An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVI, 2020, f. 1

On L$_k$-biharmonic Lorentzian hypersurfaces in Einstein spaces

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Abstract A well-known conjecture of Bang Yen-Chen says that the only biharmonic submanifolds in the Euclidean space are minimal ones. The conjecture was proved by Chen in the case of surfaces in 3-dimensional Euclidean space. In this paper, we consider an extended version of Chen's conjecture (namely, L_k-conjecture) on Lorentzian hypersurfaces of the pseudo-Euclidean space E_1^4 (i.e. the Einstein space). A hypersurface $x : M_1^3 \to E_1^4$ is called L_k-biharmonic (for $k = 0, 1, 2$) if it satisfies the condition $L_k^2 x = 0$, where L_k is the linearized operator associated to the first variation of $(k+1)$-th mean curvature vector field on M_1^3. Since $L_0 = \Delta$, the matter of L_k-biharmonicity is a natural generalization of biharmonicity. On any L_k-biharmonic Lorentzian hypersurface in E_1^4 with distinct principal curvatures, assuming H_k to be constant we get that H_{k+1} is constant. Furthermore, we show that L_k-biharmonic timelike hypersurfaces in E_1^4 with constant H_k (under some additional hypotheses) are k-minimal.

Keywords Lorentzian hypersurface · L_k-biharmonicity · k-minimal

Mathematics Subject Classification (2010) 53C40 · 53C42 · 58G25

1 Introduction

In PDE-theory, biharmonic maps play role as solutions of 4-order strongly elliptic semilinear equations. In differential geometry, the variational problem associated to the bienergy functional on the set of Riemannian metrics on a domain has given rise to the biharmonic stress-energy tensor. Also, biharmonic Bezier surfaces is studied in computational geometry. From physical point of views, biharmonic surfaces in the Euclidean 3-space appear in the theory of elastics and also in fluid mechanics. The matter of biharmonic maps is interested by a well-known conjecture of Bang-Yen Chen (in 1987) which says that biharmonic submanifolds in the Euclidean space are minimal. Chen himself has proved his conjecture on surface in Euclidean 3-space. Later on, Dimitrić showed that any biharmonic hypersurface in E^m with at most two distinct principal curvatures is minimal [8]. In 1995, Hasanis and

Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-spaces [9]. In 1998, F. Defever studied another version of Chen's conjecture on the hypersurfaces of $\mathbb{E}^4$ [7]. In 2013, Akutagawa and Maeta [1] proved the Chen conjecture under some additional hypotheses. On the other hand, Chen had found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen (for instance, in [5,6]) and also L.J. Alias, S.M.B. Kashani and others. In [10], Kashani introduced the notion of L_k-finite type Euclidean hypersurfaces. One can see main results in Chapter 11 of Chen's book [5].

The linearized operator L_k (containing the special case $L_0 = \Delta$) on a (pseudo-) Euclidean hypersurface stands for the linear part of the first variation of its $(k+1)$-th mean curvature (see, for instance, [2,11,15,16,18]). By definition, we have $L_k(f) = tr(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes the k-th Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. Recently, in [14], we have studied on the L_k-biharmonic spacelike hypersurfaces in Minkowski 4-space E_1^4. In this paper, we study on L_k-biharmonic Lorentzian hypersurfaces in E_1^4 with three mutually distinct principal curvatures and constant mean curvature.

2 Preliminaries

Firstly, some preliminaries are recalled from [2,11,12] and [13]-[17]. The 4-dimensional Minkowski space, denoted by E_1^4, is the real vector space R^4 equipped with the scalar product $< x, y >:= -x_1y_1 + \Sigma_{i=2}^4 x_iy_i$, for every $x, y \in R^4$. Any nondegenerate hypersurface M_p^3 of E_1^4, can be endowed with a Riemannian or Lorentzian induced metric of index $p = 0$ or $p = 1$, respectively. Our study will be on Lorentzian hypersurface of E_1^4, determined by an isometric immersion $x : M_1^3 \to E_1^4$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^3 and E_1^4, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y + < SX, Y > \mathbf{n},$$

where $\mathbf{n}$ is a (local) unit normal vector field on M and S is the shape operator of M relative to $\mathbf{n}$. Every non-zero vector $X \in E_1^4$ is called time-like, light-like or space-like if $< X, X >$ is negative, zero or positive, respectively. Also, the vector 0 is spacelike by definition.

Definition 2.1 For a 3-dimensional Lorentzian vector space V_1^3, a basis $\mathcal{B} := \{e_1, e_2, e_3\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i \delta_i^j$ for $i, j = 1, 2, 3$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3$. As usual, δ_i^j stands for the Kronecker function. The basis $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies $< e_1, e_1 >=< e_2, e_2 >= 0$, $< e_1, e_2 >= -1$ and $< e_i, e_3 >= \delta_i^3$, for $i = 1, 2, 3$.

As it is well-known, the shape operator of the Lorentzian hypersurface M_1^3, as a self-adjoint linear map on the tangent space of M_1^3, can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. In the cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^3, the matrix representation of the induced metric on M_1^3 is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{and} \qquad B_4 = \begin{pmatrix} \kappa & \lambda & 0 \\ -\lambda & \kappa & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For the cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^3, the induced metric on M_1^3 has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{pmatrix},$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.1 In the cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3\}$ by orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3\}$ where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \kappa - \frac{1}{2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\sqrt{2}/2 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{pmatrix}$$

After these changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

Notation: According to four possible matrix representations of the shape operator of M_1^3, we define its principal curvatures, denoted by unified notations κ_i for $i = 1, 2, 3$, as follow.

In case I, we put $\kappa_i := \lambda_i$, for $i = 1, 2, 3$, where λ_i's are the eigenvalues of B_1.

In case II, where the matrix representation of S is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_3 := \lambda$.

In case III, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$.

Finally, in the case IV, where the shape operator has matrix representation $\tilde{B}_4$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta$.

The characteristic polynomial of S on M_1^3 is of the form

$$Q(t) = \prod_{i=1}^{3}(t - \kappa_i) = \sum_{j=0}^{3}(-1)^j s_j t^{3-j},$$

where, $s_0 := 1$, $s_1 = \sum_{j=1}^{3} \kappa_j$, $s_2 := \sum_{1 \leq i_1 < i_2 \leq 3} \kappa_{i_1}\kappa_{i_2}$ and $s_3 := \kappa_1\kappa_2\kappa_3$.

