An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVI, 2020, f. 1

Approximation of continuous functions by generalized deferred Voronoi-Nörlund means of partial sums of their Fourier series

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Abstract In this paper, we have proved some results on the degree of approximation in sup-norm of continuous functions by generalized deferred Voronoi-Nörlund means of partial sums of their Fourier series.

Keywords Deferred Voronoi-Nörlund transformation · Degree of approximation · Fourier series and modulus of continuity

Mathematics Subject Classification (2010) 42A10 · 41A25 · 40Gxx

1 Introduction

Let $f(x)$ be a 2π- periodic continuous function. Let $S_n(f;x)$ denote the n-th partial sum of its Fourier series at x

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

and let $\omega(\delta) = \omega(\delta, f)$ denote the modulus of continuity of f.

Let $A := (a_{n,k})$ $(k, n = 0, 1, \dots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f) := T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x) \quad (n = 0, 1, \dots).$$

The deviation

$$\|T_{n,A}(f) - f\| = \sup_{0 \le x \le 2\pi} |T_{n,A}(f;x) - f(x)|$$

was estimated by P. Chandra [3] and [4] for monotonic sequences $\{a_{n,k}\}$.

Years later, these results are generalized by L. Leindler [9] who considered the sequences of Rest Bounded Variation and of Head Bounded Variation.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{c}$ has only finite nonzero terms, and the last nonzero term is c_N.

Chandra's results are proved by L. Leindler replacing monotonicity condition with $\mathbf{c} \in RBVS$ or $\mathbf{c} \in HBVS$.

Later on, B. Wei and D. Yu [12] have generalized Leindler's results, and thus Chandra's results, without assumptions that $\mathbf{c} \in RBVS$ or $\mathbf{c} \in HBVS$. They verified there that Leindler's results are consequences of their results. Before we recall those results we shall suppose that

$$a_{n,k} \geq 0, \quad \sum_{k=0}^{n} a_{nk} = 1, \tag{1.1}$$

and $\omega(t)$ is such that

$$\int_u^{\pi} t^{-2}\omega(t)dt = \mathcal{O}(H(u)), \quad (u \to 0^+), \tag{1.2}$$

where $H(u) \geq 0$, and

$$\int_0^t H(u)du = \mathcal{O}(tH(t)), \quad (t \to 0^+). \tag{1.3}$$

B. Wei and D. Yu's results read as follows:

Theorem 1.1 *Let (1.1) hold. Suppose that $\omega(t)$ satisfies (1.2). Then*

$$\|T_{n,A}(f) - f\| = \mathcal{O}\left(\omega(\pi/n) + \sum_{k=0}^{n} |\triangle a_{nk}| H(\pi/n)\right).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f)-f\| = \mathcal{O}\bigg(\sum_{k=0}^{n}|\triangle a_{nk}|H\bigg(\sum_{k=0}^{n}|\triangle a_{nk}|\bigg)\bigg),$$

$$\|T_{n,A}(f)-f\| = \mathcal{O}\bigg(\sum_{k=0}^{n}|\triangle a_{nk}|H(\pi/n)\bigg).$$

Theorem 1.2 *Let $(a_{n,k})$ satisfies (1.1). Then*

$$\|T_{n,A}(f)-f\|$$
$$= \mathcal{O}\bigg(\omega(\pi/n) + \sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{\mu=0}^{k+1}a_{n\mu} + \sum_{k=1}^{n}\omega(\pi/k)\sum_{\mu=k}^{n}|\triangle a_{n\mu}|\bigg).$$

Now we recall some further generalizations of the above results.
For an arbitrary sequence $\{b_n\}$ we denote

$$\triangle_m^0 b_{n,m} := b_{n,m}, \quad \triangle_m^1 b_{n,m} := \triangle_m b_{n,m} = b_{n,m} - b_{n,m+1}$$

and

$$\triangle_m^{\nu} b_{n,m} := \triangle\left(\triangle_m^{\nu-1} b_{n,m}\right), \quad \nu = 1,2,3,\dots\ .$$

Theorem 1.3 *[7] Let $(a_{n,k})$ satisfies conditions*

$$a_{n,k} \geq 0 \quad \textit{and} \quad \sum_{k=0}^{n} a_{nk} = \frac{1}{2^r-1}, \quad r \in \{1,2,\dots\}. \tag{1.4}$$

Assume that $\omega(t)$ satisfies condition (1.2). Then

$$\|T_{n,A}(f)-f\| = \mathcal{O}_r\bigg(\omega(\pi/n) + \sum_{k=0}^{n}G_{nk;r}H(\pi/n)\bigg).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f)-f\| = \mathcal{O}_r\bigg(\sum_{k=0}^{n}G_{nk;r}H\bigg(\sum_{k=0}^{n}G_{nk;r}\bigg)\bigg),$$

and

$$\|T_{n,A}(f)-f\| = \mathcal{O}_r\bigg(\sum_{k=0}^{n}G_{nk;r}H(\pi/n)\bigg),$$

where $G_{nk;r} := \sum_{j=1}^{\beta}|\triangle^j a_{n,k}|$, $\beta \in \{1,2,\dots\}$ and τ denotes the integer part of $\frac{\pi}{t}$.

