An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXV, 2019, f. 1

$\mathscr{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold

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Abstract In this paper, we study $\mathscr{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}$. The necessary and sufficient condition is obtained for the distributions allied to the characterization of a $\mathscr{PR}$-pseudo-slant submanifold being integrable and totally geodesic foliation. In addition, we have defined $\mathscr{PR}$-pseudo-slant warped product submanifold of $\widetilde{M}$ and gave some illustrations. Finally, we extracted the constraints for a submanifold of $\widetilde{M}$ to be a $\mathscr{PR}$-pseudo-slant warped product of the form $F \times_f N_\lambda$.

Keywords Warped product · paracontact manifold · pseudo-slant submanifold

Mathematics Subject Classification (2010) 53B25 · 53B30 · 53C25 · 53D15

1 Introduction

The warped product submanifolds of a pseudo-Riemannian manifold served as a fruitful platform in exploring, identifying and to solving problems in mathematical physics, especially in different models of spacetime, black holes, Ricci flow and Hamiltonian spaces (c.f. [1, 8, 13, 17, 23]). In [20], Bishop and O'Neill initiated the theory of warped product submanifold as a generalization of pseudo-Riemannian product manifolds. The study has attained momentum when Chen [9] introduced the geometric depiction of warped product CR-submanifolds in Kahlerian manifold $\widetilde{N}$ through differential point of view and proved the non-existence of proper warped product CR-submanifolds in the form $N_\perp \times {}_f N_T$ such that N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\widetilde{N}$.

Apart from warped product submanifolds, there is a major generalization of both holomorphic and totally real submanifolds, called slant submanifolds. Chen began the concept of slant submanifolds in complex geometry [10]. Later on, slant and semi-slant submanifolds in contact Riemannian geometry are studied by Lotta and Cabrerizo, respectively [6, 7, 19]. Since then several geometers have contributed many important characterization to the geometry of warped product slant submanifolds in an almost contact, complex and lorentzian manifolds (c.f., [12,24,25]). This geometric setting may not found suitable in mathematical physics particularly, in the theory of space time and black holes, where the metric is not necessarily positive definite. Thus, the geometry of warped product slant submanifolds with indefinite metric became a topic of investigation. Recently, Chen-Munteanu, brought our attention to the geometry of $\mathscr{P}\mathscr{R}$-warped products in para-Kähler manifolds [11]. Motivated by the work of [11], the authors have studied $\mathscr{P}\mathscr{R}$-warped product in paracontact manifold which can be viewed as the counterpart of para-Kähler manifold [22].

This paper is organized as follows. In Sect. 2, the basic informations about almost paracontact metric manifolds, nearly paracosymplectic manifold and slant submanifold are given. Sect. 3 concerned with $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold of a nearly paracosymplectic manifold $\widetilde{M}$. The integrability and totally geodesic foliation conditions for the distributions involved with the definition are drawn. In Sect. 4, we define $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifold M and investigate the existence and nonexistence results for such submanifolds of $\widetilde{M}$. Further, we give some examples of a $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold of the form $F \times_f N_\lambda$, where F is anti-invariant and N_λ is proper slant submanifold of a nearly paracosymplectic manifold $\widetilde{M}$. Finally, we derive the necessary and sufficient condition for a submanifold of $\widetilde{M}$ to be a $\mathscr{P}\mathscr{R}$-pseudo-slant warped product and product of the form $F \times_f N_\lambda$.

2 Preliminaries

2.1 Almost paracontact metric manifolds

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ is said to have an almost paracontact structure (φ, ξ, η), if there exist on $\widetilde{M}$ a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2.1}$$

where Id is the identity transformation and the tensor field φ induces an almost paracomplex structure on the distribution $D = \ker(\eta)$, that the eigen distributions $D^{\pm}$ corresponding to the eigenvalues ± 1 have equal dimensions n. From the equation (2.1), it can be easily deduced that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0 \quad \text{and} \quad \operatorname{rank}(\varphi) = 2n. \tag{2.2}$$

If the manifold $\widetilde{M}$ has an almost paracontact structure (φ, ξ, η) and admits a nondegenerate pseudo-Riemannian metric g on $\widetilde{M}$ such that

$$g(X,Y) = -g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.3}$$

where signature of g is necessarily $(n+1,n)$ for any vector fields X and Y; then the quadruple (φ,ξ,η,g) is called an almost paracontact metric structure and the manifold $\widetilde{M}$ equipped with paracontact metric structure is called an almost paracontact metric manifold [21,26]. With respect to g, η is metrically dual to ξ, that is

$$g(X,\xi)=\eta(X). \tag{2.4}$$

With the consequences of Eqs. (2.1), (2.2) and (2.3) we deduce

$$g(\varphi X,Y)=-g(X,\varphi Y), \tag{2.5}$$

for any $X,Y\in\Gamma(T\widetilde{M})$. Here $\Gamma(T\widetilde{M})$ is the tangent bundle of $\widetilde{M}$. Finally, the fundamental 2-form Φ on $\widetilde{M}$ is given by

$$g(X,\varphi Y)=\Phi(X,Y). \tag{2.6}$$

For any $X,Y\in\Gamma(T\widetilde{M})$, the covariant derivative of tensor field φ is defined as

$$(\widetilde{\nabla}_X\varphi)Y=\widetilde{\nabla}_X\varphi Y-\varphi\widetilde{\nabla}_XY, \tag{2.7}$$

where $\widetilde{\nabla}$ is Levi-Civita connection on $\widetilde{M}$.