For $k = 1, 2, 3$, the *k-th mean curvature* H_k of M is defined by $H_k = \frac{1}{\binom{3}{k}} s_k$. When H_k is identically null, M_1^n is said to be $(k-1)$-*minimal*.

Definition 2.2 (i) A timelike hypersurface $x : M_1^3 \to E_1^4$, with diagonalizable shape operator, is said to be *isoparametric* if all of its principal curvatures are constant.
(ii) A timelike hypersurface $x : M_1^3 \to E_1^4$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of its shape operator has constant coefficients.

Remark 2.2 Here we remember Theorem 4.10 from [12], which assures us that there is no isoparametric timelike hypersurface of E_1^4 with complex principal curvatures.

The Newton transformation on the hypersurface, $P_k : \chi(M) \to \chi(M)$, is defined by

$$P_0 = I, \quad P_k = s_k I - S \circ P_{k-1}, \qquad (k = 1, 2, 3), \tag{2.1}$$

where, I is the identity map. The explicit formula $P_k = \sum_{i=0}^{k}(-1)^i s_{k-i} S^i$ (where $S^0 = I$) gives that, P_k is self-adjoint and it commutes with S (see [2,15]).

Now, we set

$$\mu_{j;k} = \sum_{l=0}^{k}(-1)^l \tbinom{n}{k-l} H_{k-l}\kappa_j^l. \qquad (1 \leq j \leq 3,\ 1 \leq k < 3) \tag{2.2}$$

Corresponding to the four possible forms $\tilde{B}_i$ (for $1 \leq i \leq 4$) of S, the Newton transformation P_j has different representations. In the case I, where $S = \tilde{B}_1$, we have $P_j = diag[\mu_{1;j}, \mu_{2;j}, \mu_{3;j}]$, for $j = 1, 2$.

When $S = B_2$ (in the case II), we have

$$P_1 = \begin{pmatrix} \kappa + \lambda - \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \kappa + \lambda + \frac{1}{2} & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} (\kappa - \frac{1}{2})\lambda & -\frac{1}{2}\lambda & 0 \\ \frac{1}{2}\lambda & (\kappa + \frac{1}{2})\lambda & 0 \\ 0 & 0 & \kappa^2 \end{pmatrix}.$$

In the case III, we have $S = B_3$, and

$$P_1=\begin{pmatrix} 2\kappa & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa \end{pmatrix}, \qquad P_2=\begin{pmatrix} \kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\kappa \\ \frac{1}{2} & \kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}\kappa \\ \frac{\sqrt{2}}{2}\kappa & \frac{\sqrt{2}}{2}\kappa & \kappa^2 \end{pmatrix}.$$

In the case IV, $S=B_4$,

$$P_1=\begin{pmatrix} \kappa+\eta & -\lambda & 0 \\ \lambda & \kappa+\eta & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2=\begin{pmatrix} \kappa\eta & -\lambda\eta & 0 \\ \lambda\eta & \kappa\eta & 0 \\ 0 & 0 & \kappa^2+\lambda^2 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities, similar to those in [2,15].

$$tr(P_1)=6H_1,\ tr(P_2)=3H_2,\ tr(P_1\circ S)=6H_2,\ tr(P_2\circ S)=3H_3, \quad (2.3)$$

$$trS^2=9H_1^2-6H_2,\ tr(P_1\circ S^2)=9H_1H_2-3H_3,\ tr(P_2\circ S^2)=3H_1H_3. \quad (2.4)$$

The *linearized operator* arised from the first variation of the $(j+1)$th mean curvature of M denoted by $L_j:\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ is defined by the formula $L_j(f):=tr(P_j\circ\nabla^2 f)$, where, $<\nabla^2 f(X),Y>=<\nabla_X\nabla f,Y>$ for every $X,Y\in\chi(M)$. For a Lorentzian hypersurface $x:M_1^3\to E_1^4$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a}\in E_1^4$ we use the decomposition $\mathbf{a}=\mathbf{a}^T+\mathbf{a}^N$ where $\mathbf{a}^T\in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N\perp TM$, and we have the following formulae from [2,15].

$$\nabla<x,\mathbf{a}>=\mathbf{a}^T,\ \nabla<\mathbf{n},\mathbf{a}>=-S\mathbf{a}^T, L_1x=6H_2\mathbf{n}, L_2x=3H_3\mathbf{n} \quad (2.5)$$

$$\begin{aligned} &L_1\mathbf{n}=-3\nabla H_2-3[3H_1H_2-H_3]\mathbf{n},\\ &L_2\mathbf{n}=-\nabla H_3-[3H_1H_3]\mathbf{n}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} &L_1^2x=6[2P_2\nabla H_2-9H_2\nabla H_2]+6[L_1H_2-9H_1H_2^2+3H_2H_3]\mathbf{n},\\ &L_2^2x=-9H_3\nabla H_3+3(L_2H_3-3H_1H_3^2)\mathbf{n}. \end{aligned} \quad (2.7)$$

Assume that a hypersurface $x:M_1^3\to E_1^4$ satisfies the condition $L_k^2x=0$ for an integer $k\in\{0,1,2\}$, then it is said to be L_k-*biharmonic*. In the case $k=0$, we have $L_0=\Delta$ and L_0-biharmonicity is the same ordinary harmonicity which has been studied in [3,4]. By equalities (2.7), a hypersurface $x:M_1^3\to E_1^4$ is L_1-biharmonic if and only if it satisfies two following conditions:

$$\begin{aligned} &(i)\ L_1H_2=3(3H_1H_2^2-H_2H_3)=H_2tr(S^2\circ P_1),\\ &(ii)\ P_2\nabla H_2=\frac{9}{2}H_2\nabla H_2. \end{aligned} \quad (2.8)$$

Also, $x:M_1^3\to E_1^4$ is L_2-biharmonic if and only if it satisfies two following conditions:

$$(i)\ L_2H_3=3H_1H_3^2=H_3tr(S^2\circ P_2),\quad (ii)\ H_3\nabla H_3=0. \quad (2.9)$$

The structure equations of E_1^4 are given by

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.10}$$

$$d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}. \tag{2.11}$$

With restriction to M, we have $\omega_4 = 0$ and then,

$$0 = d\omega_4 = \sum_{i=1}^{3} \omega_{4,i} \wedge \omega_i. \tag{2.12}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{4,i} = \sum_{j=1}^{3} h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.13}$$

This gives the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_4$. The mean curvature H is given by $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. From (2.10) - (2.13) we obtain the structure equations of M as follow.