Theorem 1.4 *[7] Let $(a_{n,k})$ satisfies (1.4). Then*

$$\|T_{n,A}(f)-f\|$$
$$=\mathcal{O}_r\left(\omega(\pi/n)+\sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{\mu=0}^{k+1}a_{n\mu}+\sum_{k=1}^{n}\omega(\pi/k)\sum_{\mu=k}^{n}G_{n\mu;r}\right).\quad (1.5)$$

For our further investigation let $a := (a_n)$ and $b := (b_n)$ be sequences of non-negative integers with conditions

$$a_n < b_n, \quad n = 1, 2, \dots \tag{1.6}$$

and

$$\lim_{n\to\infty} b_n = +\infty. \tag{1.7}$$

The deferred Cesàro mean (see [1]) determined by a and b is defined as

$$D_n := D_a^b := \frac{S_{a_n+1}+S_{a_n+2}+\cdots+S_{b_n}}{b_n-a_n}$$

where (S_m) is a sequence of real or complex numbers.

Since each D_a^b with conditions (1.6) and (1.7) satisfies the Silverman-Toeplitz conditions, each transformation D_a^b is regular. It also should be noted that D_a^b involves transformation of deferred terms of (S_m), except the case when $a_n = 0$ for all n. Moreover, D_{n-1}^n is the identity transformation and D_0^n is the well-known $(C, 1)$ transformation. The interested reader can be found in [1] some fundamental properties of transformation D_a^b.

Recently, the authors of [6] have introduced some a deferred transformation with conditions (1.6) and (1.7). Indeed, let (p_n) be a sequence of positive real numbers. Then it is written

$$D_a^bN_n(f;x)=\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n}p_{b_n-m}S_m(f;x),$$

where

$$P_0^{b_n-a_n-1}:=\sum_{m=0}^{b_n-a_n-1}p_m\neq 0.$$

The transformation is regular under condition (1.6) and (1.7) (see [6], p. 13), and it is called the deferred Voronoi-Nörlund means, denoted by (D_a^bN, p). Note that, in the special case, when $b_n = n$ and $a_n = 0$ for all $n \geq 0$, the transformation $D_a^bN_n(f;x)$ gives the classical well-known Voronoi-Nörlund means (transformation). Moreover, for $p_m = 1$ for all $n \geq 0$, it leads to the deferred Cesàro means

$$D_a^b(f;x)=\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}S_m(f;x)$$

of $S_m(f;x)$.

We also note here that, if $b_n = n$, $a_n = 0$, and $p_n = 1$ for all $n \geq 0$ for this transformation, then it coincide with the Cesàro means $(C,1)$. In particular case of this, when $a_n = 0$, (b_n) is a strictly increasing sequence of positive integers with $b_0 = 0$ and $p_n = 1$, then they give us the Cesàro submethod which is obtained by deleting a set of rows from the Cesàro matrix, see for details in [2].

The aim of the present paper is to estimate the quantity $\|D_a^b N_n(f) - f\|$ which is not included in the above mentioned theorems. Also, we shall show that some new results, which are consequences of ours as a special case, as well as those that already has been proved previously.

We emphasize here that throughout of this paper we write $u = \mathcal{O}_r(v)$ if there exists a positive constant C, depending on r, such that $u \leq Cv$, and we assume that all transformations under consideration are regular transformations even in the cases when they are not written explicitly.

2 Helpful lemmas

To prove the main results we need some lemmas.

Lemma 2.1 *[3] If (1.2) and (1.3) hold then*

$$\int_0^{\pi/n} \omega(t)dt = \mathcal{O}\left(n^{-2}H(\pi/n)\right).$$

Lemma 2.2 *[4] If (1.2) and (1.3) hold then*

$$\int_0^{r} t^{-1}\omega(t)dt = \mathcal{O}\left(rH(r)\right), \quad (r \to +0).$$

Lemma 2.3 *For any sequence (p_n), $n = 0,1,2,\dots$ of non-negative numbers, it holds uniformly in $0 < t \leq \pi$, that*

$$\sum_{m=a_n+1}^{b_n} p_{b_n-m} \sin\left(k+\frac{1}{2}\right)t$$
$$= \mathcal{O}_r\Bigg(\sum_{m=a_n+1}^{\tau} p_{b_n-m} + \frac{1}{t}\sum_{k=\tau}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\Bigg), \quad (2.1)$$

where τ denotes the integer part of $\frac{\pi}{t}$ and $r \in \{1,2,\dots\}$.