Definition 2.1 *For all $X,Y\in\Gamma(T\widetilde{M})$ an almost paracontact metric manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ is called [14,15,18].*

- *nearly para Sasakian if*

$$(\widetilde{\nabla}_X\varphi)Y+(\widetilde{\nabla}_Y\varphi)X=2g(X,Y)\xi+(\eta(X)Y+\eta(Y)X).$$

- *paracosymplectic if the forms η and Φ are parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$,* i.*e.,*

$$\widetilde{\nabla}\eta=0 \quad \text{and} \quad \widetilde{\nabla}\Phi=0.$$

- *nearly paracosymplectic if φ is Killing,* i.*e.,*

$$(\widetilde{\nabla}_X\varphi)Y+(\widetilde{\nabla}_Y\varphi)X=0 \tag{2.8}$$

or equivalently,

$$(\widetilde{\nabla}_X\varphi)X=0. \tag{2.9}$$

Following Blair [4,5] we can write:

Proposition 2.2 *On a nearly paracosymplectic manifold the vector field ξ is Killing.*

2.2 Geometry of slant submanifolds

Let M be a submanifold immersed in a $(2n+1)$-dimensional almost paracontact manifold $\widetilde{M}$; we denote by the same symbol g the induced non-degenerate metric on M. If $\Gamma(TM)$ denotes the tangent bundle of submanifold M and $\Gamma(TM^{\perp})$ the set of vector fields normal to M then Gauss and Weingarten formulas are given by respectively

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.10}$$

$$\widetilde{\nabla}_X \zeta = -A_{\zeta} X + \nabla^{\perp}_X \zeta. \tag{2.11}$$

for any $X,Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$, where ∇ is the induced connection, $\nabla^{\perp}$ is the normal connection on the normal bundle $\Gamma(TM^{\perp})$, h is the second fundamental form, and the shape operator A_{ζ} associated with the normal section ζ is given by

$$g\left(A_{\zeta}X,Y\right) = g\left(h(X,Y),\zeta\right). \tag{2.12}$$

If we write, for all $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$ that

$$\varphi X = tX + nX, \tag{2.13}$$

$$\varphi\zeta = t'\zeta + n'\zeta, \tag{2.14}$$

where tX (resp., nX) is tangential (resp., normal) part of φX and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $\varphi\zeta$. Then the submanifold M is said to be *invariant* if n is identically zero and *anti-invariant* if t is identically zero. From Eqs. (2.5) and (2.13), we obtain that

$$g(X,tY) = -g(tX,Y). \tag{2.15}$$

The mean curvature vector H of M is given by $H = \frac{1}{n}\operatorname{trace} h$. A submanifold M is said to be [11]

- *totally geodesic* if its second fundamental form vanishes identically.
- *umbilical* in the direction of a normal vector field ζ on M, if $A_{\zeta} = \delta Id$, for certain function δ on M, here such ζ is called a umbilical section.
- *totally umbilical* if M is umbilical with respect to every (local) normal vector field.

Let $\mathscr{T}_X Y$ be the tangential and $\mathscr{N}_X Y$ be the normal part of $(\widetilde{\nabla}_X \varphi)Y$ then

$$(\widetilde{\nabla}_X \varphi)Y = \mathscr{T}_X Y + \mathscr{N}_X Y,\ \forall X,Y \in \Gamma(TM). \tag{2.16}$$

For later use we can verify the property of $\mathscr{T}$, given by

$$g(\mathscr{T}_X Y, N) = -g(Y, \mathscr{T}_X N), \tag{2.17}$$

$\forall X,Y,N \in \Gamma(TM)$. On a submanifold M of a nearly parcosymplectic manifold, by Eqs. (2.8) and (2.16), we have

$$\mathscr{T}_X Y + \mathscr{T}_Y X = 0, \tag{2.18}$$

$$\mathscr{N}_X Y + \mathscr{N}_Y X = 0, \tag{2.19}$$

for any $X,Y \in \Gamma(TM)$.
Following the notion of slant submanifold in [2,7], we give the following definition:

Definition 2.3 *Let M be an isometrically immersed submanifold of an almost paracontact manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ and $\mathfrak{D}_\lambda$ be the non-degenerate distribution on M. Then $\mathfrak{D}_\lambda$ is said to be slant distribution of $\widetilde{M}$ if there exists a constant $\lambda \geq 0$ such that*

$$t^2 = \lambda\,(Id - \eta \otimes \xi)\,, \quad g(tX,Y) = -g(X,tY)$$

for any nonzero vectors $X,Y \in \mathfrak{D}_\lambda$ at $p \in M$ and not proportional to ξ_p. Here, λ is a slant coefficient of M.

Remark 2.1 It is important to note that the invariant and anti-invariant immersions with slant coefficient $\lambda = 1$ and $\lambda = 0$ respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper slant* immersion.

3 $\mathscr{P}\mathscr{R}$-pseudo-slant submanifolds

In this section, we define *$\mathscr{P}\mathscr{R}$-pseudo-slant* submanifolds of an almost paracontact pseudo-Riemannain metric manifold and derive characterization results for the same.

Definition 3.1 *Let M be an isometrically immersed submanifold of an almost paracontact metric manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ such that the characteristic vector field $\xi \in \Gamma(TM)$. Then M is said to be a $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathfrak{D}^\perp,\mathfrak{D}_\lambda)$ satisfies the following conditions:*

(i) $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}_\lambda \oplus <\xi>$,
(ii) the distribution $\mathfrak{D}^\perp$ is anti-invariant under φ, i.e.*, $\varphi(\mathfrak{D}^\perp) \subseteq \Gamma(TM)^\perp$ and*
(iii) the distribution $\mathfrak{D}_\lambda$ is slant distribution with slant coefficient λ.

We say that a $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold is proper if $\mathfrak{D}^\perp \neq \{0\}$, $\mathfrak{D}_\lambda \neq \{0\}$ and $\lambda \neq 0,1$. A $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold is said to be mixed totally geodesic if $h(X,Z) = 0$ for all $X \in \Gamma(\mathfrak{D}_\lambda)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus <\xi>)$.