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.14}$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.15}$$

for $i, j = 1, 2, 3$, and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.16}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk}. \tag{2.17}$$

Thus, by exterior differentiation of (2.13), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.18}$$

Now we recall the definition of an L_k-finite type hypersurface from [10], which is the basic notion of the paper.

Definition 2.3 *An isometrically immersed hypersurface* $x : M_1^3 \to E_1^4$ *is said to be of* L_k*-finite type if* x *has a finite decomposition* $x = \sum_{i=0}^{m} x_i$*, for some positive integer* m*, satisfying the condition* $L_k x_i = \beta_i x_i$*, where ,* $\beta_i \in R$ *and* $x_i : M^3 \to E_1^4$ *is smooth maps, for* $i = 1, 2, \cdots, m$*, and* x_0 *is constant point. If all* β_i*'s are mutually different,* M^n *is said to be of* L_k*-m-type. An* L_k*-m-type hypersurface is said to be null if for at least one* i $(1 \leq i \leq m)$ *we have* $\beta_i = 0$*.*

3 Examples

For instance, we see some examples of L_k-biharmonic Lorentzian hypersurfaces in E_1^4.

Example 3.1 Assume that $C_1^3(r)$ be the product $S_1^2(r) \times E^1 \subset E_1^4$ where $r > 0$. It has another representation as

$$C_1^3(r) = \{(y_1, ..., y_4) \in E_1^4 | -y_1^2 + y_2^2 + y_3^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$, $\kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{2}{3}r^{-1}$, $H_2 = \frac{1}{3}r^{-2}$ and $H_3 = 0$. One can see that $L_2^2 x = 0$ and $L_k^2 x \neq 0$ for $k = 0, 1$.

Example 3.2 Assume that $\tilde{C}_1^3(r)$ be the product $S_1^1(r) \times E^2 \subset E_1^4$ where $r > 0$. It has another representation as

$$\tilde{C}_1^3(r) = \{(y_1, ..., y_4) \in E_1^4 | -y_1^2 + y_2^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, 0, 0, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \frac{1}{r}$, $\kappa_2 = \kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{1}{4r}$, and $H_2 = H_3 = 0$. One can see that $L_0^2 x \neq 0$ and $L_k^2 x = 0$ for $k = 1, 2$.

Example 3.3 Let $\bar{C}_1^3(r)$ be the product $E_1^2 \times S^1(r) \subset L^5$ where $r > 0$. It can be represented as

$$\bar{C}_1^3(r) = \{(y_1, ..., y_4) \in E_1^4 | y_4^2 + y_5^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(0, 0, 0, y_4, y_5)$. It has two distinct principal curvatures $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = \frac{1}{r}$, and the constant higher order mean curvatures $H_1 = \frac{1}{4r}$, and $H_k = 0$ for $k = 2, 3$. So, also, one can see that $L_0^2 x \neq 0$ and $L_k^2 x = 0$ for $k = 1, 2$.

Example 3.4 Consider the pseudo-sphere

$$S_1^3(r) = \{(y_1, ..., y_4) \in E_1^4 | -y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2\},$$

(for $r > 0$) with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, y_4)$. It has three principal curvatures $\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{r}$ and constant higher order mean curvatures $H_k = \frac{1}{r^k}$, for $k = 1, 2, 3$. So, one can see that $L_k^2 x \neq 0$ for $k = 0, 1, 2$.

4 L_k-biharmonic Lorentzian hypersurfaces of E_1^4

The next lemma can be proved by following a similar one in [18].

Lemma 4.1 *Let M_1^3 be a timelike hypersurface in E_1^4 of type I with principal curvatures of constant multiplicities. Then the distribution of the space of principal directions corresponding to the principal curvatures is completely integrable. In addition, if a principal curvature be of multiplicity greater than one, then it will be constant on each integral submanifold of the corresponding distribution.*

Theorem 4.2 *Let k be a nonnegative integer less than 3. There is no L_k-biharmonic L_k-finite type Lorentzian hypersurface of the Minkowski 4-space.*

Proof. Assume that $x : M_1^3 \to E_1^4$ be an L_k-biharmonic L_k-finite type Lorentzian hypersurface of E_1^4. By assumption, the map x has a finite decomposition as

$$x = x_0 + x_1 + \cdots + x_m, \tag{4.1}$$

where $m \geq 1$, $L_k x_0 = 0$ and $L_k x_i = \lambda_i x_i$ for nonzero distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ of L_k. By operating L_k^s on both sides of equality (4.1), we obtain

$$0 = L_k^s x = \lambda_1^s x_1 + \cdots + \lambda_m^s x_m, \tag{4.2}$$

for $s = 2, 3, 4, \ldots$. Since $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of L_k, system (4.2) is impossible. Then, the claim is proved.

□

4.1 L_k-biharmonic hypersurfaces with shape operator of type I

Proposition 4.3 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type I having three distinct real principal curvatures and constant k-th mean curvature. Then, the $(k+1)$-th mean curvature of M has to be constant.*

Proof. Assuming H_{k+1} to be non-constant, we consider the open subset $\mathcal{U} = \{p \in M_1^3 : \nabla H_{k+1}^2(p) \neq 0\}$ of M_1^3. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_1, such that $Se_i = \lambda_i e_i$ and then, $P_j e_i = \mu_{i,j} e_i$ for $i = 1, 2, 3$ and $j = 1, 2$.

In the case $k = 0$, the main claim is proved in [3,4]. We prove the claim in the cases $k = 1, 2$. Using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{3} \epsilon_i e_i(H_{k+1})e_i$, from conditions (2.8)(ii) and (2.9)(ii) we get

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2}H_2) = 0, \tag{4.3}$$

in the case $k = 1$, and $e_i(H_3^2) = 0$ in the case $k = 2$, for $i = 1, 2, 3$. So, there is nothing to prove in the case $k = 2$. We continue proof in the case $k = 1$. Each point of $\mathcal{U}$ has an open neighborhood on which we have $e_i(H_2) \neq 0$ for at least one i. So, without loss of generality, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2}H_2$, (locally) on $\mathcal{U}$, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$. Now, we prove three simple claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$ on $\mathcal{U}$.