It also holds that

$$\sum_{m=a_n+1}^{b_n} p_{b_n-m} \sin\left(k+\frac{1}{2}\right)t = \mathcal{O}_r\left(\frac{1}{t}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\right). \quad (2.2)$$

Proof. At first, we denote

$$\Lambda_{s,b_n}(t) := \sum_{m=s}^{b_n} p_{b_n-m} \sin\left(m+\frac{1}{2}\right) t \sin\frac{t}{2}.$$

Then, for $a_n+1 \leq s \leq b_n$, we clearly have

$$\begin{aligned}
&\Lambda_{s,b_n}(t)\\
&= \frac{1}{2}\left[p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-1} \triangle_m p_{b_n-m}\cos(m+1)t - p_0\cos(b_n+1)t\right]\\
&= \frac{1}{2}\left[p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-1} \triangle_m p_{b_n-m}\cos\left(m+\frac{1}{2}\right)t\cos\frac{t}{2}\right.\\
&\qquad \left. + \sum_{m=s}^{b_n-1} \triangle_m p_{b_n-m}\sin\left(m+\frac{1}{2}\right)t\sin\frac{t}{2} - p_0\cos(b_n+1)t\right]\\
&= \frac{1}{2}\left[p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-1} \triangle_m p_{b_n-m}\cos\left(m+\frac{1}{2}\right)t\cos\frac{t}{2} - p_0\cos(b_n+1)t\right]\\
&\quad + \frac{1}{2^2}\left[\triangle_m p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-2} \triangle_m^2 p_{b_n-m}\cos(m+1)t - \triangle_m p_1\cos b_n t\right].
\end{aligned}$$

In the same way, doing this transformation $r-$times, we obtain

$$\begin{aligned}
&\Lambda_{s,b_n}(t)\\
&= \frac{1}{2}\left[p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-1} \triangle_m p_{b_n-m}\cos\left(m+\frac{1}{2}\right)t\cos\frac{t}{2} - p_0\cos(b_n+1)t\right]\\
&\quad + \frac{1}{2^2}\left[\triangle_m p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-2} \triangle_m^2 p_{b_n-m}\cos(m+1)t - \triangle_m p_1\cos b_n t\right] + \cdots +\\
&\quad + \frac{1}{2^r}\left[\triangle_m^{r-1} p_{b_n-s}\cos(st) - \sum_{m=s}^{b_n-r} \triangle_m^r p_{b_n-m}\cos(m+1)t - \triangle_m^{r-1} p_{r-1}\cos(b_n+2-r)t\right],
\end{aligned}$$

where $r \in \{1, 2, \dots\}$.

Whence,

$$
\begin{aligned}
&|\Lambda_{s,b_n}(t)|\\
&\leq \frac{1}{2}\left[p_{b_n-s} + \sum_{m=s}^{b_n-1} |\triangle_m p_{b_n-m}| + p_0\right]\\
&\quad + \frac{1}{2^2}\left[\, |\triangle_m p_{b_n-s}| + \sum_{m=s}^{b_n-2} \left|\triangle_m^2 p_{b_n-m}\right| + |\triangle_m p_1| \,\right] + \cdots\\
&\quad + \frac{1}{2^r}\left[\, \left|\triangle_m^{r-1} p_{b_n-s}\right| + \sum_{m=s}^{b_n-r} |\triangle_m^r p_{b_n-m}| + \left|\triangle_m^{r-1} p_{r-1}\right| \,\right]\\
&= \frac{1}{2}\left[p_{b_n-s} + \sum_{m=s}^{b_n} |\triangle_m p_{b_n-m}|\,\right] + \frac{1}{2^2}\left[\, |\triangle_m p_{b_n-s}| + \sum_{m=s}^{b_n} \left|\triangle_m^2 p_{b_n-m}\right| \,\right] + \cdots\\
&\quad + \frac{1}{2^r}\left[\, \left|\triangle_m^{r-1} p_{b_n-s}\right| + \sum_{m=s}^{b_n} |\triangle_m^r p_{b_n-m}| \,\right],
\end{aligned}
$$

because we agree with convenient

$$-p_1 = 0,\ \triangle^1(-p_1) = 0, \ldots, \triangle^{r-1}(-p_1) = 0.$$

But since (p_m) is a sequence of positive numbers, then the following inequalities

$$
\begin{aligned}
&p_{b_n-s} \leq \sum_{m=s}^{b_n} |\triangle_m p_{b_n-m}|\,,\\
&|\triangle_s p_{b_n-s}| \leq \sum_{m=s}^{b_n} \left|\triangle_m^2 p_{b_n-m}\right|\,, \ldots,\ \left|\triangle_s^{r-1} p_{b_n-s}\right| \leq \sum_{m=s}^{b_n} |\triangle_m^r p_{b_n-m}| \qquad (2.3)
\end{aligned}
$$

hold true for $a_n + 1 \leq s \leq b_n$.

