}F(t,\xi)dt}{\theta_4^2}\right\} < \frac{1-K}{1-K+c^2\Big(1+K\Big)DM}\frac{\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt}{\eta^2}.$$

Then, for every

$$\lambda\in\Lambda''=\Big(\frac{\frac{1-K+c^2(1+K)DM}{2c^2}\eta^2}{\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt},\frac{1-K}{2c^2}\min\Big\{\frac{\theta_1^2}{\int_0^T\max_{|\xi|\le\theta_1}F(t,\xi)dt},\ \frac{\theta_4^2}{2\int_0^T\max_{|\xi|\le\theta_4}F(t,\xi)dt}\Big\}\Big)$$

and for every non-negative function $G:[0,T]\times R^N\to R$ which is measurable with respect to t, for all $u\in R^N$, continuously differentiable in u, for almost every $t\in[0,T]$, satisfying (1.1), there exists $\delta_2>0$ given by (3.3) such that, for each $\mu\in[0,\delta_2)$, the system (P^F_λ) possesses at least three classical solutions $u^1=(u^1_1,\dots,u^1_N)$, $u^2=(u^2_1,\dots,u^2_N)$, and $u^3=(u^3_1,\dots,u^3_N)$ such that

$$\max_{t\in[0,T]}|u^1(t)|<\theta_1,\ \max_{t\in[0,T]}|u^2(t)|<\frac{1}{\sqrt2}\theta_4 \text{ and } \max_{t\in[0,T]}|u^3(t)|<\theta_4.$$

Proof. Choose $\theta_2=\frac{1}{\sqrt2}\theta_4$ and $\theta_3=\theta_4$. So, from (A_2) one has

$$\frac{\int_0^T\max_{|\xi|\le\theta_2}F(t,\xi)dt}{\theta_2^2}=\frac{2\int_0^T\max_{|\xi|\le\frac{\theta_4}{\sqrt2}}F(t,\xi)dt}{\theta_4^2}\le\frac{2\int_0^T\max_{|\xi|\le\theta_4}F(t,\xi)dt}{\theta_4^2} < \frac{1-K}{1-K+c^2\Big(1+K\Big)DM}\frac{\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt}{\eta^2} \tag{3.4}$$

and

$$\frac{\int_0^T \max_{|\xi|\leq\theta_3} F(t,\xi)dt}{\theta_3^2-\theta_2^2} = \frac{2\int_0^T \max_{|\xi|\leq\theta_4} F(t,\xi)dt}{\theta_4^2}$$
$$< \frac{1-K}{1-K+c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2}. \tag{3.5}$$

Moreover, taking into account that $\theta_1 < \eta$, by using (A_2) we have

$$\frac{1-K}{c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt - \int_0^T \max_{|\xi|\leq\theta_1} F(t,\xi)dt}{\eta^2}$$
$$> \frac{1-K}{c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2}$$
$$- \frac{1-K}{c^2\Big(1+K\Big)DM} \frac{\int_0^T \max_{|\xi|\leq\theta_1} F(t,\xi)dt}{\theta_1^2}$$
$$> \frac{1-K}{c^2\Big(1+K\Big)DM} \Bigg(\frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2}$$
$$- \frac{1-K}{1-K+c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2} \Bigg)$$
$$= \frac{1-K}{1-K+c^2\Big(1+K\Big)DM} \frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2}.$$

Hence, from (A_2), (3.4) and (3.5), it is easy to see that the assumption (A_1) of Theorem 3.1 is satisfied, and it follows the conclusion. □

We now present the following example to illustrate Theorem 3.2.

