An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVI, 2020, f. 1

The Generalized order-m (k-Pell) numbers

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Abstract Considering the usual k-Pell numbers we define a new sequence which is called Generalized order-m (k-Pell) sequence. Some properties of this sequence and a formula for sums of k-Pell numbers are obtained. A generalized Binet's formula and some identities involving this sequence are provided and also a combinatorial representation of the Generalized order-m (k-Pell) number is deduced.

Keywords k-Pell numbers · Binet's formula · Matrix method

Mathematics Subject Classification (2010) MSC 11B37 · 05A15 · 11C20 · 15A18

1 Introduction

For a positive integer k, Catarino in [1] studied a generalization of the Pell sequence, as well as in [2] and [3], where Catarino and Vasco studied generalizations of the Pell-Lucas and the Modified Pell sequences, respectively. This generalization of the Pell sequence is defined by a recurrence relation of second order given by

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \tag{1.1}$$

with initial terms $P_{k,0} = 0$ and $P_{k,1} = 1$ and called by k-Pell sequence.

The idea in this paper is to consider a new generalization of the k-Pell sequence using the idea of Er in [6], where the author has defined k sequences of the Generalized order-k Fibonacci for some positive integer k. Also the Generalized order-k Lucas sequence was studied in [14]. In the literature several research has been dedicated to this kind of generalizations of some sequences of positive integers, see, for example, [8], [9], [10], [11], [12], [13] and [16], among others.

The k-Pell sequence is a special case of a sequence which is defined as a linear combination (see [7]), as follows:

$$a_{n+m} = c_1 a_{n+m-1} + c_2 a_{n+m-2} + \cdots + c_m a_n, \tag{1.2}$$

where $c_1, c_2, \ldots, c_m$ are real constants. In [7], Kalman derives a number of closed-form formulas for the generalized sequence by the companion matrix method. Also in [11] the matrix method is used for the case of the Generalized order-k Pell numbers.

Consider a matrix T of order $m \times m$ such that the first line is consisting of the constants $c_i,\ i = 1, \ldots, m$, the entries $t_{i+1,i} = 1$, for $i = 1, \ldots, m-1$ and the remaining entries are equal to zero. Also consider a matrix $A_n = (a_n, \ldots, a_{n+m-1})^T$ associated with (1.2). It is easy to show that $TA_n = A_{n+1}$.

This paper is organized in the following way: in Section 2 we introduce the Generalized order-m (k-Pell) sequence and some properties of such sequence are investigated by the use of the respective companion matrix; in Section 3 we refer a generalized Binet's formula and some identities involving this sequence are provided; in Section 4, a formula for sums of k-Pell numbers is studied and in the last Section we present a combinatorial representation of the mth Generalized order-m (k-Pell) number.

2 The Generalized order-m (k-Pell) numbers and some properties

For positive integers m and k, we define m sequences called by Generalized order-m (k-Pell) sequences of numbers which are defined by

$$P^i_{k,n} = 2P^i_{k,n-1} + k\left(P^i_{k,n-2} + \cdots + P^i_{k,n-m}\right) \tag{2.1}$$

for $n > 0$ and $1 \leq i \leq m$, with initial conditions

$$P^i_{k,n} = \begin{cases} 1 & \text{if} \quad n = 1-i \\ 0 & \text{otherwise} \end{cases}, \tag{2.2}$$

for $1 - m \leq n \leq 0$, where $P^i_{k,n}$ is the nth term of the ith sequence.

In the following table we present a few Generalized order-m (k-Pell) numbers.

$n \backslash m$	$m=1$	$m=2$		$m=3$		
$n \backslash i$	$i=1$	$i=1$	$i=2$	$i=1$	$i=2$	$i=3$
$n=-2$				0	0	1
$n=-1$		0	1	0	1	0
$n=0$	1	1	0	1	0	0
$n=1$	2	2	k	2	k	k
$n=2$	4	$4+k$	$2k$	$4+k$	$3k$	$2k$
$n=3$	8	$8+4k$	$4k+k^2$	$8+5k$	$6k+k^2$	$4k+k^2$
$n=4$	16	$16+12k+k^2$	$8k+4k^2$	$16+16k+k^2$	$12k+6k^2$	$8k+5k^2$

Table 1 The Generalized order-m (k-Pell) sequences for $m = 1, 2, 3$ and different values of i, $(1 \leq i \leq m)$.