Now by (2.3) and taking into account that $b_n \geq \tau$, we have

$$
\begin{aligned}
&|\Lambda_{a_n+1,b_n}(t)| \\
\leq &\sum_{m=a_n+1}^{\tau} p_{b_n-m} + \left|\sum_{m=\tau}^{b_n} p_{b_n-m} \sin\left(m+\frac{1}{2}\right)t\right| \\
\leq &\sum_{m=a_n+1}^{\tau} p_{b_n-m} \\
&+\mathcal{O}_r\left(\frac{1}{t}\left(\sum_{m=\tau}^{b_n} |\triangle_m p_{b_n-m}| + \sum_{m=\tau}^{b_n} \left|\triangle_m^2 p_{b_n-m}\right| + \cdots + \sum_{m=\tau}^{b_n} |\triangle_m^r p_{b_n-m}|\right)\right) \\
= &\mathcal{O}_r\left(\sum_{m=a_n+1}^{\tau} p_{b_n-m} + \frac{1}{t}\sum_{k=\tau}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\right),
\end{aligned}
$$

where O_r depends only on r.

This completely proves (2.1).

Similarly we have

$$
\begin{aligned}
|\Lambda_{a_n+1,b_n}(t)| = &\mathcal{O}_r\left(\frac{1}{t}\left(\left(p_{b_n-a_n-1} + \sum_{m=a_n+1}^{b_n} |\triangle_m p_{b_n-m}|\right)\right.\right. \\
&+\left(|\triangle_m p_{b_n-a_n-1}| + \sum_{m=a_n+1}^{b_n} \left|\triangle_m^2 p_{b_n-m}\right|\right) \\
&\left.\left.+\cdots+\left(\left|\triangle_m^{r-1} p_{b_n-a_n-1}\right| + \sum_{m=a_n+1}^{b_n} |\triangle_m^r p_{b_n-m}|\right)\right)\right) \\
= &\mathcal{O}_r\left(\frac{1}{t}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\right),
\end{aligned}
$$

which completes (2.2), and with this, the proof of the lemma.

□

Remark 2.1 If we take $r = 1$, $b_n = n$, $a_n = 0$ for all n, and (p_n) is non-decreasing sequence, then Lemma 2.3 reduces to the Lemma (5.11) proved in [10].

3 Main Results

At first, we prove the following.

Theorem 3.1 *Assume that $\omega(t)$ satisfies condition (1.2). Then*

$$\|D_a^b N_n(f)-f\| \tag{3.1}$$
$$= \mathcal{O}_r\left(\frac{b_n p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\omega\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)+\Omega(a_n,b_n;r)H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|D_a^b N_n(f)-f\| = \mathcal{O}_r\Big(\Omega(a_n,b_n;r)H\Big(\Omega(a_n,b_n;r)\Big)\Big) \tag{3.2}$$

and

$$\|D_a^b N_n(f)-f\| \tag{3.3}$$
$$= \mathcal{O}_r\left(\left(\left(\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)^2 b_n(b_n-a_n)+1\right)\Omega(a_n,b_n;r)H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right),$$

where

$$\Omega(a_n,b_n;r):=\frac{\pi}{2(2^r-1)P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left|\triangle_m^i p_{b_n-m}\right|.$$

Proof. First we set

$$\phi_x(t):=\frac{f(x+t)+f(x-t)-2f(x)}{2},$$

Next, we can write

$$D_a^b N_n(f;x)-f(x)$$
$$=\frac{2}{\pi(P_0^{b_n-a_n-1})}\int_0^\pi \phi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{m=a_n+1}^{b_n} p_{b_n-m}\sin\left(m+\frac{1}{2}\right)t\,dt \tag{3.4}$$

By (3.4) we have

$$\|D_a^b N_n(f)-f\|\le\frac{2}{\pi P_0^{b_n-a_n-1}}\left(\int_0^{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}}+\int_{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}}^{\pi}\right):=J_1(n)+J_2(n). \tag{3.5}$$

