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General weak amenability of module extension Banach algebras

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Abstract The main aim of the current paper is to find necessary and sufficient conditions for a module extension Banach algebra $\mathcal{A}\oplus X$ to be $(\Phi_\varphi, \Psi_\psi)$-weakly amenable, where φ, ψ are homomorphisms on Banach algebra $\mathcal{A}$ and Φ_φ, Ψ_ψ are homomorphisms on $\mathcal{A}\oplus X$.

Keywords Amenability · Module extension Banach algebra · Weak amenability · Homomorphism

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1 Introduction

The notion of amenability for Banach algebras was first introduced by B. E. Johnson [7] in 1972. An amenable Banach algebra $\mathcal{A}$ has been known to satisfy $H^1(\mathcal{A}, X^*) = \{0\}$ for each Banach $\mathcal{A}$-bimodule X, where X^* is the dual module of X and $H^1(\mathcal{A}, X^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients of X^*. Since the notion of amenability was considered, several generalizations of this concept such as weak amenability, approximate amenability, character amenability were introduced for Banach algebras; these notions can be found in [1], [6] and [10].

The notion of weak amenability for Banach algebras was introduced by Bade, Curtis and Dales in [1]. A Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation from $\mathcal{A}$ into $\mathcal{A}^*$-module is inner, or equivalently if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. They considered this concept only for commutative Banach algebras. Next, Johnson defined the weak amenability for arbitrary Banach algebras [8]. Furthermore, the weak amenability of module extension Banach algebras has been studied by Zhang in [13]. In the papers [9] and [4], the authors have defined and studied two concepts of the amenability and weak amenability for Banach algebras by applying homomorphisms which are different from the classical cases.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote the metric space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$ by Hom$(\mathcal{A},\mathcal{B})$, where the metric derived from the bounded linear operators from $\mathcal{A}$ into $\mathcal{B}$, and denote Hom$(\mathcal{A},\mathcal{A})$ by Hom$(\mathcal{A})$. Let X be an $\mathcal{A}$-bimodule, and let $\sigma,\tau \in$Hom$(\mathcal{A})$. A bounded linear map $D:\mathcal{A}\longrightarrow X$ is called a (σ,τ)-*derivation* if

$$D(ab)=D(a)\cdot\sigma(b)+\tau(a)\cdot D(b)\quad (a,b\in\mathcal{A}).$$

Moreover, a bounded linear map $d:\mathcal{A}\longrightarrow X$ is called a (σ,τ)-*inner derivation* if there exists $x\in X$ such that

$$d(a)=x\cdot\sigma(a)-\tau(a)\cdot x\quad (a\in\mathcal{A}).$$

Let $\sigma,\tau\in$Hom$(\mathcal{A},\mathcal{B})$. Then, $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule with the following module actions:

$$a\cdot b=\tau(a)b,\quad b\cdot a=b\sigma(a)\qquad (a\in\mathcal{A},b\in\mathcal{B})$$

We denote the above $\mathcal{A}$-bimodule by $\mathcal{B}_{\sigma,\tau}$ and denote $\mathcal{B}_{\sigma,\tau}$ by $\mathcal{B}_\sigma$ if $\sigma=\tau$. A Banach algebra $\mathcal{A}$ is called (σ,τ)-*weakly amenable* if $H^1(\mathcal{A},(\mathcal{A}_{(\sigma,\tau)})^*)=\{0\}$. These concepts have been introduced and studied in [4] and [9]. For the module versions of these notions refer to [2] and [3].

The main motivation for this paper is taken from [13]. In this paper, we obtain all conditions under which a module extension Banach algebra $\mathcal{A}\oplus X$ can be (Φ_φ,Ψ_ψ)-weakly amenable, where $\varphi,\psi\in$Hom$(\mathcal{A})$ and $\Phi_\varphi,\Psi_\psi\in$Hom$(\mathcal{A}\oplus X)$.