If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (4.3) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $\lambda_3(\lambda_2 - \lambda_1) = 0$ or $\lambda_2(\lambda_1 - \lambda_3) = 0$. But, since λ_i's are assumed to be mutually distinct, we get $\lambda_3 = 0$ or $\lambda_2 = 0$, which gives $H_2 = 0$ on $\mathcal{U}$. The result is in contradiction with the definition of $\mathcal{U}$.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$ on $\mathcal{U}$.

Since H assumed to be constant on M, we have $e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2)$. On the other hand, from two recent results $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$ we get $e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2(3H_2 - \frac{9}{2}H_2) = 0$, which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)e_2\lambda_1 = 0$, and then we have

$$\begin{aligned}\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 &= \lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3)e_2\lambda_1 \\ &= (-\lambda_1 + \lambda_2 + \lambda_3)e_2\lambda_1 = 0.\end{aligned}$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2 + \lambda_3$ which gives the contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$. So, Claim 2 is confirmed.

Claim 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$ on $\mathcal{U}$.

Using the notations

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \quad (i, j = 1, 2, 3), \tag{4.4}$$

and the compatibility condition $\nabla_{e_k} < e_i, e_j >= 0$, we have

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \quad (i, j, k = 1, 2, 3) \tag{4.5}$$

and applying the Codazzi equation (see [13], page 115, Corollary 34(2))

$$(\nabla_V S)W = \nabla_W S)V, \quad (\forall V, W \in \chi(M)) \tag{4.6}$$

on the basis $\{e_1, e_2, e_3\}$, we get for distinct $i, j, k = 1, 2, 3$

$$(a) \;\; e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega^j_{ji}, \qquad (b) \;\; (\lambda_i - \lambda_j)\omega^j_{ki} = (\lambda_k - \lambda_j)\omega^j_{ik}. \tag{4.7}$$

Also, by a straightforward computation of components of the identity

$$\nabla_{e_i} S)e_j - \nabla_{e_j} S)e_i \equiv 0$$

for distinct $i, j = 1, 2, 3$, we get $[e_2,\ e_3](H_2) = 0$ and

$$\omega^1_{12} = \omega^1_{13} = \omega^2_{13} = \omega^3_{21} = \omega^1_{32} = 0,$$

$$\omega^2_{21} = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2},\ \omega^3_{31} = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3},\ \omega^2_{23} = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2},\ \omega^3_{32} = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{4.8}$$

Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to $\nabla_{e_1} e_k = 0$ for $k = 1, 2, 3$, and

$$\begin{aligned} \nabla_{e_2} e_1 &= \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2, \quad \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} e_3, \quad \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_1, \\ \nabla_{e_3} e_2 &= \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} e_3, \qquad \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} e_2, \\ \nabla_{e_3} e_3 &= \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3 - \lambda_2} e_2. \end{aligned} \tag{4.9}$$

Now, the Gauss equation for $< R(e_2, e_3)e_1,\ e_2 >$ and $< R(e_2, e_3)e_1,\ e_3 >$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right), \tag{4.10}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right). \tag{4.11}$$

Also, the Gauss equation for $< R(e_1, e_2)e_1,\ e_2 >$ and $< R(e_3, e_1)e_1,\ e_3 >$ give the following relations

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) + \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right)^2 = \lambda_1\lambda_2, \quad e_1\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) + \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\right)^2 = \lambda_1\lambda_3. \tag{4.12}$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}. \tag{4.13}$$

On the other hand, we consider the condition (2.8)(ii) which (using Claim 1) gives

$$-\mu_{1,1}e_1e_1(H_2)+\left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2)-9H_2^2(H-\frac{3}{2}\lambda_1)=0. \tag{4.14}$$

Differentiating on both sides of (4.14) along e_2 and e_3 (by using (4.10) and (4.11), respectively) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{4.15}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}-\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{4.16}$$

Using (4.9), we find that

$$[e_1,\ e_2]=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{4.17}$$

Applying both sides of the equality (4.17) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (4.15), (4.12), and (4.13), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{4.18}$$

(4.18) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{4.19}$$

From equation (4.19), by differentiating on its both sides along e_1 and applying (4.12), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (4.9), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{4.20}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3)=e_3(\lambda_2)=0$ which, by applying the Gauss equation for $< R(e_2,e_3)e_1,e_3 >$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{4.21}$$

Finally, using (4.12), differentiating (4.21) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0, \tag{4.22}$$

which implies $\lambda_2\lambda_3 = 0$ (since we have seen $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)\neq 0$). Therefore, we obtain $H_2 = 0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M^3.

□

Theorem 4.4 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type I having three distinct real principal curvatures and constant k-th mean curvature. Then, it is k-minimal.*

Proof. When $k = 0$, the result is derived from [3,4]. In the case $k = 1$, by assumption H_1 is assumed to be constant and then, by Theorem 4.3 it is proved that H_2 has to be constant. We claim that H_2 is null. If $H_2 \neq 0$, by using (2.8)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of case I in the Einstein space has at most one nonzero principal curvature, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_2 \equiv 0$ and M_1^3 is 1-minimal.

In the case $k = 2$, if $H_3 \neq 0$, by using (2.9)(ii) we obtain that H_3 is constant and by (2.9)(i), H_1 is null. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of case I in the Einstein space has at most one nonzero principal curvature, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_3 \equiv 0$ and M_1^3 is 2-minimal.

□

Proposition 4.5 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type I which has exactly two distinct principal curvatures and constant k-th mean curvature, then its $(k+1)$-th mean curvature is constant.*

Proof. By assumption, M_1^3 has two distinct principal curvatures λ_1 and λ_2 of multiplicities 1 and 2, respectively. Defining the open subset $\mathcal{U}$ of M_1^3 as $\mathcal{U} := \{p \in M_1^3 : \nabla H_{k+1}^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \emptyset$, we consider $\{e_1, e_2, e_3\}$ as a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$. By assumption, we have

$$\lambda_1 = \lambda_2 = \lambda, \qquad \lambda_3 = \mu.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\mu, \quad \mu_{3,2} = \lambda^2, \quad 3H_2 = \lambda^2 + 2\lambda\mu. \tag{4.23}$$

In the case $k = 1$, by condition (2.8)(ii), we have

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{4.24}$$

Then, using the polar decomposition

$$\nabla H_2 = \sum_{i=1}^{3} \epsilon_i < \nabla H_2, e_i > e_i, \tag{4.25}$$

we see that (4.24) is equivalent to

$$\epsilon_i < \nabla H_2, e_i > (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

on $\mathcal{U}$ for $i = 1, 2, 3$. Hence, for every i such that $< \nabla H_2, e_i > \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{4.26}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

State 1. $< \nabla H_2, e_i > \neq 0$, for $i = 1$ or $i = 2$. By equalities (4.23) and (4.26), we obtain

$$\lambda\mu = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(2\mu + \frac{3}{2}\lambda) = 0. \tag{4.27}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\mu = -\frac{3}{4}\lambda$, $H_2 = -\frac{1}{6}\lambda^2$.