Since,

$$\frac{1}{P_0^{b_n-a_n-1}}\left|\sum_{m=a_n+1}^{b_n} p_{b_n-m}\sin\left(m+\frac{1}{2}\right)t\right|\le b_n t,$$

then by Lemma 2.2 we obtain

$$J_1(n) = \mathcal{O}(b_n) \int_0^{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}} \omega(t)dt = \mathcal{O}_r\left(\frac{b_n p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\omega\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right). \tag{3.6}$$

Moreover, by (2.2) and (1.2) we have

$$\begin{aligned} J_2(n) &= \mathcal{O}_r\left(\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\right)\int_{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}}^{\pi} t^{-2}\omega(t)dt \\ &= \mathcal{O}_r\left(\Omega(a_n,b_n;r)H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right). \end{aligned} \tag{3.7}$$

Now (3.1) follows from (3.5)–(3.7).

Further, taking into account that

$$\begin{aligned} \Omega(a_n,b_n;r) &:= \frac{\pi}{2(2^r-1)P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left|\triangle_m^i p_{b_n-m}\right| \\ &\le \frac{\pi}{2(2^r-1)P_0^{b_n-a_n-1}}(2+2^2+\cdots+2^r)\sum_{m=a_n+1}^{b_n} p_{b_n-m} = \pi, \end{aligned}$$

we get

$$\|D_a^b N_n(f)-f\| \le \frac{2}{\pi P_0^{b_n-a_n-1}}\left(\int_0^{\Omega(a_n,b_n;r)} + \int_{\Omega(a_n,b_n;r)}^{\pi}\right) := B_1(n)+B_2(n). \tag{3.8}$$

It is obvious that

$$\left|\sum_{m=a_n+1}^{b_n} p_{b_n-m}\sin\left(m+\frac{1}{2}\right)t\right| \le P_0^{b_n-a_n-1}.$$

Thus, by Lemma 2.2 we have

$$B_1(n) = \mathcal{O}_r(1)\int_0^{\Omega(a_n,b_n;r)} t^{-1}\omega(t)dt = \mathcal{O}_r\Big(\Omega(a_n,b_n;r)H\Big(\Omega(a_n,b_n;r)\Big)\Big). \tag{3.9}$$

Now, using (2.2) and (1.2) we obtain

$$\begin{aligned} B_2(n) &= \mathcal{O}_r\left(\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n}\sum_{i=1}^{r}\left(\left|\triangle_m^i p_{b_n-m}\right|\right)\int_{\Omega(a_n,b_n;r)}^{\pi} t^{-2}\omega(t)dt\right) \\ &= \mathcal{O}_r\Big(\Omega(a_n,b_n;r)H\Big(\Omega(a_n,b_n;r)\Big)\Big). \end{aligned} \tag{3.10}$$

From (3.8), (3.9), and (3.10) follows (3.2).

Now we prove (3.3). Applying Lemma 2.1 we find that

$$J_1(n) = \mathcal{O}(b_n) \int_0^{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}} \omega(t)dt = \mathcal{O}_r\left(\frac{b_n (p_{b_n-a_n-1})^2}{\left(P_0^{b_n-a_n-1}\right)^2} H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right).$$

Since $p_{b_n-m} = 0$ for $m > b_n$, then it implies

$$p_{b_n-j} \le \sum_{m=j}^{b_n} |\triangle p_{b_n-m}| \le \sum_{m=a_n+1}^{b_n} \sum_{i=1}^{r} \left|\triangle_m^i p_{b_n-m}\right|,$$

for $j = a_n+1, a_n+2, \ldots, b_n$, and hence

$$\begin{aligned} 1 &= \frac{1}{P_0^{b_n-a_n-1}} \sum_{j=a_n+1}^{b_n} p_{b_n-j} \\ &\le \frac{b_n-a_n}{P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} \sum_{i=1}^{r} \left|\triangle_m^i p_{b_n-m}\right|, \end{aligned}$$

i.e.

$$\Omega(a_n, b_n; r) \ge \frac{\pi}{2(2^r-1)(b_n-a_n)}.$$

Subsequently,

$$J_1(n) = \mathcal{O}_r\left(\left(\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)^2 b_n(b_n-a_n)\Omega(a_n, b_n; r) H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right). \tag{3.11}$$

Therefore, by (3.5), (3.7), and (3.11), the relation (3.3) is proved.

The proof is completed.