2 Main results

Let $\mathcal{A}$ be a Banach algebra and X be an $\mathcal{A}$-bimodule. Consider $\mathcal{A}\oplus X$ as a Banach space with the norm

$$\|(a,x)\|=\|a\|+\|x\|\qquad (a\in\mathcal{A},x\in x).$$

Then, $\mathcal{A}\oplus X$ is a Banach algebra with the multiplication as follows:

$$(a_1,x_1)\cdot(a_2,x_2)=(a_1a_2,x_1\cdot a_2+a_1\cdot x_2).$$

$\mathcal{A}\oplus X$ is called a *module extension* Banach algebra. It is known that $(\mathcal{A}\oplus X)^*=(0+X)^\perp\dot{+}(\mathcal{A}\oplus 0)^\perp$, where $\dot{+}$ denotes the direct $\mathcal{A}$-bimodule l_∞-sum, and $(0\oplus X)^\perp$(resp. $(\mathcal{A}\oplus 0)^\perp$) is isometrically isomorphic to $\mathcal{A}^*$ (resp. X^*) as $\mathcal{A}$-bimodule. For convenience, we simply identify the corresponding terms and write

$$(\mathcal{A}\oplus X)^*=\mathcal{A}^*\dot{+}X^*.$$

Definition 2.1 *Let $\mathcal{A}$ be a Banach algebra, X and Y be left (right) $\mathcal{A}$-module and $\phi\in$Hom$(\mathcal{A})$. A mapping $\Gamma : X \longrightarrow Y$ is said to be a left (right) ϕ-module morphism if*

$$\Gamma(a\cdot x)=\phi(a)\cdot\Gamma(x)\qquad(\Gamma(x\cdot a)=\Gamma(x)\cdot\phi(a)).$$

In addition, Γ is called a ϕ-bimodule morphism if Γ is a left ϕ-module and right ϕ-module morphism.

Let $\phi\in$Hom$(\mathcal{A})$ and Φ be a ϕ-bimodule morphism from X into itself. It is easily verified that $T:\mathcal{A}\oplus X\longrightarrow\mathcal{A}\oplus X$ defined by $T(a,x)=(\phi(a),\Phi(x))$ belongs to Hom$(\mathcal{A}\oplus X)$. Throughout this paper, such homomorphism is denoted by Φ_ϕ.

Here, we indicate the following lemma which is the fundamental tool to achieve our goal in this paper.

Lemma 2.2 *Every (Φ_φ,Ψ_ψ)-derivation $D:\mathcal{A}\oplus X\longrightarrow(A\oplus X)^*$ enjoys the presentation*

$$D(a,x)=\big(D_{\mathcal{A}}(a)+\Gamma_{\mathcal{A}}(x),D_X(a)+\Gamma_X(x)\big)$$

where

(1) *$D_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathcal{A}^*$ and $D_X:\mathcal{A}\longrightarrow X^*$ are (φ,ψ)-derivations.*
(2) *$\Gamma_{\mathcal{A}}:X\longrightarrow\mathcal{A}^*$ is a bounded linear map such that*

$$\Gamma_{\mathcal{A}}(a\cdot x)=D_X(a)\cdot\Phi(x)+\psi(a)\cdot\Gamma_{\mathcal{A}}(x),$$
$$\Gamma_{\mathcal{A}}(x\cdot a)=\Gamma_{\mathcal{A}}(x)\cdot\varphi(a)+\Psi(x)\cdot D_X(a).$$

(3) *$\Gamma_X:X\longrightarrow X^*$ is a left ψ-module and a right φ-module morphism such that $\Gamma_X(x)\cdot\Phi(y)+\Psi(x)\cdot\Gamma_X(y)=0$.*