State 2. $< \nabla H_2, e_3 > \neq 0$. By equalities (4.23) and (4.26), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(3\mu + \frac{1}{2}\lambda) = 0. \tag{4.28}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\mu = -\frac{1}{6}\lambda$, $H_2 = \frac{2}{9}\lambda^2$.

Both states requires the same calculation, so we consider for instance State 2.

By Lemma 4.1, let us denote the maximal integral submanifold through $x \in \mathcal{U}$, corresponding to λ by $\mathcal{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_{i=1}^{3} \lambda_{,i}\omega_i \qquad\qquad d\mu = \sum_{j=1}^{3} \mu_{,j}\omega_j. \tag{4.29}$$

Then, Lemma 4.1 implies that $\lambda_{,1} = \lambda_{,2} = 0$. We can assume that $\lambda > 0$ on $\mathcal{U}$, then we have (in State 2)

$$\mu = \frac{-1}{6}\lambda < 0. \tag{4.30}$$

From (2.17), we obtain

$$\sum_{k=1}^{3} h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}, \tag{4.31}$$

for $i, j, k = 1, 2, 3$. Here, we adopt the notational convention that $a, b, c = 1, 2$.
From (4.29) and (4.31), we have

$$\begin{aligned} &h_{12k} = h_{21k} = 0, \\ &h_{aab} = 0, \qquad h_{aa3} = \lambda_{,3}, \\ &h_{33a} = 0, \qquad h_{333} = \mu_{,3}. \end{aligned} \tag{4.32}$$

Combining this with (2.18) and the formula

$$\sum_{i=1}^{3} h_{a3i}\omega_i = dh_{a3} + \sum_{i=1}^{3} h_{i3}\omega_{ia} + \sum_{i=1}^{3} h_{ai}\omega_{i3} = (\lambda - \mu)\omega_{a3},$$

we obtain from (4.30)

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda - \mu}\omega_a = \frac{6\lambda_{,3}}{7\lambda}\omega_a. \tag{4.33}$$

Therefore we have

$$d\omega_3 = \sum_{a=1}^{2} \omega_{3a} \wedge \omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ. We may put $\omega_3 = ds$.
Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_{,3} ds, \qquad \lambda_{,3} = \lambda'(s),$$

so from (4.33), we get

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda - \mu}\omega_a = \frac{6\lambda'(s)}{7\lambda}\omega_a. \tag{4.34}$$

According to the structure equations of E_1^4 and (4.34), we may compute

$$\begin{aligned}
(i):\quad d\omega_{a3} &= \sum_{b=1}^{2}\omega_{ab}\wedge\omega_{b3} + \omega_{a4}\wedge\omega_{43} = \left(\frac{6\lambda'}{7\lambda}\right)\sum_{b=1}^{2}\omega_{ab}\wedge\omega_b - \lambda\mu\omega_a\wedge ds,\\
(ii):\quad d\omega_{a3} &= d\left\{\frac{6\lambda'}{7\lambda}\omega_a\right\} = \left(\frac{6\lambda'}{7\lambda}\right)' ds\wedge\omega_a + \left(\frac{6\lambda'}{7\lambda}\right)d\omega_a\\
&= \left\{-\left(\frac{6\lambda'}{7\lambda}\right)' + \left(\frac{6\lambda'}{7\lambda}\right)^2\right\}\omega_a\wedge ds + \left(\frac{6\lambda'}{7\lambda}\right)\sum_{b=1}^{2}\omega_{ab}\wedge\omega_b.
\end{aligned} \tag{4.35}$$

Comparing equalities (4.35)(i) and (4.35)(ii), we get $\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \lambda\mu = 0$, which, by combining with (4.30), gives

$$\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \left(\frac{-1}{6}\right)\lambda^2 = 0. \tag{4.36}$$

Defining function $\beta(s) := \left(\frac{1}{\lambda(s)}\right)^{\frac{6}{7}}$ for $s \in \mathcal{R}$, from (4.36) we get $\beta'' = \left(\frac{1}{6}\right)\beta^{\frac{-8}{6}}$, which, by integrating, gives $(\beta')^2 = -\beta^{\frac{-2}{6}} + c$, where c is the constant of integration. The last equation is equivalent to

$$(\lambda')^2 = -\left(\frac{7}{6}\right)^2\lambda^4 + c\left(\frac{7}{6}\right)^2\lambda^{\frac{26}{7}}. \tag{4.37}$$

Now, in order to compare two sides of condition (2.8)(i), we need to compute $\nabla_{e_i}\nabla H_2$ and $P_1(e_i)$ for $i = 1, 2, 3$. From (4.27) we have $\nabla H_2 = \frac{4}{9}\lambda\lambda' e_3$, which by using (4.34), gives

$$\begin{aligned}
\nabla_{e_a}\nabla H_2 &= \frac{4}{9}\lambda\lambda'\nabla_{e_a}e_3 = \frac{4}{9}\lambda^r\lambda'\sum_b \omega_{3b}(e_a)e_b = -\frac{8}{21}\lambda'^2 e_a,\\
\nabla_{e_3}\nabla H_2 &= \frac{4}{9}\nabla_{e_3}(\lambda\lambda' e_3) = \frac{4}{9}\lambda'^2 e_3 + \frac{4}{9}\lambda\lambda'' e_3.
\end{aligned} \tag{4.38}$$

By using (4.23) and (4.30), we compute $P_1(e_a)$ and $P_1(e_3)$.

$$P_1(e_1) = \frac{5}{6}\lambda e_1, \quad P_1(e_2) = \frac{5}{6}\lambda e_2 \quad P_1(e_3) = 2\lambda e_3. \tag{4.39}$$