□

Our second main result is the following:

Theorem 3.2 *Let (1.2) and (1.3) be satisfied. Then*

$$\begin{aligned} \|D_a^b N_n(f) - f\| = \mathcal{O}_r\Bigg\{ &\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}} \Bigg[b_n \omega\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right) \\ &+ \sum_{j=1}^{\frac{P_0^{b_n-a_n-1}}{p_{b_n-a_n-1}}} j^{-1} \omega\left(\frac{\pi}{j}\right) \sum_{m=a_n+1}^{j+1} p_{b_n-m} \\ &+ \sum_{j=1}^{\frac{P_0^{b_n-a_n-1}}{p_{b_n-a_n-1}}} \omega\left(\frac{\pi}{j}\right) \sum_{m=j}^{b_n} \sum_{i=1}^{r} \left|\triangle_m^i p_{b_n-m}\right| \Bigg] \Bigg\}. \end{aligned} \tag{3.12}$$

Proof. According to (2.1) and the monotonicity of $\omega(t)$, we have

$$
\begin{aligned}
J_2(n) &\leq \frac{1}{2P_0^{b_n-a_n-1}} \int_{\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}}^{\pi} \frac{\phi_x(t)}{t} \sum_{m=a_n+1}^{b_n} p_{b_n-m} \sin\left(m+\frac{1}{2}\right) t dt \\
&= \mathcal{O}_r\left[\frac{1}{P_0^{b_n-a_n-1}} \sum_{j=1}^{\frac{P_0^{b_n-a_n-1}}{p_{b_n-a_n-1}}-1} \int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}} t^{-1}\omega(t)\right. \\
&\qquad \left.\times\left(\sum_{m=a_n+1}^{\tau} p_{b_n-m} + \frac{1}{t}\sum_{m=\tau}^{b_n}\sum_{i=1}^{r}\left|\triangle_m^i p_{b_n-m}\right|\right)dt\right] \\
&= \mathcal{O}_r\left\{\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\left[\sum_{j=1}^{\frac{P_0^{b_n-a_n-1}}{p_{b_n-a_n-1}}} j^{-1}\omega\left(\frac{\pi}{j}\right)\sum_{m=a_n+1}^{j+1} p_{b_n-m}\right.\right. \\
&\qquad \left.\left.+\sum_{j=1}^{\frac{P_0^{b_n-a_n-1}}{p_{b_n-a_n-1}}} \omega\left(\frac{\pi}{j}\right)\sum_{m=j}^{b_n}\sum_{i=1}^{r}\left|\triangle_m^i p_{b_n-m}\right|\right]\right\}.
\end{aligned}
\tag{3.13}
$$

Combining (3.5), (3.6), and (3.13), we get (3.12).
The proof is completed.

□

4 Corollaries of the main results

Now let us see corollaries which derive form Theorem 3.1. If (p_n) is non-decreasing sequence and $r=1$, then

$$
\Omega(a_n, b_n; 1) := \frac{\pi}{2P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} (p_{b_n-m} - p_{b_n-m-1}) = \frac{\pi p_{b_n-a_n-1}}{2P_0^{b_n-a_n-1}}.
$$

So, we have the following.

Corollary 4.1 *Assume that $\omega(t)$ satisfies condition (1.2), (p_n) is nondecreasing sequence, and $r=1$. Then*

$$
\|D_a^b N_n(f) - f\| = \mathcal{O}\left[\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\left(b_n\omega\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right) + H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right)\right].
$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$
\|D_a^b N_n(f) - f\| = \mathcal{O}\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}} H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right)
$$

and

$$\|D_a^b N_n(f) - f\|$$
$$= \mathcal{O}\left(\left(\left(\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)^2 b_n(b_n-a_n)+1\right)\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}} H\left(\frac{\pi p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}\right)\right).$$

If we take $p_n = 1$ for all n, then the generalized deferred Voronoi-Nörlund transformation $D_a^b N_n(f;x)$ reduces to the deferred Cesàro mean

$$D_a^b(f;x) = \frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} S_m(f;x),$$

and

$$\Omega(a_n, b_n; 1) := \frac{\pi}{2(b_n - a_n)},$$

and there from we can write the following corollary.

Corollary 4.2 *Assume that $\omega(t)$ satisfies condition (1.2). Then*

$$\|D_a^b(f) - f\| = \mathcal{O}\left[\frac{\pi b_n}{b_n - a_n}\omega\left(\frac{\pi}{b_n - a_n}\right) + \frac{\pi}{b_n - a_n} H\left(\frac{\pi}{b_n - a_n}\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|D_a^b(f) - f\| = \mathcal{O}\left(\frac{\pi}{b_n - a_n} H\left(\frac{\pi}{b_n - a_n}\right)\right),$$

and

$$\|D_a^b(f) - f\| = \mathcal{O}\left(\frac{b_n}{b_n - a_n} + 1\right)\frac{\pi}{b_n - a_n} H\left(\frac{\pi}{b_n - a_n}\right).$$

It is known in [1] that

$$(C,1) \subset D_a^b \quad \text{if and only if} \quad \frac{b_n}{b_n - a_n} = \mathcal{O}(1).$$