Now, we can give the following important corollary as a straight forward consequence of the definition of slant submanifold of an almost paracontact manifold:

Proposition 3.2 *Let M be a slant submanifold of an almost paracontact metric manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TM)$. Then*

$$g(tX, tY) = \lambda g(\varphi X, \varphi Y), \tag{3.1}$$
$$g(nX, nY) = (1-\lambda) g(\varphi X, \varphi Y), \tag{3.2}$$

for any $X, Y \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. Since $g(tX, tY) = -g(X, t^2Y) = -\lambda g(X, \varphi^2 Y) = \lambda g(\varphi X, \varphi Y)$. Therefore, by the use of Eq. (2.3) and definition 2.3, we get Eq. (3.1). Eq. (3.2) follows from Eqs. (2.13) and (3.1). This completes the proof. □

Proposition 3.3 *Let M be an immersed submanifold of an almost paracontact metric manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then for any $X, Y \in \Gamma(\mathfrak{D}_\lambda)$, we have*

(i) $t'nX = (1-\lambda)(X - \eta(X)\xi)$ *and*
(ii) $n'nX = -ntX$.

Proof. We have from Eq. (2.14) that $\varphi nX = t'nX + n'nX$. Then, taking inner product with Y and applying Eqs. (2.1) and (3.2) we derive the formula-(i). For formula-(ii), we have from Eq. (2.13) and definition 2.3 that

$$\varphi tX = t^2X + ntX = \lambda(X - \eta(X)\xi) + ntX. \tag{3.3}$$

On the other hand, using Eq. (2.14) and formula-(i) it can be written that

$$\varphi nX = t'nX + n'nX = (1-\lambda)(X - \eta(X)\xi) + n'nX. \tag{3.4}$$

From Eqs. (3.3) and (3.4) we get $\varphi tX + \varphi nX = (X - \eta(X)\xi) + ntX + n'nX$. Formula-(ii) can be achieved by employing Eqs. (2.1) and (2.13) in previous expression. This completes the proof. □

Now, we give the necessary and sufficient condition for integrability and totally geodesic foliation of distributions equipped with the submanifold M.

Theorem 3.4 *Let M be a proper $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$. Then the distribution $\mathfrak{D}_\lambda$ is integrable if and only if*

$$2\lambda g(\widetilde{\nabla}_X Y, Z) = g(A_{ntY}X, Z) + g(A_{ntX}Y, Z) - g(A_{\varphi Z}tY, X) - g(A_{\varphi Z}tX, Y), \tag{3.5}$$

for any $Z \in \Gamma(\mathfrak{D}^\perp \oplus \xi)$ and $X, Y \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. From the fact that ξ is Killing and Eq. (2.3), we have

$$g([X,Y], Z) = -g(\varphi \widetilde{\nabla}_X Y, \varphi Z) + \eta(\widetilde{\nabla}_X Y)\eta(Z) - g(\widetilde{\nabla}_Y X, Z). \tag{3.6}$$

Using Eqs. (2.8) and (2.7) in Eq. (3.6), we obtain that

$$g([X,Y],Z) = -g(\widetilde{\nabla}_X tY, \varphi Z) - g(\widetilde{\nabla}_X nY, \varphi Z) - g((\widetilde{\nabla}_Y \varphi)X, \varphi Z) - g(\widetilde{\nabla}_Y X, Z). \tag{3.7}$$

Applying connection property and Eq. (2.10) in equation (3.7), we get

$$g([X,Y],Z) = -g(h(X,tY), \varphi Z) + g(\widetilde{\nabla}_X \varphi Z, nY) \\ - g(\widetilde{\nabla}_Y \varphi X - \varphi \widetilde{\nabla}_Y X, \varphi Z) - g(\widetilde{\nabla}_Y X, Z).$$

By the use of Eqs. (2.1), (2.3), (2.13) and covariant differentiation of φ, the above expression reduces to

$$g([X,Y],Z) = -g(h(X,tY), \varphi Z) + g((\widetilde{\nabla}_X \varphi)Z + \varphi \widetilde{\nabla}_X Z, nY) \\ - g(\widetilde{\nabla}_Y (tX + nX), \varphi Z) - 2g(\widetilde{\nabla}_Y X, Z). \tag{3.8}$$

Employing Eqs. (2.14), (2.16) and the fact that structure is nearly paracosymplectic in (3.8), we receive that

$$g([X,Y],Z) = -g(h(X,tY), \varphi Z) - g(\widetilde{\nabla}_X Z, t'nY + n'nY) \\ - g(h(Y,tX), \varphi Z) - g(\widetilde{\nabla}_Y Z, t'nX + n'nX) - 2g(\widetilde{\nabla}_Y X, Z). \tag{3.9}$$

In light of proposiotion 3.3 and connection property, equation (3.9) yields

$$g([X,Y],Z) = -g(h(X,tY), \varphi Z) + (1-\lambda)g(\widetilde{\nabla}_X Y, Z) + g(\widetilde{\nabla}_X Z, ntY) \\ - g(h(Y,tX), \varphi Z) + (1-\lambda)g(\widetilde{\nabla}_Y X, Z) + g(\widetilde{\nabla}_Y Z, ntX) - 2g(\widetilde{\nabla}_Y X, Z).$$

From above equation, we conclude that

$$2\lambda g(\widetilde{\nabla}_X Y, Z) = -g(h(X,tY), \varphi Z) + g(\widetilde{\nabla}_X Z, ntY) \\ - g(h(Y,tX), \varphi Z) + g(\widetilde{\nabla}_Y Z, ntX). \tag{3.10}$$

By the virtue of Eqs. (2.12) and (3.10), we obtain equation (3.5). This completes the proof of the theorem. □