Example 3.1 Consider the following system

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \mu \nabla G(t, u(t)) \\ +\nabla H(u(t)), \qquad a.e.\ t \in [0, 3], \\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \qquad i = 1, 2,\ j = 1, 2, \\ u(0) - u(3) = \dot{u}(0) - \dot{u}(3) = 0 \end{cases} \tag{3.6}$$

where $A(t)$ is a second-order identity matrix, $H(x_1, x_2) = \frac{1}{10^4}\sin(x_1 + x_2)$ for all $x_1, x_2 \in R$, $t_1 = 1$, $t_2 = 2$, $I_{ij}(\xi) = \frac{1}{10^4}\sin(\xi)$ for all $\xi \in R$, $i = j = 1, 2$ and

$$F(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2)^4, & \text{if } x_1^2 + x_2^2 \le 1, \\ \frac{1}{x_1^2+x_2^2}, & \text{if } x_1^2 + x_2^2 > 1. \end{cases}$$

By simple calculations, we obtain $m = 1$, $M = 2$, $c = \sqrt{6}$, $D = \frac{14}{3}$ and $K = 6\left(\frac{12\sqrt{2}}{10^4} + \frac{4}{10^4}\right)$. Taking $\theta_1 = 10^{-4}$, $\theta_4 = 10^2$ and $\eta = 1$, then all conditions in Theorem 3.2 are satisfied. Therefore, it follows that for each

$$\lambda \in \left(\frac{1 - K + 56(1 + K)}{12}, \frac{(1 - K)10^4}{72}\right)$$

for every non-negative continuously differentiable function $G : R^2 \to R$ there exists $\delta^*_{\lambda,G} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,G}[$, the system (3.6) possesses at least three classical solutions $u^1 = (u^1_1, u^1_2)$, $u^2 = (u^2_1, u^2_2)$, and $u^3 = (u^3_1, u^3_2)$ such that

$$\max_{t\in[0,3]} |u^1(t)| < 10^{-4},\ \max_{t\in[0,3]} |u^2(t)| < \frac{1}{\sqrt{2}} 10^2 \text{ and } \max_{t\in[0,3]} |u^3(t)| < 10^2.$$

Following the idea in [3, Corollary 3.1] we present the below result as a consequence of Theorem 3.2 for the system (P^F_λ) when $\mu = 0$.

Theorem 3.3 *Let $F : [0, T] \times R^N \to R$ be a continuous function such that*

$$\exists \xi_i,\ F_{\xi_i}(t, \xi_1, \ldots, \xi_i, \ldots, \xi_N)\xi_i > 0,\ i = 1, \ldots, N$$

and

$$F(t, \xi_1, \ldots, 0, \ldots, \xi_N) \ge 0$$

for all $(t, \xi_1, \ldots, \xi_N) \in [0, T] \times R^N$. Assume that

(A_3) $$\lim_{\xi\to 0^+} \frac{F(t, \xi)}{\sum_{i=1}^N |\xi_i|^2} = \lim_{\xi\to+\infty} \frac{F(t, \xi)}{\sum_{i=1}^N |\xi_i|^2} = 0.$$

Then, for every $\lambda > \overline{\lambda}$ where

$$\overline{\lambda} = \frac{1 - K + c^2(1 + K)DM}{2c^2} \max\left\{ \inf_{\eta>0} \frac{\eta^2}{\int_{t_1}^{t_p} F(t, \eta\varepsilon)dt}, \inf_{\eta<0} \frac{(-\eta)^2}{\int_{t_1}^{t_p} F(t, \eta\varepsilon)dt} \right\},$$

the system (P^F_λ), in the case $\mu = 0$ possesses at least four distinct non-trivial classical solutions.

Proof. Set

$$F_1(t,x) = \begin{cases} F(t,x_1,\ldots,x_N), & \text{if } (t,x_1,\ldots,x_N) \in [0,T]\times[0,+\infty)^N, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_2(t,x) = \begin{cases} F(t,-x_1,\ldots,-x_N), & \text{if } (t,x_1,\ldots,x_N) \in [0,T]\times[0,+\infty)^N, \\ 0, & \text{otherwise.} \end{cases}$$

Fix $\lambda > \lambda^*$, and let $\eta > 0$ such that

$$\lambda > \frac{1-K+c^2(1+K)DM}{2c^2}\frac{\eta^2}{\displaystyle\int_{t_1}^{t_p} F_1(t,\eta\varepsilon)dt}.$$