Proof. Let $D:\mathcal{A}\oplus X\longrightarrow(\mathcal{A}\oplus X)^*$ defined by $D(a,x)=(\ell(a,x),k(a,x))$ be a (Φ_φ,Ψ_ψ)-derivation, where $\ell:\mathcal{A}\oplus X\longrightarrow\mathcal{A}^*$ and $k:\mathcal{A}\oplus X\longrightarrow X^*$ are two mappings. We also consider $\Delta_1:\mathcal{A}^*\dot{+}X^*\longrightarrow\mathcal{A}^*$, $\Delta_2:\mathcal{A}^*\dot{+}X^*\longrightarrow X^*$, as projection and $i_1:\mathcal{A}\longrightarrow\mathcal{A}\oplus X$, $i_2:X\to\mathcal{A}\oplus X$ as inclusion mappings. Put $D_{\mathcal{A}}=\Delta_1\circ D\circ i_1:\mathcal{A}\longrightarrow\mathcal{A}^*$. Then, $D_{\mathcal{A}}$ is a (φ,ψ)-derivation. Indeed, for each $a,b\in\mathcal{A}$, we get

$$\begin{aligned}D_{\mathcal{A}}(ab)&=\Delta_1\circ D\circ i_1(ab)=\Delta_1\circ D(ab,0)=\Delta_1\circ D\big((a,0)(b,0)\big)\\&=\Delta_1\big(D(a,0)\cdot(\varphi(b),0)+(\psi(a),0)\cdot D(b,0)\big)\\&=\Delta_1\Big(\big(\ell(a,0),k(a,0)\big)\cdot\big(\varphi(b),0\big)+\big(\psi(a),0\big)\cdot\big(\ell(b,0),k(b,0)\big)\Big)\\&=\Delta_1\Big(\big(\ell(a,0)\cdot\varphi(b),k(a,0)\cdot\varphi(b)\big)+\big(\psi(a)\cdot\ell(b,0),\psi(a)\cdot k(b,0)\big)\Big)\\&=\ell(a,0)\cdot\varphi(b)+\psi(a)\cdot\ell(b,0)\\&=\Delta_1\circ D\circ i_1(a)\cdot\varphi(b)+\psi(a)\cdot\Delta_1\circ D\circ i_1(b)\\&=D_{\mathcal{A}}(a)\cdot\varphi(b)+\psi(a)\cdot D_{\mathcal{A}}(b).\end{aligned}$$

Let $D_X = \varDelta_2 \circ D \circ i_1$. Similar to the above mentioned equations, one can check that D_X is a (φ, ψ)-derivation. Define $\varGamma_{\mathcal{A}} : X \longrightarrow \mathcal{A}^*$ by $\varGamma_{\mathcal{A}}(x) = \varDelta_1 \circ D \circ i_2(x)$. Now, for each $a \in \mathcal{A}$ and $x \in X$ we have

$$\begin{aligned}
\varGamma_{\mathcal{A}}(a \cdot x) &= \varDelta_1 \circ D \circ i_2(a \cdot x) = \varDelta_1 \circ D\big((a,0)(0,x)\big) \\
&= \varDelta_1\Big(D(a,0) \cdot (0, \varPhi(x)) + \big(\psi(a), 0\big) \cdot D(0,x)\Big) \\
&= \varDelta_1\Big(\big(\ell(a,0), k(a,0)\big) \cdot \big(0, \varPhi(x)\big) + \big(\psi(a), 0\big) \cdot \big(\ell(0,x), k(0,x)\big)\Big) \\
&= \varDelta_1\Big(\big(k(a,0) \cdot \varPhi(x), 0\big) + \big(\psi(a) \cdot \ell(0,x), \psi(a) \cdot k(0,x)\big)\Big) \\
&= k(a,0) \cdot \varPhi(x) + \psi(a) \cdot \ell(0,x) \\
&= D_X(a) \cdot \varPhi(x) + \psi(a) \cdot \varGamma_{\mathcal{A}}(x),
\end{aligned}$$