Theorem 3.5 *Let M be a proper $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$. Then the distribution $(\mathfrak{D}^\perp \oplus \xi)$ defines a totally geodesic foliation if and only if*

$$2g(A_{ntX}Z, W) = g(A_{\varphi W}Z, tX) + g(A_{\varphi Z}W, tX), \tag{3.11}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \xi)$ and $X \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. We have from Eq. (2.3) and the fact that X and ξ are orthogonal that

$$g(\widetilde{\nabla}_Z W, X) = -g(\varphi\widetilde{\nabla}_Z W, \varphi X). \tag{3.12}$$

Using Eqs. (2.5), (2.13) and (2.7) in Eq. (3.12), we get

$$\begin{aligned} g(\widetilde{\nabla}_Z W, X) &= -g(\varphi\widetilde{\nabla}_Z W, tX) - g(\varphi\widetilde{\nabla}_Z W, nX) \\ &= -g(\widetilde{\nabla}_Z \varphi W - (\widetilde{\nabla}_Z \varphi)W, tX) + g(\widetilde{\nabla}_Z W, \varphi nX). \end{aligned} \tag{3.13}$$

Employing Eq. (2.16) and Proposition 3.3 in (3.13), we obtain that

$$\begin{aligned} g(\widetilde{\nabla}_Z W, X) = {} & g(\widetilde{\nabla}_Z tX, \varphi W) + g((\widetilde{\nabla}_Z \varphi)W, tX) + (1-\lambda)g(\widetilde{\nabla}_Z W, X) \\ & - g(\widetilde{\nabla}_Z W, ntX). \end{aligned}$$

Above expression on using nearly paracosymplectic structure and Eq. (2.7) reduces to

$$\begin{aligned} g(\widetilde{\nabla}_Z W, X) = {} & g(\widetilde{\nabla}_Z tX, \varphi W) - g(\widetilde{\nabla}_W \varphi Z - \varphi\widetilde{\nabla}_W Z, tX) + (1-\lambda)g(\widetilde{\nabla}_Z W, X) \\ & - g(\widetilde{\nabla}_Z W, ntX). \end{aligned} \tag{3.14}$$

From above expression, we conclude that

$$\begin{aligned} \lambda g(\widetilde{\nabla}_Z W, X) = {} & g(\widetilde{\nabla}_Z tX, \varphi W) - g(\widetilde{\nabla}_W \varphi Z, tX) - g(\widetilde{\nabla}_W Z, t^2X + ntX) \\ & - g(\widetilde{\nabla}_Z W, ntX). \end{aligned} \tag{3.15}$$

By the use of Eq. (2.12) and the fact that $\eta(X) = 0$, we obtain from Eq. (3.15) that

$$\begin{aligned} \lambda g(\widetilde{\nabla}_Z W, X) = {} & g(h(Z, tX), \varphi W) + g(h(W, tX), \varphi Z) - \lambda g(\widetilde{\nabla}_W Z, X) \\ & - 2g(h(W, Z), ntX). \end{aligned}$$

By the virtue of Eq. (2.12) the above expression yields (3.11). This completes the proof of the theorem. □

4 $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifolds

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and f be a positive smooth function on B. Consider the product manifold $B \times F$ with canonical projections

$$\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F. \tag{4.1}$$

Then the manifold $M = B \times_f F$ is said to be *warped product* if it is equipped with the following warped metric

$$g(X, Y) = g_B(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_F(\sigma_*(X), \sigma_*(Y)) \tag{4.2}$$

for all $X,Y \in TM$ and $*$ stands for derivation map, or equivalently,

$$g = g_B + f^2 g_F. \tag{4.3}$$

The function f is called *the warping function* and a warped product manifold M is said to be trivial if f is constant. In view of simplicity, we will determine a vector field X on B with its lift $\widetilde{X}$ and a vector field Z on F with its lift $\widetilde{Z}$ on $M = B \times_f F$ (see also [3,11,22]). Now, we recall the following proposition for the warped product manifolds [3]:

Proposition 4.1 *For $X,Y \in \Gamma(TB)$ and $Z \in \Gamma(TF)$, we obtain on warped product manifold $M = B \times_f F$ that*

(i) $\nabla_X Y \in \Gamma(TB)$,
(ii) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$,

where ∇ denotes the Levi-Civita connections on M.

For a warped product $M = B \times_f F$, B is totally geodesic and F is totally umbilical in M [3].

Definition 4.2 *A $\mathscr{PR}$- pseudo-slant submanifold is called a $\mathscr{PR}$-pseudo-slant warped product if it is a warped product of the form: $N_\lambda \times_f F$ or $F \times_f N_\lambda$, where F is an anti-invariant submanifold, N_λ is a proper slant submanifold of an almost paracontact manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ with slant coefficient λ and f is a non-constant positive function on the first factor. If f is constant then the product of the form: $F \times_f N_\lambda$ or $N_\lambda \times_f F$ is called $\mathscr{PR}$- pseudo-slant product.*

In this section, we shall examine the existence and non existence of proper $\mathscr{PR}$-pseudo-slant warped product submanifolds of a nearly paracosymplectic manifold when $\xi \in \Gamma(TN_\lambda)$ or $\xi \in \Gamma(TF)$.

Proposition 4.3 *There doesn't exist a $\mathscr{PR}$-pseudo-slant warped product submanifold $M = N_\lambda \times_f F$ of a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ such that the characteristic vector field ξ is tangent to F.*

Proof. From Proposition 2.2 and Eq. (2.10) we have $\nabla_X \xi + h(X,\xi) = 0$. Comparing tangential part and using Proposition 4.1 we obtain that $X \ln f = 0$. Thus f is constant, since $X \in \Gamma(TN_\lambda)$ is non null vector field. This completes the proof. □

Theorem 4.4 *Let $M = N_\lambda \times_f F$ is a $\mathscr{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ such that $\xi \in \Gamma(TN_\lambda)$. Then M is a $\mathscr{PR}$-pseudo-slant product if and only if the shape operator satisfies*

$$A_{nZ}X = A_{nX}Z,$$

for any $X \in \Gamma(TN_\lambda)$ and $Z \in \Gamma(TF)$.