Note that when $i = 1$ and $m = 2$, the Generalized order-2 (k-Pell) sequence $\{P^1_{k,n}\}$ is reduced to the classical k-Pell sequence defined in (1.1). The relation between the Generalized order-2 (k-Pell) sequence and the k-Pell sequence is given by $P^1_{k,-1} = P_{k,0},\ P^1_{k,0} = P_{k,1},\ P^1_{k,1} = P_{k,2},\ \ldots$, i.e.,

$$P^1_{k,j} = P_{k,j+1}, \quad j \geq -1. \tag{2.3}$$

Also the sequence obtained for $m = i = 1$ coincides with a geometric progression which the common ratio is equal to 2.

By definition of Generalized order-m (k-Pell) sequence, we can write the following vector recurrence relation:

$$\begin{bmatrix} P^i_{k,n+1} \\ P^i_{k,n} \\ P^i_{k,n-1} \\ \vdots \\ P^i_{k,n-m+2} \end{bmatrix} = T \begin{bmatrix} P^i_{k,n} \\ P^i_{k,n-1} \\ P^i_{k,n-2} \\ \vdots \\ P^i_{k,n-m+1} \end{bmatrix}, \tag{2.4}$$

where T is a $m \times m$ companion matrix, called by Generalized order-m (k-Pell) matrix and given by

$$T = \begin{bmatrix} 2 & k & k & \cdots & k & k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.5}$$

We define a m-square matrix $B_n = [b_{i,j}]$ in order to work with the m sequences of the Generalized order-m (k-Pell) numbers, as follows:

$$B_n = \begin{bmatrix} P^1_{k,n} & P^2_{k,n} & \cdots & P^m_{k,n} \\ P^1_{k,n-1} & P^2_{k,n-1} & \cdots & P^m_{k,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ P^1_{k,n-m+1} & P^2_{k,n-m+1} & \cdots & P^m_{k,n-m+1} \end{bmatrix}. \tag{2.6}$$

It is easy to show that $B_n = TB_{n-1}$ and that $B_1 = T$. Then we have the following result:

Lemma 2.1 *Let T and B_n be matrices as stated in (2.5) and (2.6), respectively. Then for all integers $n \geq 0$*

$$B_n = T^n.$$

Proof. Since $B_n = TB_{n-1}$, then, by an inductive argument, we can write that $B_n = T^{n-1}B_1$. According the fact that $B_1 = T$ we obtain $B_n = T^n$, as required.

□

As a consequence of this lemma we have the following corollary:

Corollary 2.2 *Let B_n the matrix defined in (2.6). Then*

$$detB_n = \begin{cases} 2^n & if \quad m = 1 \\ (-k)^n & if \quad m \quad is\ even \\ k^n & if \quad m \quad is\ odd \quad (m \neq 1) \end{cases} \tag{2.7}$$

Proof. By Lemma 2.1, $detB_n = det\left(T^n\right) = \left(detT\right)^n$. By the Laplace expansion of the determinant of T with respect to last column, it is easy to get the result required.

□

Now we give some properties involving the Generalized order-m (k-Pell) numbers.

Theorem 2.3 *Let $P^i_{k,n}$ be the nth term of the ith Generalized order-m (k-Pell) sequence, with $1 \leq i \leq m$. Then for $l \geq 0$ and $n \geq 1$ we have*

$$P^i_{k,n+l} = \sum_{j=1}^{m} P^j_{k,l} P^i_{k,n-j+1}.$$

Proof. By Lemma 2.1, $B_n = T^n$ and then $B_n = B_{n-1}B_1 = B_1B_{n-1}$ and, generalizing, we can write $B_{n+l} = B_nB_l = B_lB_n$. Therefore, an element of B_{n+l} is the product of a row of B_n by a column of B_l, and the result required follows.

□

In the following result we present some relations involving the terms of the Generalized order-m (k-Pell) sequence.

Lemma 2.4 *For positive integers k and m and a natural number n the following identities hold:*

1. $P^1_{k,n+1} = 2P^1_{k,n} + P^2_{k,n}$;
2. $P^2_{k,n+1} = kP^1_{k,n} + P^3_{k,n}$;
3. $P^i_{k,n+1} = kP^1_{k,n} + P^{i+1}_{k,n},\ 3 \leq i \leq m-1,\ m \geq 4$;
4. $P^m_{k,n+1} = kP^1_{k,n},\ m \neq 1$.