From (4.38) and (4.39), we get

$$L_1 H_2 = 6H_2 \left(\frac{-10(\lambda')^2}{21\lambda} + \frac{2(\lambda')^2}{3\lambda} + \frac{2}{3}\lambda'' \right). \tag{4.40}$$

From (2.8)(i), we have $L_1 H_2 = H_2 tr(S^2 \circ P_1) = 2H_2 \frac{11}{6}\lambda^3$. Using also (4.40) one obtains

$$\lambda\lambda'' + \left(1 + \frac{-5}{7}\right)\lambda'^2 - 2\frac{33}{12}\lambda^4 = 0. \tag{4.41}$$

On the other hand, the equality (4.36) is equivalent to

$$\lambda\lambda'' = \frac{13}{7}\lambda'^2 + \frac{-7}{36}\lambda^4. \tag{4.42}$$

Now, substituting (4.42) and (4.41), we obtain

$$\frac{15}{7}\lambda'^2 + \frac{191}{36}\lambda^4 = 0. \tag{4.43}$$

From equations (4.37), (4.43) and (4.27), we get that H_2 is locally constant on $\mathcal{U}$, which is a contradiction with the definition of $\mathcal{U}$. Hence H_2 is constant on M.

By a similar discussion, one can get the same result in State 1.

In the case $k = 0$, the main claim is proved in [3,4]. In the case $k = 2$, from condition (2.9)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove.

□

Theorem 4.6 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type I which has exactly two distinct principal curvatures and constant k-th mean curvature, then it is k-minimal.*

Proof. When $k = 0$, the result is derived from [3,4]. In the case $k = 1$, by assumption H_1 is assumed to be constant and then, by the last proposition, it is proved that H_2 has to be constant. We claim that H_2 is null. If $H_2 \neq 0$, by using (2.8)(i) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of type I in the Einstein space has at most one nonzero principal curvature, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_2 \equiv 0$ and M_1^3 is 1-minimal.

In the case $k = 2$, if $H_3 \neq 0$, by using (2.9)(ii) we obtain that H_3 is constant and by (2.9)(i), H_1 is null (i.e. $2\kappa + \mu = 0$). Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric and by Corollary 2.7 in [12], it can have at most one nonzero principal curvature, then we get $\mu = -2\kappa = 0$, which contradicts with the assumption that, two principal curvatures of M are assumed to be mutually distinct. So $H_3 \equiv 0$ and M_1^3 is 2-minimal.

□

4.2 L_k-biharmonic hypersurfaces with shape operator of type II

Proposition 4.7 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type II which has exactly two distinct principal curvatures and constant k-th mean curvature, then it's $(k+1)$-th mean curvature is constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and then, we have $P_2 e_1 = (\kappa - \frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $P_2 e_2 = -\frac{1}{2}\lambda e_1 + (\kappa + \frac{1}{2})\lambda e_2$ and $P_2 e_3 = \kappa^2 e_3$.

When $k = 0$, the result is derived from [3,4]. In the case $k = 1$, by condition (2.8)(ii), using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, we get

$$\begin{aligned} &(i) \quad \epsilon_1 e_1(H_2)[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2] = \epsilon_2 e_2(H_2)\frac{\lambda}{2} \\ &(ii) \quad \epsilon_2 e_2(H_2)[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\epsilon_1 e_1(H_2)\frac{\lambda}{2}, \\ &(iii) \quad \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0. \end{aligned} \tag{4.44}$$

Now, we prove three simple claims.

Claims: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$ on $\mathcal{U}$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.44)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned} &(i) \ (\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\frac{\lambda}{2}, \\ &(ii) \ \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\frac{\lambda}{2}, \end{aligned} \tag{4.45}$$

which, by substituting (i) in (ii), gives $\frac{\lambda}{2}(1+u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from (4.45)(i).

Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$, then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, H_2 has to be constant and we have $e_1(H_2) = 0$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then by (4.44)(iii) we get $\kappa^2 = \frac{9}{2}H_2$, then $\kappa(\kappa+6\lambda) = 0$, which gives $\kappa = 0$ or $\kappa = -6\lambda$. If $\kappa = 0$, then $H_2 = 0$, and if $\kappa = -6\lambda$ then since $H_1 = -\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2) = 0$. Which is a contradiction, so we have $e_3(H_2) = 0$.

In the case $k = 2$, from condition (2.9)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove.

□

Theorem 4.8 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type II which has exactly two distinct principal curvatures and constant k-th mean curvature, then it is k-minimal.*

Proof. When $k = 0$, the result is derived from [3,4]. When $k = 1$, by assumption H_1 is assumed to be constant and then, by Proposition 4.7 it is proved that H_2 has to be constant. We claim that H_2 is null. Since the shape operator is of type II, there exist two possible cases as:
Case 1: M_1^3 has a principal curvature κ of multiplicity 3;
Case 2: M_1^3 has two principal curvatures κ and λ of multiplicities 2 and 1, respectively.

In Case 1, we have $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$. By (2.8)(i), we have $3H_1H_2^2 = H_2H_3$, which gives $\kappa^5 = 0$, and then $H_2 = 0$.

In Case 2, we have $H_1 = \frac{1}{3}(2\kappa + \lambda)$, $H_2 = \frac{1}{3}(\kappa^2 + 2\kappa\lambda)$ and $H_3 = \kappa^2\lambda$. We assume that $H_2 \neq 0$ and continue in two subcases as follow. Since $H_2 \neq 0$, then $\kappa \neq 0$ and by using (2.8)(ii) we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of type II in the Einstein space has at most one nonzero principal curvature, so we get $\lambda = 0$. Then $H_1 = \frac{2}{3}\kappa$, $H_2 = \frac{1}{3}\kappa^2$ and $H_3 = 0$, hence, by (2.8)(i), we get $\kappa = 0$, which contradicts with the assumption of this case. Therefore $H_2 = 0$.

When $k = 2$, from condition (2.9)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which implies that H_3 is constant. Assuming $H_3 \neq 0$, by condition (2.9)(i) we get H_1 is null (i.e. $2\kappa + \mu = 0$). Therefore, all of mean curvatures H_i (for $i = 1, 2, 3$) are constant, which means that M_1^3 is isoparametric and by Corollary 2.7 in [12], it can have at most one nonzero principal curvature, then we get $\mu = -2\kappa = 0$. This contradicts the assumption that, two principal curvatures of M are mutually distinct. So $H_3 \equiv 0$ and M_1^3 is 2-minimal.