and similarly

$$\begin{aligned}
\varGamma_{\mathcal{A}}(x \cdot a) &= \varDelta_1 \circ D\big((0,x) \cdot (a,0)\big) \\
&= \varDelta_1\Big(D(0,x) \cdot \big(\varphi(a), 0\big) + \big(0, \varPsi(x)\big) \cdot D(a,0)\Big) \\
&= \varDelta_1\Big(\big(\ell(0,x), k(0,x)\big) \cdot \big(\varphi(a), 0\big) + \big(0, \varPsi(x)\big) \cdot \big(\ell(a,0), k(a,0)\big)\Big) \\
&= \varDelta_1\big(\ell(0,x) \cdot \varphi(a) + \varPsi(x) \cdot k(a,0), k(0,x) \cdot \varphi(a)\big) \\
&= \ell(0,x) \cdot \varphi(a) + \varPsi(x) \cdot k(a,0) \\
&= \varGamma_{\mathcal{A}}(x) \cdot \varphi(a) + \varPsi(x) \cdot D_X(a).
\end{aligned}$$

Once more, the mapping $\varGamma_X : X \longrightarrow X^*$ defined through $\varGamma_X(x) = \varDelta_2 \circ D \circ i_2$ is a left ψ-module and a right φ-module morphism. In other words, for each $a \in \mathcal{A}$ and $x \in X$ we obtain

$$\begin{aligned}
D(a,x) &= D(a,0) + D(0,x) \\
&= \big(\ell(a,0), k(a,0)\big) + \big(\ell(0,x) + k(0,x)\big) \\
&= \big(\ell(a,0) + \ell(0,x), k(a,0) + k(0,x)\big) \\
&= \big(D_{\mathcal{A}}(a) + \varGamma_{\mathcal{A}}(x), D_X(a) + \varGamma_X(x)\big).
\end{aligned}$$

Since $D(a,x)$ is a $(\varPhi_\varphi, \varPsi_\psi)$-derivation, we find

$$\begin{aligned}
&\big(D_{\mathcal{A}}(ab) + \varGamma_{\mathcal{A}}(a \cdot y + x \cdot b), D_X(ab) + \varGamma_X(a \cdot y + x \cdot b)\big) \\
&= \big(D_{\mathcal{A}}(a) + \varGamma_{\mathcal{A}}(x), D_X(a) + \varGamma_X(x)\big) \cdot \big(\varphi(b), \varPhi(y)\big) \\
&+ \big(\psi(a), \varPsi(x)\big) \cdot \big(D_{\mathcal{A}}(b) + \varGamma_{\mathcal{A}}(y), D_X(b) + \varGamma_X(y)\big) \\
&= \Big(D_{\mathcal{A}}(a) \cdot \varphi(b) + \psi(a) \cdot D_{\mathcal{A}}(b) + \varGamma_{\mathcal{A}}(x) \cdot \varphi(b) + \varPsi(x) \cdot D_X(b) \\
&+ \psi(a) \cdot \varGamma_{\mathcal{A}}(y) + D_X(a) \cdot \varPhi(y) + \varGamma_X(x) \cdot \varPhi(y) + \varPsi(x) \cdot \varGamma_X(y), \\
&D_X(a) \cdot \varphi(b) + \psi(a) \cdot D_X(b) + \varGamma_X(x) \cdot \varphi(b) + \psi(a) \cdot \varGamma_X(y)\Big).
\end{aligned}$$

Now, the above relation implies that $\Gamma_X(x)\cdot\Phi(y)+\Psi(x)\cdot\Gamma_X(y)=0$, which completes the proof.

$\square$

The upcoming result is our main aim of the current paper, which presents necessary and sufficient conditions for the (Φ_φ,Ψ_ψ)-weak amenability of a module extension Banach algebra $\mathcal{A}\oplus X$.