Subsequently, if we take $p_n = 1$ for all n, and assuming that $\frac{b_n}{b_n - a_n} = \mathcal{O}(1)$, then we obtain:

Corollary 4.3 *Assume that $\omega(t)$ satisfies condition (1.2) and $\frac{b_n}{b_n - a_n} = \mathcal{O}(1)$ for all n. Then*

$$\|D_a^b(f) - f\| = \mathcal{O}\left[\omega\left(\frac{\pi}{b_n - a_n}\right) + \frac{\pi}{b_n - a_n} H\left(\frac{\pi}{b_n - a_n}\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|D_a^b(f) - f\| = \mathcal{O}\left(\frac{\pi}{b_n - a_n} H\left(\frac{\pi}{b_n - a_n}\right)\right).$$

Let us suppose that $\mathbb{F}$ is a subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of a strictly increasing sequence of positive integers, say $\mathbb{F} = (\lambda(n))_1^\infty$. Following [2] the Cesàro sub-method C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \ \ (n = 1, 2, \dots),$$

where (x_k) is a sequence of a real or complex numbers. Whence, C_λ -transformation yields a sub-sequence of the Cesàro transformation $(C, 1)$, and therefore it is regular for any λ. In fact, C_λ-transformation is obtained by deleting a set of rows from Cesàro matrix.

The case of C_λ-transformation is also included in the generalized deferred Voro-noi-Nörlund transformation $D_a^b N_n(f; x)$. Namely, it is sufficient to we take in it $p_n = 1$ and $a_n = 0$ for all n, and thus,

$$D_0^b(f; x) \equiv C_\lambda(f; x)$$

and therefore we have the next corollary.

Corollary 4.4 *Assume that $\omega(t)$ satisfies condition (1.2). Then*

$$\|D_0^b(f) - f\| = \mathcal{O}\left[\omega\left(\frac{\pi}{b_n}\right) + \frac{1}{b_n} H\left(\frac{\pi}{b_n}\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|D_0^b(f) - f\| = \mathcal{O}\left(\frac{\pi}{b_n} H\left(\frac{\pi}{b_n}\right)\right).$$

The interested reader (see [13], p.80) can find the following transformation

$$D_{n-1}^{n+k-1} = \sigma_{n,k} = \left(1 + \frac{n}{k}\right)\sigma_{n+k-1} - \frac{n}{k}\sigma_{n-1},$$

which is called First Delayed Arithmetic Mean. It is also known that

$$(C, 1) \subset D_{n-1}^{n+k-1} \text{ if and only if } \frac{n}{k} = \mathcal{O}(1) \text{ as } k \to \infty \text{ with } n.$$

Taking $k = 2n$, the condition $\frac{n}{k} = \mathcal{O}(1)$ is satisfied, and then the first Delayed Arithmetic Mean takes the form

$$D_{n-1}^{3n-1} = \sigma_{n,2n} = \frac{1}{2}\left(3\sigma_{3n-1} - \sigma_{n-1}\right),$$

which is called the Second Type Delayed Arithmetic Mean (see [11], p. 566).

The degree of approximation of continuous functions f by Second Type Delayed Arithmetic Mean of partial sums of their Fourier series is given in next corollary.

Corollary 4.5 *Assume that $\omega(t)$ satisfies condition (1.2). Then*

$$\|\sigma_{n,2n}(f) - f\| = \mathcal{O}\left[\omega\left(\frac{\pi}{2n}\right) + \frac{1}{2n}H\left(\frac{\pi}{2n}\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|\sigma_{n,2n}(f) - f\| = \mathcal{O}\left(\frac{\pi}{2n}H\left(\frac{\pi}{2n}\right)\right).$$

Also, if we take $b_n = n$ and $a_n = 0$ for all $n \geq 0$, then the generalized deferred Voronoi-Nörlund mean reduces to the classical Voronoi-Nörlund mean, and therefore Theorem 3.1 implies:

Corollary 4.6 *Assume that $\omega(t)$ satisfies condition (1.2), (p_n) is nondecreasing sequence, and $r = 1$. Then*

$$\|N_n(f) - f\| = \mathcal{O}\left[\frac{p_{n-1}}{P_{n-1}}\left(n\omega\left(\frac{\pi p_{n-1}}{P_{n-1}}\right) + H\left(\frac{\pi p_{n-1}}{P_{n-1}}\right)\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then ([5], [9])

$$\|N_n(f) - f\| = \mathcal{O}\left(\frac{\pi p_{n-1}}{P_{n-1}}H\left(\frac{\pi p_{n-1}}{P_{n-1}}\right)\right)$$

and

$$\|N_n(f) - f\| = \mathcal{O}\left(\left(\left(\frac{np_{n-1}}{P_{n-1}}\right)^2 + 1\right)\frac{\pi p_{n-1}}{P_{n-1}}H\left(\frac{\pi p_{n-1}}{P_{n-1}}\right)\right).$$