Proof. We can write using Eqs. (2.5), (2.12) and Gauss formula that $g(A_{nZ}tX,Z) = -g(\varphi\widetilde{\nabla}_Z tX,Z)$. Now, previous equality in light of Eqs. (2.7), (2.16) and (2.17), simplifies to

$$g(A_{nZ}tX,Z) = -g(\mathscr{T}_Z tX,Z) + g(t^2X,\widetilde{\nabla}_Z Z) + g(ntX,\widetilde{\nabla}_Z Z). \tag{4.4}$$

Again using Eqs. (2.10), (2.17) and definition of slant submanifold in Eq. (4.4), we obatin that

$$g(A_{nZ}tX,Z) = g(tX,\mathscr{T}_Z Z) + \lambda g(X,\widetilde{\nabla}_Z Z) - \lambda\eta(X)g(\xi,\widetilde{\nabla}_Z Z) + g(ntX,h(Z,Z)). \tag{4.5}$$

Above equation, using connection property of $\widetilde{\nabla}$ and fact that the manifold is nearly paracosymplectic, ξ is Killing, reduces to

$$g(A_{nZ}tX,Z) = -\lambda g(\nabla_Z X,Z) + g(ntX,h(Z,Z)). \tag{4.6}$$

By the use of Eq. (2.12) and Propostion 4.1 in Eq. (4.6), we get

$$g(h(tX,Z),nZ) = -\lambda(X\ln f)g(Z,Z) + g(ntX,h(Z,Z)). \tag{4.7}$$

Further, interchanging X by tX and using fact $h(Z,\xi) = 0$, definition of slant submanifold in above equation, we reach at

$$\lambda g(h(X,Z),nZ) = -\lambda(tX\ln f)g(Z,Z) + \lambda g(nX,h(Z,Z)). \tag{4.8}$$

Then Eq. (4.8), using Eq. (2.12) can be written as

$$(tX\ln f)g(Z,Z) = g(A_{nX}Z - A_{nZ}X,Z). \tag{4.9}$$

Thus, from above equation we complete the proof of the theorem, since X and Z are non-degenerate vector fields. □

Remark 4.1 From Proposition 4.3 and Theorem 4.4, one can conclude that the $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifold of the form $M = N_\lambda \times_f F$

- does not exists when $\xi \in \Gamma(TF)$,
- exists when $\xi \in \Gamma(TN_\lambda)$.

Here, we present an example to support the above Remark 4.1 for the existence of $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifold of the form $M = N_\lambda \times_f F$ as $\xi \in \Gamma(TN_\lambda)$.

Example 4.1 Let $\widetilde{M} = \mathbb{R}^6 \times \mathbb{R}_+ \subset \mathbb{R}^7$ be a 7-dimensional manifold with the standard Cartesian coordinates $(x_1,x_2,x_3,y_1,y_2,y_3,z)$. Define the nearly paracosymplectic pseudo-Riemannian metric structure (φ,ξ,η,g) on $\widetilde{M}$ by

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i},\ \varphi\left(\frac{\partial}{\partial t}\right) = 0,\ \xi = \frac{\partial}{\partial z},\ \eta = dz, \tag{4.10}$$
$$g = \sum(dx_i)^2 - \sum(dy_i)^2 + (dz)^2,\ \forall i \in \{1,2,3\}.$$

Now, let M is an isometrically immersed smooth submanifold in $\mathscr{R}^7$ defined by

$$\chi(u,\alpha,v,z) = (u, u\cos\alpha, u\sin\alpha, v, k_1, k_2, z) \tag{4.11}$$

where k_1, k_2 are constants and $u \in \mathbb{R}_+,\ \alpha \in (0,\pi/2)$. Then the TM spanned by the vectors

$$\begin{aligned} Z_u &= \frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3}, \\ Z_\alpha &= -u\sin\alpha\frac{\partial}{\partial x_2} + u\cos\alpha\frac{\partial}{\partial x_3}, \\ Z_v &= \frac{\partial}{\partial y_1},\ Z_z = \frac{\partial}{\partial z}, \end{aligned} \tag{4.12}$$

where $Z_u, Z_\alpha, Z_v, Z_z \in \Gamma(TM)$. Using Eq. (4.10), we obtain that

$$\begin{aligned} \varphi(Z_u) &= \frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_2} + \sin\alpha\frac{\partial}{\partial y_3}, \\ \varphi(Z_\alpha) &= u\sin\alpha\frac{\partial}{\partial y_2} + u\cos\alpha\frac{\partial}{\partial y_3}, \\ \varphi(Z_v) &= \frac{\partial}{\partial x_1},\ \varphi(Z_z) = 0, \end{aligned} \tag{4.13}$$

From Eqs. (4.12) and (4.13) we can find that $\mathfrak{D}_\lambda$ is a proper slant distribution defined by span$\{Z_u, Z_v, Z_z\}$ with slant coefficient $\lambda = \frac{1}{\sqrt{2}}$ and $\mathfrak{D}^\perp$ is an anti-invariant distribution defined by span$\{Z_\alpha\}$ with dimension not equal to zero, where $\xi = Z_z$ and $\eta(Z_z) = 1$. So, M turn into a proper $\mathscr{PR}$-pseudo-slant submanifold. Here, the induced pseudo-Riemannian non-degenerate metric tensor g of M is specified by $g = g_{N_\lambda} + f^2 g_F$. Thus, M is a 4-dimensional $\mathscr{PR}$-pseudo-slant warped product submanifold of $\mathscr{R}^7$ with wrapping function $f = u$.