Theorem 2.3 *Let $\mathcal{A}$ be a Banach algebra. Then, $H^1\big(\mathcal{A}\oplus X,\big((\mathcal{A}\oplus X)_{(\Phi_\varphi,\Psi_\psi)}\big)^*\big)=\{0\}$ if and only if the following conditions hold:*

(1) $H^1(\mathcal{A},(\mathcal{A}_{(\varphi,\psi)})^*)=\{0\}$;
(2) *If $D:\mathcal{A}\longrightarrow X^*$ is a (φ,ψ)-derivation, then there is $y^*\in X^*$ such that $D(a)=y^*\cdot\varphi(a)-\psi(a)\cdot y^*$ and $y^*\cdot\Phi(x)-\Psi(x)\cdot y^*=0$.*
(3) *For each bounded linear map $\Gamma:X\longrightarrow\mathcal{A}^*$ in which Γ is a left ψ-module morphism and right φ-module morphism, there is $h^*\in X^*$ such that*
$$\Gamma(x)=h^*\cdot\Phi(x)-\Psi(x)\cdot h^*,\qquad h^*\cdot\varphi(a)-\psi(a)\cdot h^*=0.$$
(4) *The only continuous left ψ-module and right φ-module morphism $\Lambda:X\longrightarrow X^*$, for which $\Lambda(x)\cdot\Phi(y)+\Psi(x)\cdot\Lambda(y)=0$ is $\Lambda=0$.*

Proof. At first, we prove necessity part. Assume that $\mathcal{A}\oplus X$ is (Φ_φ,Ψ_ψ)-weakly amenable. Let $D_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathcal{A}^*$ be a (φ,ψ)-derivation. By Lemma 2.2, the mapping $D:\mathcal{A}\oplus X\longrightarrow(\mathcal{A}\oplus X)^*$ defined by $D(a,x)=\big(D_{\mathcal{A}}(a),0\big)$ is a (Φ_φ,Ψ_ψ)-derivation, and consequently $D_{\mathcal{A}}$ is inner. Let now $D:\mathcal{A}\longrightarrow X^*$ be a (φ,ψ)-derivation. Then, $D(a,x)=\big(0,D(a)\big)$ is a (Φ_φ,Ψ_ψ)-derivation. Moreover, by hypothesis D is inner. Hence, there exists $(a^*,y^*)\in\mathcal{A}^*\dot{+}X^*$ such that
$$D(a,x)=(a^*,y^*)\cdot(\varphi(a),\Phi(x))-(\psi(a),\Psi(x))\cdot(a^*,y^*).$$
This means that $D(a)=y^*\cdot\varphi(a)-\psi(a)\cdot y^*$. Furthermore
$$D(a,x)=D(a,0)=(a^*,y^*)\cdot\big(\varphi(a),0\big)-\big(\psi(a),0\big)\cdot(a^*,y^*),$$
and so for each $a\in\mathcal{A}$, we have $a^*\cdot\varphi(a)-\psi(a)\cdot a^*=0$. Besides
$$D(a,x)=\big(a^*\cdot\varphi(a)+y^*\cdot\Phi(x)-\psi(a)\cdot a^*-\Psi(x)\cdot y^*,y^*\cdot\varphi(a)-\psi(a)\cdot y^*\big).$$
This implies that $y^*\cdot\Phi(x)-\Psi(x)\cdot y^*=0$ for all $x\in X$.

For the proof of part (3), suppose that $\Gamma:X\longrightarrow\mathcal{A}^*$ is a left ψ-module and right φ-module morphism. Consider $D:\mathcal{A}\oplus X\longrightarrow(\mathcal{A}\oplus X)^*$, defined via $D(a,x)=\big(\Gamma(x),0\big)$. We wish to show that D is a (Φ_φ,Ψ_ψ)-derivation. For each $a,b\in\mathcal{A}$ and $x,y\in X$, we get
$$\begin{aligned}D\big((a,x)(b,y)\big)&=D(ab,a\cdot y+x\cdot b)=\big(\Gamma(a\cdot y+x\cdot b),0\big)\\&=\big(\psi(a)\cdot\Gamma(y)+\Gamma(x)\cdot\varphi(b),0\big)\\&=\big(\Gamma(x),0\big)\cdot\big(\varphi(b),\Phi(y)\big)+\big(\psi(a),\Psi(x)\big)\cdot\big(\Gamma(y),0\big)\\&=D(a,x)\cdot\big(\varphi(b),\Phi(y)\big)+\big(\psi(a),\Psi(x)\big)\cdot D(b,y).\end{aligned}$$