Moreover, if we take $p_n = 1$, $b_n = n$ and $a_n = 0$ for all $n \geq 0$, then the deferred Voronoi-Nörlund mean reduces to the classical Cesàro mean, and Theorem 3.1 also implies:

Corollary 4.7 *Assume that $\omega(t)$ satisfies condition (1.2) and $r = 1$. Then*

$$\|\sigma_n(f) - f\| = \mathcal{O}\left[\left(\omega\left(\frac{\pi}{n}\right) + \frac{1}{n}H\left(\frac{\pi}{n}\right)\right)\right].$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|\sigma_n(f) - f\| = \mathcal{O}\left(\frac{\pi}{n}H\left(\frac{\pi}{n}\right)\right).$$

Remark 4.1 All results presented here having used condition, (p_n) is a nondecreasing sequence, can be replaced by condition $(p_n) \in RBVS$ with $K(\mathbf{c}) = 1$.

Note, that similar corollaries with Corollaries 4.1- 4.7 can be derived from Theorem 3.2. Based on their importance, we write only few of them. Indeed, as we saw above, if we take $b_n = n$ and $a_n = 0$ for all $n \geq 0$, then the generalized deferred Voronoi-Nörlund mean reduces to the classical Voronoi-Nörlund mean, and therefore Theorem 3.2 implies:

Corollary 4.8 *Let (1.2) and (1.3) be satisfied and $r = 1$. Then*

$$\|N_n(f) - f\| = \mathcal{O}_r\left[\frac{np_{n-1}}{P_{n-1}}\omega\left(\frac{\pi p_{n-1}}{P_{n-1}}\right)\right.$$
$$\left. + \frac{p_{n-1}}{P_{n-1}}\sum_{j=1}^{\frac{P_{n-1}}{p_{n-1}}} j^{-1}\omega\left(\frac{\pi}{j}\right)\sum_{m=1}^{j+1} p_{n-m} + \sum_{j=1}^{\frac{P_{n-1}}{p_{n-1}}}\omega\left(\frac{\pi}{j}\right)p_{n-j}\right].$$

If we take $p_n = 1$ for all n, and assuming that $\frac{b_n}{b_n - a_n} = \mathcal{O}(1)$, then we obtain:

Corollary 4.9 *Let (1.2) and (1.3) be satisfied. Then*

$$\|D_a^b(f) - f\| = \mathcal{O}_r\left\{\omega\left(\frac{\pi}{b_n - a_n}\right)\right.$$
$$\left. + \frac{1}{b_n - a_n}\sum_{j=1}^{b_n - a_n}\left[j^{-1}(j + 1 - a_n) + (b_n - j - 1)\right]\omega\left(\frac{\pi}{j}\right)\right\}.$$

If we take $b_n = 3n - 1$ and $a_n = n - 1$, in Corollary 4.9, then we get:

Corollary 4.10 *Let (1.2) and (1.3) be satisfied. Then*

$$\|\sigma_{n,2n}(f) - f\|$$
$$= \mathcal{O}_r\left\{\omega\left(\frac{\pi}{2n}\right) + \frac{1}{2n}\sum_{j=1}^{2n}\left[j^{-1}(j + 2 - n) + (3n - j - 2)\right]\omega\left(\frac{\pi}{j}\right)\right\}.$$

5 Conclusions

Since the time of Bernstein and Jackson, approximations of periodic and continuous functions by trigonometric polynomials took place in the unit of approximation theory. This is owing to periodicity and continuity of the functions. Approximations of such functions has been realized, in different norms and particularly in sup-norm, by various transformations of partial sums of their Fourier series (see [3]-[5] and [7]-[9], [12]). In many interesting results we have encounter various type of conditions on the considered transformations (belonging various classes of the real sequences).

Here, in this paper we have proved some new results pertaining to the degree of approximation in sup-norm of periodic continuous functions by regular generalized deferred Voronoi-Nörlund transformations. In particular, these transformation can be reduced to the deferred Cesàro means, to the classical Voronoi-Nörlund transformations, and several others. The results obtained here can be taken, not only as counterparts of some earlier results proved by others, but also as their significant generalizations. Among others, our results include the degree of approximation of periodic continuous functions in terms of modulus of continuity by the Second Type Delayed Arithmetic Mean of partial sums of their Fourier series.

Acknowledgements The author would like to thank the referee for her/his useful suggestions.

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Received: 26.I.2019 / Accepted: 26.III.2019

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