Now, we study the existence and non-existence of $\mathscr{PR}$-pseudo-slant warped product submanifold of the form $F \times_f N_\lambda$ of a nearly paracosymplectic manifold.

Proposition 4.5 *There doesn't exist a $\mathscr{PR}$-pseudo-slant warped product submanifold $M = F \times_f N_\lambda$ of a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ such that the characteristic vector field ξ is tangent to N_λ.*

Proof. The proof can be achieved by following the same step as in Proposition 4.3. □

Here, we first give an example illustrating proper $\mathscr{PR}$-pseudo-slant submanifolds of the form $M = F \times_f N_\lambda$ of a nearly paracosymplectic manifold $\widetilde{M}$ and then prove an important lemma for later use.

Example 4.2 Let $\widetilde{M} = \mathbb{R}^4 \times \mathbb{R}_+ \subset \mathbb{R}^5$ be a 5-dimensional manifold with the standard Cartesian coordinates (x_1, x_2, y_1, y_2, z). Define the nearly paracosymplectic pseudo-Riemannian metric structure (φ, ξ, η, g) on $\widetilde{M}$ by

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \varphi\left(\frac{\partial}{\partial z}\right) = 0, \xi = \frac{\partial}{\partial z}, \eta = dz, \tag{4.14}$$
$$g = \sum (dx_i)^2 - \sum (dy_i)^2 + (dz)^2, \ \forall i \in \{1,2\}.$$

Now, let M is an isometrically immersed smooth submanifold in $\mathscr{R}^5$ defined by

$$x_1 = v\cosh\alpha, x_2 = v\cosh\beta, y_1 = v\sinh\alpha, y_2 = v\sinh\beta, z = u, \tag{4.15}$$

where $v \in \mathbb{R}_+ - \{1\}$. Then the TM spanned by the vectors

$$\begin{aligned} Z_1 &= \cosh\alpha\frac{\partial}{\partial x_1} + \cosh\beta\frac{\partial}{\partial x_2} + \sinh\alpha\frac{\partial}{\partial y_1} + \sinh\beta\frac{\partial}{\partial y_2}, \\ Z_2 &= v\sinh\alpha\frac{\partial}{\partial x_1} + v\cosh\alpha\frac{\partial}{\partial y_1}, \\ Z_3 &= v\sinh\beta\frac{\partial}{\partial x_2} + v\cosh\beta\frac{\partial}{\partial y_2}, \ Z_4 = \frac{\partial}{\partial z}, \end{aligned} \tag{4.16}$$

where $Z_1, Z_2, Z_3, Z_4 \in \Gamma(TM)$. Therefore from Eq. (4.14), we find that

$$\begin{aligned} \varphi(Z_1) &= \sinh\alpha\frac{\partial}{\partial x_1} + \sinh\beta\frac{\partial}{\partial x_2} + \cosh\alpha\frac{\partial}{\partial y_1} + \cosh\beta\frac{\partial}{\partial y_2}, \\ \varphi(Z_2) &= v\cosh\alpha\frac{\partial}{\partial x_1} + v\sinh\alpha\frac{\partial}{\partial y_1}, \\ \varphi(Z_3) &= v\cosh\beta\frac{\partial}{\partial x_2} + v\sinh\beta\frac{\partial}{\partial y_2}, \ \varphi(Z_4) = 0. \end{aligned} \tag{4.17}$$

From Eqs. (4.16) and (4.17), we obtain that $\mathfrak{D}_\lambda$ is a proper slant distribution given by span$\{Z_1, Z_2, Z_3\}$ with slant coefficient $\lambda = \frac{1}{2}$ and $\mathfrak{D}^\perp$ is an anti-invariant distribution given by span$\{Z_4\}$ with dimension not equal to zero, where $\xi = Z_4$ and $\varphi(Z_4) = 0$, and $\eta(Z_4) = 1$. Therefore, M is a proper $\mathscr{P}\mathscr{R}$-pseudo-slant submanifold of a nearly paracosymplectic manifold $\widetilde{M}$. Here, the induced pseudo-Riemannian metric tensor g of M is given by $g = g_F + f^2 g_{N_\lambda}$. Hence M is a 4-dimensional $\mathscr{P}\mathscr{R}$-pseudo-slant product submanifold of $\mathscr{R}^5$ with $f = v$.

Lemma 4.6 *If $M = F \times_f N_\lambda$ be a $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TF)$ then for all X tangent to N_λ and Z tangent to F, we have*

(a) $2g(A_{ntX}X, Z) = g(A_{\varphi Z}X, tX) + g(A_{nX}Z, tX) - (Z\ln f)\lambda||X||^2$ *and*
(b) $g(A_{nX}Z, tX) = 2g(A_{\varphi Z}X, tX) - g(A_{ntX}X, Z)$.

Proof. From Gauss formula and Eq. (2.12), we can write that

$$g(A_{ntX}X,Z) = g(\widetilde{\nabla}_Z X - \nabla_Z X, ntX). \tag{4.18}$$

Using Eqs. (2.13), (2.7), definition 2.3 and the fact that ξ is orthogonal to X, we get

$$\begin{aligned} g(A_{ntX}X,Z) &= g(\widetilde{\nabla}_Z X, \varphi tX - t^2X) = -g(\varphi\widetilde{\nabla}_Z X, tX) - g(\widetilde{\nabla}_Z X, t^2X) \\ &= -g(\widetilde{\nabla}_Z \varphi X - (\widetilde{\nabla}_Z \varphi)X, tX) - \lambda g(\widetilde{\nabla}_Z X, X). \end{aligned}$$