□

4.3 L_k-biharmonic hypersurfaces with shape operator of type III

Proposition 4.9 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \rightarrow E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type III which has constant k-th mean curvature, then it's $(k+1)$-th mean curvature is constant.*

Proof. When $k = 0$, the result is derived from [3,4]. In the case $k = 1$, suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1 = \kappa e_1 + \frac{\sqrt{2}}{2}e_3$, $Se_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Se_3 = -\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and then, we have $P_2e_1 = (\kappa^2 - \frac{1}{2})e_1 - \frac{1}{2}e_2 - \frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2 = \frac{1}{2}e_1 + (\kappa^2 + \frac{1}{2})e_2 + \frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3 = \frac{\sqrt{2}}{2}\kappa e_1 + \frac{\sqrt{2}}{2}\kappa e_2 + \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2)e_i$, from condition (2.8)(ii) we get

$$
\begin{aligned}
&(i)\ \epsilon_1 e_1(H_2)[(\kappa^2 - \frac{1}{2}) - \frac{9}{2}H_2] + \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0,\\
&(ii)\ \frac{-1}{2}\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)[(\kappa^2 + \frac{1}{2}) - \frac{9}{2}H_2] + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0,\\
&(iii)\ \epsilon_1 e_1(H_2)\frac{-\sqrt{2}}{2}\kappa + \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa + \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0,
\end{aligned}
\tag{4.46}
$$

Now, we prove three claims.

Claims: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$ on $\mathcal{U}$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.46)(i, ii, iii) by $\epsilon_1 e_1(H_2)$, and using the identity $H_2 = \kappa^2$ in type III, we get

$$
\begin{aligned}
&(i)\ -\frac{1}{2} - \frac{7}{2}\kappa^2 + \frac{1}{2}u_1 + \frac{\sqrt{2}}{2}u_2\kappa = 0,\\
&(ii)\ \frac{-1}{2} + u_1(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}u_2\kappa = 0,\\
&(iii)\ \frac{-\sqrt{2}}{2}\kappa + \frac{\sqrt{2}}{2}u_1\kappa - \frac{7}{2}\kappa^2)u_2 = 0,
\end{aligned}
\tag{4.47}
$$

where $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, which, by comparing (i) and (ii), gives $\kappa^2(u_1 - 1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = 1$, which, using (4.47)(iii), gives $u_2 = 0$. Substituting $u_1 = 1$ and $u_2 = 0$ in (4.47)(i), we obtain again $\kappa = 0$, which is a contradiction. Hence $e_1(H_2) \equiv 0$.

Therefore, using the result $e_1(H_2)\equiv 0$, the system of equations (4.46) gives

$$\begin{aligned}
&(i)\ \ \frac{1}{2}\epsilon_2 e_2(H_2)+\frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa=0,\\
&(ii)\ \ \epsilon_2 e_2(H_2)(\frac{1}{2}-\frac{7}{2}\kappa^2)+\frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa=0,\\
&(iii)\ \ \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa-\epsilon_3 e_3(H_2)\frac{7}{2}\kappa^2=0.
\end{aligned} \tag{4.48}$$

Comparing parts $(i),(ii)$ and (iii) of (4.48), we get $\kappa e_2(H_2)=0$ and $\kappa e_3(H_2)=0$, hence, using (i), gives $e_2(H_2)=0$. Then, the second claim (i.e. $e_2(H_2)=0$) is proved.

Now, using the results $e_1(H_2)=e_2(H_2)=0$, we get $\kappa e_3(H_2)=0$, which, using $H_2=\kappa^2$, implies $\kappa e_3(\kappa^2)=0$ and then $e_3(\kappa^3)=0$, and finally $e_3(H_2)=0$.

In the case $k=2$, from condition (2.9)(ii) we get $e_i(H_3^2)=0$ for $i=1,2,3$, which means that there is nothing to prove.

□

Theorem 4.10 *Let k be a nonnegative integer less than 3 and $x:M_1^3\to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with the shape operator of type III which has constant k-th mean curvature. Then, it is k-minimal. Furthermore, all of mean curvatures of M_1^3 are null.*

Proof. When $k=0$, the result is derived from [3,4]. In the case $k=1$, by Proposition 4.9, the 2-th mean curvature of M_1^3 is constant, which, by (2.8)(i), gives $L_1H_2=9H_1H_2^2-3H_2H_3=0$, and then $3H_1H_2^2=H_2H_3$, which, using $H_1=\kappa$, $H_2=\kappa^2$ and $H_3=\kappa^3$, gives $\kappa^5=0$, and then $H_1=H_2=H_3=0$.

In the case $k=2$, by Proposition 4.9, the 3-th mean curvature of M_1^3 is constant, which, by (2.9)(i), gives $L_1H_3=3H_1H_3^2=0$, which, using $H_1=\kappa$, $H_2=\kappa^2$ and $H_3=\kappa^3$, gives $\kappa^7=0$, and then $H_1=H_2=H_3=0$.

□

4.4 L_k-biharmonic hypersurfaces with shape operator of type IV

Proposition 4.11 *Let k be a nonnegative integer less than 3 and $x:M_1^3\to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type IV which has constant k-th mean curvature and a constant real principal curvature. Then, its 2-th and 3-th mean curvatures are constant.*

Proof. When $k=0$, the result is derived from [3,4]. In the case $k=1$, suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U}=\{p\in M:\nabla H_2^2(p)\neq 0\}$, we try to show $\mathcal{U}=\emptyset$. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_4, such that $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta e_3$ and then, we have $P_2 e_1 = \kappa\eta e_1 + \lambda\eta e_2$, $P_2 e_2 = -\lambda\eta e_1 + \kappa\eta e_2$ and $P_2 e_3 = (\kappa^2 + \lambda^2)e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.8)(ii) we get

$$\begin{aligned}
&(i)\ \ \epsilon_1 e_1(H_2)(\kappa\eta - \frac{9}{2}H_2) = \epsilon_2 e_2(H_2)\lambda\eta,\\
&(ii)\ \ \epsilon_2 e_2(H_2)(\kappa\eta - \frac{9}{2}H_2) = -\epsilon_1 e_1(H_2)\lambda\eta,\\
&(iii)\ \ \epsilon_3 e_3(H_2)(\kappa^2 + \lambda^2 - \frac{9}{2}H_2) = 0.
\end{aligned} \tag{4.49}$$

Now, we prove some simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = 0$ on $\mathcal{U}$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.49)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&(i)\qquad \kappa\eta - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\lambda\eta,\\
&(ii)\qquad \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}(\kappa\eta - \frac{9}{2}H_2) = -\lambda\eta,
\end{aligned} \tag{4.50}$$

which, by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by (4.50)(i), we have $H_2 = 0$.