By assumption D is $(\Phi_\varphi, \Psi_\psi)$-inner and thus there exist $a^* \in \mathcal{A}^*$ and $h^* \in X^*$ such that

$$D(a,x) = \big(\Gamma(x), 0\big) = (a^*, h^*) \cdot \big(\varphi(a), \Phi(x)\big) - \big(\psi(a), \Psi(x)\big) \cdot (a^*, h^*).$$

Hence, $\Gamma(x) = a^* \cdot \varphi(a) - \psi(a) \cdot a^* + h^* \cdot \Phi(x) - \Psi(x) \cdot h^*$ and $h^* \cdot \varphi(a) - \psi(a) \cdot h^* = 0$ for all $a \in \mathcal{A}, x \in X$. Since $D(a,x) = D(0,x)$, we achieve the following equation

$$\Gamma(x) = h^* \cdot \Phi(x) - \Psi(x) \cdot h^*, \qquad (x \in X).$$

For (4), assume that $\Lambda : X \longrightarrow X^*$ is a left ψ-module and right φ-module morphism satisfying $\Lambda(x) \cdot \Phi(y) + \Psi(y) \cdot \Lambda(x) = 0$. Define the mapping $D : \mathcal{A} \oplus X \longrightarrow (\mathcal{A} \oplus X)^*$ through $D(a,x) = \big(0, \Lambda(x)\big)$. By Lemma 2.2, D is a $(\Phi_\varphi, \Psi_\psi)$-derivation. Based on the above mentioned assumption, there exist $a^* \in \mathcal{A}^*$ and $t^* \in X^*$ such that

$$D(a,x) = \big(0, \Lambda(x)\big) = (a^*, t^*) \cdot \big(\varphi(a), \Phi(x)\big) - \big(\psi(a), \Psi(x)\big) \cdot (a^*, t^*).$$

Since $D(a,x) = D(0,x)$, we have

$$\big(0, \Lambda(x)\big) = (a^*, t^*) \cdot \big(0, \Phi(x)\big) - \big(0, \Psi(x)\big) \cdot (a^*, t^*).$$

It concludes that $\Lambda(x) = 0$. This completes the necessity implication.

We now prove the sufficiency part. Assume that the condition (1)-(4) hold and $D : \mathcal{A} \oplus X \longrightarrow \big((\mathcal{A} \oplus X)_{(\Phi_\varphi, \Psi_\psi)}\big)^*$ is a continuous derivation. Then, $D(a,x) = \big(D_{\mathcal{A}}(a) + \Gamma_{\mathcal{A}}(x), D_X(a) + \Gamma_X(x)\big)$. The part (3) of Lemma 2.2 necessitates that $\Gamma_X(x) \cdot \Phi(y) + \Psi(x) \cdot \Gamma_X(a) = 0$. From the condition (4), it follows that $\Gamma_X = 0$. By condition (1), there is $a^* \in \mathcal{A}^*$ such that $D_{\mathcal{A}}(a) = a^* \cdot \varphi(a) - \psi(a) \cdot a^*$ and from the condition (2), there is $y^* \in X^*$ such that

$$D_X(a) = y^* \cdot \varphi(a) - \psi(a) \cdot y^*, \qquad y^* \cdot \Phi(x) - \Psi(x) \cdot y^* = 0.$$

Besides, for each $a \in \mathcal{A}$ and $x \in X$ we have

$$\begin{aligned}
\Psi(x) \cdot D_X(a) &= \Psi(x) \cdot \big(y^* \cdot \varphi(a) - \psi(a) \cdot y^*\big) \\
&= \Psi(x) \cdot y^* \cdot \varphi(a) - \Psi(x) \cdot \psi(a) \cdot y^* \\
&= \big(y^* \cdot \Phi(x)\big) \cdot \varphi(a) - \Psi(x \cdot a) \cdot y^* \\
&= y^* \cdot \Phi(x \cdot a) - \Psi(x \cdot a) \cdot y^* \\
&= 0.
\end{aligned}$$