Above equation by applying definition of nearly paracosymplectic and Proposition 4.1 reduces to

$$g(A_{ntX}X,Z) = -g(\widetilde{\nabla}_Z \varphi X + (\widetilde{\nabla}_X \varphi)Z, tX) - (Z\ln f)\lambda||X||^2. \tag{4.19}$$

Therefore, again using Eqs. (2.13), (2.7) and Proposition 3.2 in (4.19), we obtain that

$$\begin{aligned} g(A_{ntX}X,Z) &= -g(\widetilde{\nabla}_Z nX, tX) - g(\widetilde{\nabla}_X \varphi Z - \varphi\widetilde{\nabla}_X Z, tX) \\ &= g(\widetilde{\nabla}_Z tX, nX) - g(\widetilde{\nabla}_X \varphi Z, tX) - g(\widetilde{\nabla}_X Z, \varphi tX). \end{aligned} \tag{4.20}$$

Employing Eqs. (2.13), (2.12), definition 2.3 and Proposition 4.1 in equation (4.20) we find the formula-(a). For formula-(b): we have from Eq. (2.12) that

$$g(A_{nX}tX,Z) = g(h(tX,Z), nX). \tag{4.21}$$

Applying Gauss formula, Eqs. (2.13) and definition 2.3, we find from above equation that

$$g(A_{nX}tX,Z) = -g(\varphi\widetilde{\nabla}_Z tX, X) + \lambda g(\nabla_Z X, X). \tag{4.22}$$

Using Proposition 4.1, Eqs. (2.7) and (2.8) in equation (4.22), we obtain that

$$g(A_{nX}tX,Z) = -g(\widetilde{\nabla}_Z \varphi tX, X) - g((\widetilde{\nabla}_{tX}\varphi)Z, X) + (Z\ln f)\lambda||X||^2. \tag{4.23}$$

Employing Eq. (2.7), definition 2.3 and Proposition 4.1 in equation (4.23), we get

$$g(A_{nX}tX,Z) = -g(\widetilde{\nabla}_Z ntX, X) - g(\widetilde{\nabla}_{tX}\varphi Z, X) + g(\widetilde{\nabla}_{tX} Z, \varphi X). \tag{4.24}$$

Again using Eqs. (2.13), (2.12) and Gauss-Weingarten formula, we achieve from equation (4.24) that

$$2g(h(tX,Z), nX) = g(h(Z,X), ntX) + g(h(tX,X), \varphi Z) - g(\nabla_Z tX, tX). \tag{4.25}$$

Equation (4.25) by the virtue of Proposition 4.1 and Eq. (3.1) reduces to

$$2g(h(tX,Z), nX) = g(h(Z,X), ntX) + g(h(tX,X), \varphi Z) + (Z\ln f)\lambda g(X,X). \tag{4.26}$$

Interchanging X by tX and using definition 2.3, Proposition 4.1 in equation (4.26), we have

$$2g(h(X,Z), ntX) = g(h(Z,tX), nX) + g(h(tX,X), \varphi Z) - (Z\ln f)\lambda||X||^2. \tag{4.27}$$

Thus, formula-(b) follows from Eqs. (4.26), (4.27) and (2.12). □

Lemma 4.7 *If $M = F \times_f N_\lambda$ be a $\mathcal{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TF)$ then*

$$g(\mathcal{T}_X tX, Z) = g(A_{ntX}X, Z) - g(A_{nX}Z, tX) \tag{4.28}$$

for all X is tangent to N_λ and Z is tangent to F.

Proof. By virtue of Eqs. (2.10), (2.13) and (2.7), we obtain that

$$g(A_{nX}Z, tX) = g(tX, \nabla_Z tX) - g(tX, (\widetilde{\nabla}_{Z\varphi})X) - g(tX, \varphi\widetilde{\nabla}_Z X).$$

By use of Eqs. (2.5), (2.16) and Proposition 4.1, the above equation reduces to

$$g(A_{nX}Z, tX) = (Z\ln f)g(tX, tX) - g(tX, \mathcal{T}_Z X) + g(\varphi(tX), \widetilde{\nabla}_Z X). \tag{4.29}$$

Employing Eqs. (2.3), (2.18), (3.1) and Gauss formula in Eq. (4.29), we have

$$\begin{aligned} g(A_{nX}Z, tX) = &- (Z\ln f)\lambda g(X, X) + g(tX, \mathcal{T}_X Z) + g(t^2X, \nabla_Z X) \\ &+ g(A_{ntX}X, Z). \end{aligned} \tag{4.30}$$

In light of Proposition 4.1, Eq. (2.17) and definition of slant submaifold, Eq. (4.30) yields Eq. (4.28). This completes the proof. □

Here, we prove:

Theorem 4.8 *Let $M = F \times_f N_\lambda$ is a $\mathcal{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TF)$. Then M is a $\mathcal{PR}$-pseudo-slant product if and only if $\mathcal{T}_X tX$ is tangent to N_λ, for all $X \in \Gamma(TN_\lambda)$ and $Z \in \Gamma(TF)$.*

Proof. From Eqs. (2.5), (2.10), (2.12), (2.7) and (2.16), we have

$$g(A_{nZ}X, tX) = g(\widetilde{\nabla}_{tX}X, \varphi Z) = g(\mathcal{T}_{tX}X, Z) + g(\varphi X, \widetilde{\nabla}_{tX}Z).$$

From Gauss fromula and Eqs. (2.13), (2.18), we obtain that

$$g(A_{nZ}X, tX) = -g(\mathcal{T}_X tX, Z) + g(tX, \nabla_{tX}Z) + g(A_{nX}tX, Z).$$

Then, using Eq. (3.1) and Proposition 4.1, we get

$$g(A_{nZ}X, tX) = -(Z\ln f)\lambda g(X, X) - g(\mathcal{T}_X tX, Z) + g(A_{nX}tX, Z). \tag{4.31}$$