Similarly, if $e_2(H_2) \neq 0$, then by dividing both sides of (4.49)(i, ii) by $\epsilon_2 e_2(H_2)$ we get

$$\begin{aligned}
&(i)\qquad \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}(\kappa\eta - \frac{9}{2}H_2) = \lambda\eta,\\
&(ii)\qquad \kappa\eta - \frac{9}{2}H_2 = -\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\lambda\eta,
\end{aligned} \tag{4.51}$$

which, by substituting (i) in (ii), gives $\lambda\eta(1 + (\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)})^2) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by (4.51)(ii), we have $H_2 = 0$.

Claim 2: $e_3(H_2) = 0$ on $\mathcal{U}$.

If $e_3(H_2) \neq 0$, then from equality (4.49)(iii) we have $\kappa^2 + \lambda^2 = \frac{9}{2}H_2$, which gives $\kappa^2 + \lambda^2 = -6\kappa\eta$, where $\eta = 3H_1 - 2\kappa$ and η and H_1 are assumed to be constant on $\mathcal{U}$. So, κ is also constant on $\mathcal{U}$, and then, we obtain $H_2 = \frac{-4}{3}\kappa\eta = \frac{8}{3}\kappa^2 - 4H_1\kappa$ and $H_3 = -6\kappa\eta^2 = -6\kappa(3H_1 - 2\kappa)^2$ are constant on $\mathcal{U}$.

In the case $k = 2$, from condition (2.9)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove.

□

Theorem 4.12 *Let k be a nonnegative integer less than 3 and $x : M_1^3 \to E_1^4$ be an L_k-biharmonic connected orientable timelike hypersurface with shape operator of type IV which has constant k-th mean curvature and a constant real principal curvature. Then, it is k-minimal.*

Proof. When $k = 0$, the result is derived from [3,4]. In the case $k = 1$, by Proposition 4.11, the 2-th mean curvature of M_1^3 is constant, which gives $L_1 H_2 = 0$. Then, by (2.8)(i), we have $9H_1H_2^2 - 3H_2H_3 = 0$, which gives $(7\eta - 4\kappa)\kappa^2\eta^2 = 0$.

Now, if $7\eta = 4\kappa$, then from $\kappa^2 + \lambda^2 = -6\kappa\eta$ we get $\frac{31}{7}\kappa^2 + \lambda^2 = 0$, and then $\kappa = \lambda = 0$, which gives $H_2 = H_3 = 0$. Also, if $\kappa^2\eta^2 = 0$, then we have $H_2 = H_3 = 0$.

In the case $k = 2$, from condition (2.9)(ii) we get $e_i(H_3^2) = 0$ for $i = 1, 2, 3$, which means that there is nothing to prove.

□

References

1. Akutagawa, K.; Maeta, S. – *Biharmonic properly immersed submanifolds in Euclidean spaces*, Geom. Dedicata, **164** (2013), 351-355.
2. Alías, L.J.; Gürbüz, N. – *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata **121** (2006), 113-127.
3. Arvanitoyeorgos, A.; Defever, F.; Kaimakamis, G.; Papantoniou, V.J. – *Biharmonic Lorentz hypersurfaces in E_1^4*, Pacific J. Math., **229** (2007), no. 2, 293-305.
4. Arvanitoyeorgos, A.; Defever, F.; Kaimakamis, G. – *Hypersurfaces of E_s^4 with proper mean curvature vector*, J. Math. Soc. Japan **59** (2007), no. 3, 797-809.
5. Chen, B.Y. – *Total mean curvature and submanifolds of finite type*, Series in Pure Mathematics (1), World Scientific Publishing Co. Pte. Ltd., Singapore (1984).
6. Chen, B.Y. – *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., **17** (1991), no. 2, 169-188.
7. Defever, F. – *Hypersurfaces of E^4 with harmonic mean curvature vector*, Math. Nachr., **196** (1998), 61-69.
8. Dimitrić, I. – *Submanifolds of E^m with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica, **20** (1992), no. 1, 53-65.
9. Hasanis, T.; Vlachos, T. – *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr., **172** (1995), 145-169.
10. Kashani, S.M.B. – *On some L_1-finite type (hyper)surfaces in R^{n+1}*, Bull. Korean Math. Soc., **46** (2009), no. 1, 35-43.
11. Lucas, P.; Ramírez-Ospina, H.F. – *Hypersurfaces in the Lorentz-Minkowski space satisfying $L_k\psi = A\psi + b$*, Geom. Dedicata, **153** (2011), 151-175.
12. Magid, M.A. – *Lorentzian isoparametric hypersurfaces*, Pacific J. Math., **118** (1985), no. 1, 165-197.
13. O'Neill, B. – *Semi-Riemannian geometry. With applicatins to relativity*, Pure and Applied Mathematics, 103, Acad. Press Inc., New York (1983).
14. Pashaie, F.; Mohammadpouri, A. – *L_k-biharmonic spacelike hypersurfaces in Minkowski 4-space E_1^4*, Sahand Communications in Mathematical Analysis, **5** (2017), no. 1, 21-30.

15. Pashaie, F.; Kashani, S.M.B. – *Spacelike hypersurfaces in Riemannian or Lorentzian space forms satisfying* $L_k x = Ax + b$, Bull. Iranian Math. Soc. **39** (2013), no. 1, 195-213.
16. Pashaie, F.; Kashani, S.M.B. – *Timelike hypersurfaces in the standard Lorentzian space forms satisfying* $L_k x = Ax + b$, Mediterr. J. Math. **11** (2014), no. 2, 755-773.
17. Petrov, A.Z. – *Einstein spaces*, Translated from the Russian by R. F. Kelleher. Translation edited by J. Woodrow Pergamon Press, Oxford-Edinburgh-New York (1969).
18. Reilly, R.C. – *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geometry, **8** (1973), 465-477.

Received: 12.I.2019 / Revised: 10.IV.2019 / Accepted: 19.VI.2019

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