Similarly, $D_X(a) \cdot \Phi(x) = 0$. In other words,

$$\begin{aligned} D_X(a) \cdot \Phi(x) &= \big(y^* \cdot \varphi(a) - \psi(a) \cdot y^*\big) \cdot \Phi(x) \\ &= y^* \cdot \varphi(a) \cdot \Phi(x) - \psi(a) \cdot y^* \cdot \Phi(x) \\ &= y^* \cdot \Phi(a \cdot x) - \psi(a) \cdot \big(\Psi(x) \cdot y^*\big) \\ &= y^* \cdot \Phi(a \cdot x) - \Psi(a \cdot x) \cdot y^* \\ &= 0. \end{aligned}$$

On the other hand, $\Gamma_{\mathcal{A}}$ is a left ψ-module and a right φ-module morphism. By condition (3), it follows that there is $h^* \in X^*$ such that

$$\Gamma_{\mathcal{A}}(x) = h^* \cdot \Phi(x) - \Psi(x) \cdot h^*, \qquad h^* \cdot \varphi(a) = \psi(a) \cdot h^*.$$

Here, we claim that

$$D(a,x) = \big(a^*, h^* + y^*\big) \cdot \big(\varphi(a), \Phi(x)\big) - \big(\psi(a), \Psi(x)\big) \cdot (a^*, h^* + y^*).$$

For each $a \in \mathcal{A}$ and $x \in X$, we have

$$\begin{aligned} \big(a^* \cdot \varphi(a) - \psi(a) \cdot a^* + h^* \cdot \Phi(x) - \Psi(x) \cdot h^* + y^* \cdot \Phi(x) - \Psi(x) \cdot y^*, \\ h^* \cdot \varphi(a) - \psi(a) \cdot h^* + y^* \cdot \varphi(a) - \psi(a) \cdot y^*\big) \\ = \big(D_{\mathcal{A}}(a) + \Gamma_{\mathcal{A}}(x), D_X(a)\big) \\ = \big(D_{\mathcal{A}}(a) + \Gamma_{\mathcal{A}}(x), D_X(a) + \Gamma_X(x)\big) \\ = D(a,x), \end{aligned}$$

which finishes the proof.

□

Lemma 2.4 *For a Banach algebra $\mathcal{A}$ and $\varphi \in$Hom $(\mathcal{A})$, $\varphi(\mathcal{A}) \cdot \mathcal{A} + \mathcal{A} \cdot \varphi(\mathcal{A})$ is dense in $\varphi(\mathcal{A})$ provided that $H^1(\mathcal{A}, (\mathcal{A}_{(\varphi,\varphi)})^*) = \{0\}$.*

Proof. Assume that $\varphi(\mathcal{A}) \cdot \mathcal{A} + \mathcal{A} \cdot \varphi(\mathcal{A})$ is not dense in $\varphi(\mathcal{A})$. Then, there is a nonzero $f \in \mathcal{A}^*$ such that $f\Big|_{\varphi(\mathcal{A}) \cdot \mathcal{A} + \mathcal{A} \cdot \varphi(\mathcal{A})} = 0$ and $f|_{\varphi(\mathcal{A})} \neq 0$. Define the mapping $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ via $D(a) = f\big(\varphi(a)\big)f$. Since $D(ab) = f\big(\varphi(a)\varphi(b)\big)f = 0$ and $\big(D(a) \cdot \varphi(b) + \varphi(a) \cdot D(b)\big)(c) = 0$, it follows that D is a (φ, φ)-derivation. By hypothesis, there is $a^* \in \mathcal{A}^*$ such that

$$f\big(\varphi(a)\big)f = a^* \cdot \varphi(a) - \varphi(a) \cdot a^*.$$

There is also $a \in \mathcal{A}$ such that $f\big(\varphi(a)\big) \neq 0$, and thus

$$f\big(\varphi(a)\big)f\big(\varphi(a)\big) = a^*(\varphi^2(a) - \varphi^2(a)) = 0,$$

which is a contradiction.