Replacing X by tX and, using Eqs. (2.17), (2.18), definition of slant submaifold, (3.1) in Eq. (4.31), we arrive at

$$g(A_{nZ}X, tX) = (Z\ln f)\lambda ||X||^2 + g(\mathcal{T}_X tX, Z) + g(A_{ntX}X, Z). \tag{4.32}$$

Thus, from Eqs. (4.31), (4.32) and lemma (4.28), we derive

$$g(\mathcal{T}_X tX, Z) = \frac{2}{3}(Z\ln f)\lambda ||X||^2. \tag{4.33}$$

Eq. (4.33) implies that, $(Z\ln f) = 0$ if and only if $\mathcal{T}_X tX$ is tangent to N_λ. Since, X, Z are non-null vector fields and the fact that N_λ is proper slant submanifold. This completes the proof. □

Theorem 4.9 *Let $M = F \times_f N_\lambda$ be a mixed totally geodesic $\mathscr{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$ with $\xi \in \Gamma(TF)$. Then M is a $\mathscr{PR}$-pseudo-slant warped product submanifold, for any $X \in \Gamma(TN_\lambda)$ and $Z \in \Gamma(TF)$.*

Proof. From formula-(a) and formula-(b) of lemma 4.6, we obtain that

$$3g(A_{ntX}X,Z) = 3g(A_{\varphi Z}X,tX) - (Z\ln f)\lambda||X||^2. \tag{4.34}$$

Employing Eq. (2.12) in (4.34), we find that

$$3g(h(X,Z),ntX) = 3g(A_{\varphi Z}X,tX) - (Z\ln f)\lambda||X||^2.$$

Since M is mixed totally geodesic, so we achieve from above equation that

$$g(A_{\varphi Z}X,tX) = \frac{1}{3}\{(Z\ln f)\lambda||X||^2\}. \tag{4.35}$$

This completes the proof of the theorem. □

Theorem 4.10 *Let $M \to \widetilde{M}$ be an isometric immersion of a $\mathscr{PR}$-pseudo-slant submanifold M into a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$. Then M is locally $\mathscr{PR}$-pseudo-slant warped product submanifold M of the form $F \times_f N_\lambda$ if and only if shape operator of M satisfies*

$$A_{ntX}Z - A_{\varphi Z}tX = -\frac{1}{3}(\lambda)Z(\mu)X, \forall Z,W \in \Gamma(\mathfrak{D}^\perp \oplus \xi), X \in \Gamma(\mathfrak{D}_\lambda), \tag{4.36}$$

for some function μ on M such that $Y(\mu) = 0$, $Y \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. If M is a $\mathscr{PR}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi,\xi,\eta,g)$. Then from formula-$(a)$ and formula-(b) of lemma 4.6, we derive Eq. (4.36). Since f is a function on F, setting $\mu = \ln f$ implies that $Y(\mu) = 0$. Conversely, let us assume that M is $\mathscr{PR}$-pseudo-slant submanifold of $\widetilde{M}(\varphi,\xi,\eta,g)$ such that Eq. (4.36) holds. Taking inner product of Eq. (4.36) with W and from theorem 3.5 we conclude that the integral manifold F of $(\mathfrak{D}^\perp \oplus \xi)$ defines totally geodesic foliation in M. Then by theorem 3.4, we find that the distribution $\mathfrak{D}_\lambda$ is integrable if and only if

$$2\lambda g(\widetilde{\nabla}_X Y,Z) = g(A_{ntY}X,Z) + g(A_{ntX}Y,Z) - g(A_{\varphi Z}tY,X) - g(A_{\varphi Z}tX,Y). \tag{4.37}$$

From Eq. (2.12) and fact that h is symmetric in Eq. (4.37), we arrive at

$$2\lambda g(\widetilde{\nabla}_X Y,Z) = g(A_{ntY}Z - A_{\varphi Z}tY,X) + g(A_{ntX}Z - A_{\varphi Z}tX,Y) \tag{4.38}$$

Eq. (4.36) yields by taking inner product with Y that

$$g(A_{ntX}Z - A_{\varphi Z}tX,Y) = -\frac{1}{3}g(\lambda Z(\mu)X,Y). \tag{4.39}$$

Now interchanging X by Y in Eq. (4.36) and taking inner product with X, we obtain that

$$g(A_{ntY}Z - A_{\varphi Z}tY, X) = -\frac{1}{3}g(\lambda Z(\mu)Y, X). \tag{4.40}$$

From Eqs. (4.38)-(4.40) and symmetry of h, we conclude that

$$g(h_\lambda(X,Y), Z) = -\frac{1}{3}g(Z(\mu)X, Y) = -\frac{1}{3}g(X,Y)g(\nabla\mu, Z).$$

This implies $h_\lambda(X,Y) = -\frac{1}{3}g(X,Y)\nabla\mu$, where h_λ is a second fundamental form of $\mathfrak{D}_\lambda$ in M and $\nabla\mu$ is gradient of $\mu = \ln f$. Hence, the integrable manifold of $\mathfrak{D}_\lambda$ is totally umbilical submanifold in M and its mean curvature is non-zero and parallel and $Y(\mu) = 0$ for all $Y \in \Gamma(\mathfrak{D}_\lambda)$. Thus, by [16] we achieve that M is a $\mathscr{P}\mathscr{R}$-pseudo-slant warped product submanifold of a nearly paracosymplectic manifold $\widetilde{M}(\varphi, \xi, \eta, g)$. This completes the proof of the theorem. □

Acknowledgements S. K. Srivastava: partially supported through the UGC-BSR Start-Up-Grant vide their letter no. F.30-29/2014(BSR). The authors are grateful to the anonymous reviewer's valuable comments that improved the manuscript.

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Received: 31.III.2016 / Revised: 29.VIII.2016 / Accepted: 25.X.2016

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