From

$$\lim_{\xi\to 0^+}\frac{F_1(t,\xi)}{\sum_{i=1}^N \xi_i^2} = \lim_{\xi\to+\infty}\frac{F_1(t,\xi)}{\sum_{i=1}^N \xi_i^2} = 0,$$

there is $\theta_1 > 0$ such that $\theta_1 < \sqrt{Dm}c\eta$ and

$$\frac{\displaystyle\int_0^T \max_{|\xi|\le\theta_1} F_1(t,\xi)dt}{\theta_1^2} < \frac{1-K}{2\lambda c^2}$$

and there is $\theta_4 > 0$ such that $\sqrt{\frac{2DM(1+K)}{1-K}}c\eta < \theta_4$ and

$$\frac{\displaystyle\int_0^T \max_{|\xi|\le\theta_4} F_1(t,\xi)dt}{\theta_4^2} < \frac{1-K}{4\lambda c^2}.$$

Then, (A_2) in Theorem 3.2 is fulfilled,

$$\lambda \in \Lambda'' = \Bigg(\frac{\frac{1-K+c^2(1+K)DM}{2c^2}\eta^2}{\displaystyle\int_{t_1}^{t_p} F_1(t,\eta\varepsilon)dt}, \frac{1-K}{2c^2}\min\Bigg\{\frac{\theta_1^2}{\displaystyle\int_0^T \max_{|\xi|\le\theta_1} F_1(t,\xi)dt}, \frac{\theta_4^2}{2\displaystyle\int_0^T \max_{|\xi|\le\theta_4} F_1(t,\xi)dt}\Bigg\}\Bigg).$$

Hence, the system $(P_\lambda^{F_1})$, in the case $\mu = 0$ admits two solutions $u^1 = (u_1^1,\ldots,u_N^1)$, $u^2 = (u_1^2,\ldots,u_N^2)$, which are positive solutions of the system (P_λ^F), in the case $\mu = 0$. Next, by similar arguments, from

$$\lim_{\xi\to 0^+}\frac{F_2(t,\xi)}{\sum_{i=1}^N \xi_i^2} = \lim_{\xi\to+\infty}\frac{F_2(t,\xi)}{\sum_{i=1}^N \xi_i^2} = 0,$$

we ensure the existence of two solutions $u^3=(u_1^3,\dots,u_N^3)$, $u^4=(u_1^4,\dots,u_N^4)$ for the problem $(P_\lambda^{F_2})$, in the case $\mu=0$. Clearly, $-u^3=(-u_1^3,\dots,-u_N^3)$, $-u^4=(-u_1^4,\dots,-u_N^4)$ are solutions of the system (P_λ^F), in the case $\mu=0$ and the conclusion is achieved. □

Remark 3.1 We explicitly observe that in Theorem 3.3 no symmetric condition on F is assumed. However, whenever F is an even continuous nonzero function such that $F(t,x_1,\dots,x_N)\ge 0$ for all $(t,x_1,\dots,x_N)\in[0,T]\times[0,+\infty)^N$, (A_3) can be replaced by

$$(\mathrm{A}_4)\quad \lim_{\xi\to 0^+}\frac{F(t,\xi)}{\sum_{i=1}^N \xi_i^2}=\lim_{\xi\to+\infty}\frac{F(x,\xi)}{\sum_{i=1}^N \xi_i^2}=0,$$

ensuring the existence of at least four distinct non-trivial solutions classical the system (P_λ^F), in the case $\mu=0$ for every $\lambda>\lambda^*$ where

$$\lambda^*=\frac{1-K+c^2(1+K)DM}{2c^2}\,\frac{\eta^2}{\displaystyle\int_{t_1}^{t_p}F(t,\eta\varepsilon)dt}.$$

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Received: 2.XII.2017 / Accepted: 8.III.2018

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