□

Theorem 2.5 *Let $\mathcal{A}$ be a Banach algebra and $\varphi \in$Hom$(\mathcal{A})$. If $\{\varphi(a)\cdot b - b\cdot\varphi(a) : a,b\in\mathcal{A}\}$ is not dense in $\varphi(\mathcal{A})$, then $\mathcal{A}\oplus\mathcal{A}$ is not $(\varphi_\varphi,\varphi_\varphi)$-weakly amenable.*

Proof. Since $\{\varphi(a)\cdot b - b\cdot\varphi(a) : a,b\in\mathcal{A}\}$ is not dense in $\varphi(\mathcal{A})$, there is $f\in\mathcal{A}^*$ such that $f|_{\varphi(\mathcal{A})}\neq 0$ and $f\big(\varphi(a)\cdot b - b\cdot\varphi(a)\big)=0$ for $a,b\in\mathcal{A}$. Hence, $f\cdot\varphi(a)=\varphi(a)\cdot f$. This implies that the mapping $\Gamma:\mathcal{A}\longrightarrow\mathcal{A}^*$ defined through $\Gamma(a)=\varphi(a)\cdot f = f\cdot\varphi(a)$ is a φ-bimodule morphism. Now suppose that $\mathcal{A}\oplus\mathcal{A}$ is $(\varphi_\varphi,\varphi_\varphi)$-weakly amenable. By Lemma 2.4, there are $a,b\in\mathcal{A}$ such that $f\big(\varphi(a)\cdot b\big)\neq 0$. This shows that $\Gamma\neq 0$. Therefore, condition (3) of Theorem 2.3 does not hold which is a contradiction.

□

The next result is a direct consequence of Theorem 2.5.

Corollary 2.6 *Under each one of the following conditions, $\mathcal{A}\oplus\mathcal{A}$ is not $(\varphi_\varphi,\varphi_\varphi)$-weakly amenable.*

(1) $\varphi(\mathcal{A})\subseteq\{a\in\mathcal{A} : ab=ba \ for\ all\ b\in\mathcal{A}\}$.
(2) *$\mathcal{A}$ is a commutative Banach algebra.*

Remark 2.1 It is shown in [5, Theorem 2.6] that for any locally compact group G (not necessarily commutative ones), the group algebra $L^1(G)$ is (φ,ψ)-weakly amenable for all $\varphi,\psi\in\mathrm{Hom}(L^1(G))$ if $L^1(G)$ is a two-sided Banach G-module while $L^1(G)\oplus L^1(G)$ is not $(\varphi_\varphi,\varphi_\varphi)$-weakly amenable by Corollary 2.6. It can happen even if φ is an identity homomorphism. In fact, Zhang in [13, Proposition 5.2] showed that for a commutative Banach algebra $\mathcal{A}$, $\mathcal{A}\oplus\mathcal{A}$ is not weakly amenable.

In the following example we present some results of [12] for the weak amenability of $\mathcal{A}\oplus X$.

Example 2.1 It is proved in [12, Theorem 5.4] that if span$\{ab-ba : a,b\in\mathcal{A}\}$ is dense in the Banach algebra $\mathcal{A}$, then $\mathcal{A}\oplus\mathcal{B}$ is weakly amenable, provided that $\mathcal{A}$ is weakly amenable and has a bounded approximate identity, where $\mathcal{B}$ is a Banach algebra which is also an $\mathcal{A}$-bimodule. However, for an infinite dimensional Hilbert space $\mathcal{H}$, the Banach algebras $B(\mathcal{H})\oplus B(\mathcal{H})$, $B(\mathcal{H})\oplus K(\mathcal{H})$ and $K(\mathcal{H})\oplus K(\mathcal{H})$ are always weakly amenable, where $B(\mathcal{H})$ and $K(\mathcal{H})$ are the algebra of bounded operators and compact operators on $\mathcal{H}$, respectively; for more details see [12, Proposition 5.9, Remark 5.10] and [11, Theorem 1].

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