Proof. Taking into account the definition of the matrix B_n, the identity $B_{n+1} = B_nT$ and the multiplication of matrices, these identities are easily shown . For instance, the first identity is exactly the entry of the matrix B_{n+1} that is in first line and first column. By taking the product of the first line of B_n with the first column of T we obtain the second member of this identity and given that $B_{n+1} = B_nT$ we immediately have the desired result. Using a similar process we immediately get the other identities of this Lemma.

□

3 The Generalized Binet formula

In this section we present a generalized Binet's formula and some identities involving this sequence are provided.

From companion matrices, it is known that the characteristic equation of the matrix T is

$$x^m - 2x^{m-1} - k\left(x^{m-2} + x^{m-3} + \cdots + x + 1\right) = 0,$$

which is also the characteristic equation of Generalized order-m (k-Pell) numbers. The following result will be used in the proof of theorems in this section.

Lemma 3.1 *The equation $x^{m+1} - 3x^m - (k-2)\,x^{m-1} + k$ does not have multiple roots for $m \geq 2,\ k \geq 3$.*

Proof. Let $f(x) = x^m - 2x^{m-1} - k\left(x^{m-2} + x^{m-3} + \cdots + x + 1\right)$ and

$$\begin{aligned}
g(x) &= (x-1)\,f(x)\\
&= (x-1)\left(x^m - 2x^{m-1} - k\left(x^{m-2} + x^{m-3} + \cdots + x + 1\right)\right)\\
&= x^{m+1} - 3x^m - (k-2)\,x^{m-1} + k
\end{aligned}$$

Clearly, 1 is a root of $g(x)$ but not a multiple root since $m \geq 2$ and $f(1) \neq 1$. Suppose that α is a multiple root of $g(x)$ such that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, we have that

$$g(\alpha) = \alpha^{m+1} - 3\alpha^m - (k-2)\,\alpha^{m-1} + k$$

and

$$g'(\alpha) = \alpha^{m-2}\left((m+1)\,\alpha^2 - 3m\alpha - (k-2)\,(m-1)\right).$$

Thus $g'(\alpha) = 0 \Leftrightarrow (m+1)\,\alpha^2 - 3m\alpha - (k-2)\,(m-1) = 0$ since $\alpha \neq 0$. Let denote by $\Delta = 9m^2 + 4\left(m^2-1\right)(k-2)$. Hence the two roots of derivative of g are given by

$$\alpha_1 = \frac{3m + \sqrt{\Delta}}{2\,(m+1)}, \quad \alpha_2 = \frac{3m - \sqrt{\Delta}}{2\,(m+1)}.$$

We obtain that $-g(\alpha_1) = \alpha_1^{m-1}\left(-\alpha_1^2 + 3\alpha_1 + (k-2)\right) - k$ and $-g(\alpha_2) = \alpha_2^{m-1}\left(-\alpha_2^2 + 3\alpha_2 + (k-2)\right) - k$ and if $a_{m,k} = \alpha_1^{m-1}\left(-\alpha_1^2 + 3\alpha_1 + (k-2)\right)$ and $b_{m,k} = \alpha_2^{m-1}\left(-\alpha_2^2 + 3\alpha_2 + (k-2)\right)$, then$-g(\alpha_1) = 0 \Leftrightarrow a_{m,k} - k = 0$ and $-g(\alpha_2) = 0 \Leftrightarrow b_{m,k} - k = 0$ for $k \geq 3,\ m \geq 2$.

However if we choose $k = 3$ and $m = 2$ we get $\alpha_1 = \frac{3+2\sqrt{3}}{3},\ \alpha_2 = \frac{3-2\sqrt{3}}{3}$ and $a_{2,3} = 3 + \frac{16\sqrt{3}}{9},\ b_{2,3} = 3 - \frac{16\sqrt{3}}{9}$. Hence $a_{2,3} \neq 3$ and $b_{2,3} \neq 3$, which is a contradiction because we suppose that α is a multiple root for $m \geq 2,\ k \geq 3$. Therefore, the equation $g(x) = 0$ does not have multiple roots.

□

As a consequence of Lemma 3.1, the equation

$$x^m - 2x^{m-1} - k\left(x^{m-2} + x^{m-3} + \cdots + x + 1\right) = 0$$

does not have multiple roots for $m \geq 2,\ k \geq 3$.

Let $f(\lambda)$ be the characteristic polynomial of the Generalized order-m (k-Pell) matrix T. Then $f(\lambda) = \lambda^m - 2\lambda^{m-1} - k\left(\lambda^{m-2} + \lambda^{m-3} + \cdots + \lambda + 1\right)$ and let $\lambda_1, \lambda_2,$
$\ldots, \lambda_m$ be the eigenvalues of the matrix T, which are distinct by Lemma 3.1.

Let V be the m-square matrix defined by

$$V = \begin{bmatrix} \lambda_1^{m-1} & \lambda_1^{m-2} & \lambda_1^{m-3} & \ldots & \lambda_1 & 1 \\ \lambda_2^{m-1} & \lambda_2^{m-2} & \lambda_2^{m-3} & \ldots & \lambda_2 & 1 \\ \lambda_3^{m-1} & \lambda_3^{m-2} & \lambda_3^{m-3} & \ldots & \lambda_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_m^{m-1} & \lambda_m^{m-2} & \lambda_m^{m-3} & \ldots & \lambda_m & 1 \end{bmatrix}, \tag{3.1}$$

which is a Vandermonde matrix and let c_m^i be a $m \times 1$ matrix

$$c_m^i = \begin{bmatrix} \lambda_1^{n+m-i} \\ \lambda_2^{n+m-i} \\ \lambda_3^{n+m-i} \\ \vdots \\ \lambda_m^{n+m-i} \end{bmatrix}. \tag{3.2}$$

Let $V_j^{(i)}$ be a m-square matrix obtained from V by replacing the jth column of V by c_m^i. Considering these matrices in the following result we present the generalized Binet formula for the Generalized order-m (k-Pell) numbers.

Theorem 3.2 *Let $P_{k,n}^i$ be the nth term of the ith k-Pell sequence of Generalized order-m (k-Pell) numbers, for $1 \leq i \leq m$. Then*

$$P_{k,n-i+1}^j = \frac{det\left(V_j^{(i)}\right)}{det\,(V)}. \tag{3.3}$$

Proof. The matrix T is diagonalizable since the eigenvalues of T are distinct. Now we have $\left(V^T\right)^{-1} TV^T = D = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_m)$ as the matrix V^T is invertible. So we can write $T^n V^T = V^T D^n$. Taking into account Lemma 2.1 we have the following linear system of equations:

$$\begin{cases} b_{i1}\lambda_1^{m-1} + b_{i2}\lambda_1^{m-2} + \cdots + b_{im} = \lambda_1^{n+m-i} \\ b_{i1}\lambda_2^{m-1} + b_{i2}\lambda_2^{m-2} + \cdots + b_{im} = \lambda_2^{n+m-i} \\ \vdots \\ b_{i1}\lambda_m^{m-1} + b_{i2}\lambda_m^{m-2} + \cdots + b_{im} = \lambda_m^{n+m-i} \end{cases} \tag{3.4}$$

where $B_n = [b_{ij}]$ is the m-square matrix defined in (2.6) and so $b_{ij} = P_{k,n-i+1}^j$ for each $j = 1, 2, \ldots, m$. Therefore the result easily follows. □

4 Sums of the Generalized order-m (k-Pell) numbers

In this section we present the sums of the Generalized order-m (k-Pell) numbers and for a particular case we obtain also the sums of k-Pell numbers. Let us consider $S_n = \sum_{i=0}^{n-1} P_{k,j}^1$. For $m = 1$ the sums of the Generalized order-1 (k-Pell) is given by

$$S_n = 2^n - 1 \tag{4.1}$$

since in this case the sequence is a geometric progression as we have mentioned in Section 2.

From now on consider $m \geq 2$ and by the use of item 4 of Lemma 2.4 then we have

$$S_n = \sum_{i=0}^{n-1} \left(\frac{1}{k} P_{k,i+1}^m \right) \tag{4.2}$$

$$= \frac{1}{k} \left(\sum_{i=0}^{n-1} P_{k,i+1}^m \right) \tag{4.3}$$

$$= \frac{1}{k} \left(\sum_{i=1}^{n} P_{k,i}^m \right) \tag{4.4}$$

Following what was the idea in [9] and in [16], let us consider the square $(m+1) \times (m+1)$ matrices E and W_n defined by

$$E = \begin{bmatrix} 1 & 0\ 0 \cdots 0\ 0 \\ 1 & \\ 0 & \\ 0 & T \\ \vdots & \\ 0 & \end{bmatrix} \tag{4.5}$$

and

$$W_n = \begin{bmatrix} 1 & 0\ 0 \cdots 0\ 0 \\ S_n & \\ S_{n-1} & \\ S_{n-2} & B_n \\ \vdots & \\ S_{n-m+1} & \end{bmatrix}, \tag{4.6}$$

where T and B_n are the matrices given in (2.5) and (2.6), respectively.

Once more by item 4 of Lemma 2.4, we obtain that $S_n = \frac{1}{k} \left(\sum_{i=1}^{n} P_{k,i}^m \right) = \frac{1}{k} \left(P_{k,n}^m + \sum_{i=1}^{n-1} P_{k,i}^m \right) = \frac{1}{k} \left(k P_{k,n-1}^1 \right) + S_{n-1} = P_{k,n-1}^1 + S_{n-1}$. Therefore

$$S_n = P_{k,n-1}^1 + S_{n-1} \tag{4.7}$$

and $W_{n-1}E = W_n$ and by generalization we get that $W_n = W_1 E^{n-1}$.

Taking into account the fact that $S_{-i} = 0$ for $1 \leq i \leq m$ and the definition of the Generalized order-m (k-Pell) numbers, we conclude that $W_1 = E$ and consequently $W_n = E^n$. Thus we obtain that $W_{n+1} = E^{n+1} = E^n E = W_n W_1 = E E^n = W_1 W_n$ and we conclude that the matrix W_n commutes with W_1 and vice-versa. Therefore since $W_{n+1} = W_n W_1$ then we obtain the equality

$$S_{n+1} = P^1_{k,n} + S_n \tag{4.8}$$

and since $W_{n+1} = W_1 W_n$ we have that

$$S_{n+1} = 1 + 2S_n + k\left(\sum_{i=1}^{n-1} S_{n-i}\right). \tag{4.9}$$

Taking into account (4.8) and (4.9), we obtain

$$P^1_{k,n} + S_n = 1 + 2S_n + k\left(\sum_{i=1}^{n-1} S_{n-i}\right). \tag{4.10}$$

For the particular case of $m = 2$ and $i = 1$ (case of the k-Pell sequence), taking into account (2.3) the equation (4.10) gives

$$P_{k,n+1} + S_n = 1 + 2S_n + kS_{n-1}.$$

Hence $S_n = P_{k,n+1} - kS_{n-1} - 1$ and then by (4.7) we obtain

$$S_n = P_{k,n+1} - k\left(S_n - P^1_{k,n-1}\right) - 1.$$

Finally, by the use of the fact that $kP^1_{k,n-1} = P_{k,n}$ and some algebraic calculation we obtain the formula of the sums of k-Pell numbers given in Proposition 1 of [4] and Proposition 3, item 3 of [15].

5 Combinatorial representation of the Generalized order-m (k-Pell) numbers

In this section we present a combinatorial representation of the mth Generalized order-m (k-Pell) number. We shall use the companion matrix T given by (2.5), the matrix B_n given by (2.6) and Theorem 3.1 of [5]. The idea is the explicit formula for all elements in the nth power of the companion matrix T given by Lemma 2.1.

Considering the entry $b_{p,q}$ of matrix B_n that is in the line p and column q and recall Theorem 3.1 of [5], we have in our case:

Theorem 5.1

$$b_{p,q} = \sum_{(l_1,l_2,\ldots,l_m)} \frac{l_q + l_{q+1} + \cdots + l_m}{l_1 + l_2 + \cdots + l_m} \times \binom{l_1 + l_2 + \cdots + l_m}{l_1,\ldots,l_m} 2^{l_1} k^{l_2} k^{l_3} \cdots k^{l_m},$$

where the summation is over nonnegative integers satisfying $l_1 + 2l_2 + \cdots + ml_m = n-p+q$ and the coefficient used above is defined to be 1 if $n = p-q$.

In particular for $p = 1$ and $q = m$ we obtain the following corollary

Corollary 5.2 *For positive integers k, m, n, we have*

$$b_{1,m} = \sum_{(l_1,l_2,\ldots,l_m)} \frac{l_m}{l_1+l_2+\cdots+l_m} \times \binom{l_1+l_2+\cdots+l_m}{l_1,\ldots,l_m} 2^{l_1} k^{l_2} k^{l_3} \cdots k^{l_m},$$

where the summation is over nonnegative integers satisfying $l_1+2l_2+\cdots+ ml_m = n-1+m$ and the coefficient used above is defined to be 1 if $n = 1-m$.

Therefore taking into account (2.6), a combinatorial representation of the mth Generalized order-m (k-Pell) number is provided.

Acknowledgements This research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UID/MAT/00013/2013 and UID/CED/00194/2013.

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Received: 30.III.2019 / Accepted: 23.IX.2019

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