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Projective change between Z-Shen's square metric and Kropina metric with some curvature properties

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Abstract In this paper, we show that two (α, β)-metrics, namely, Z-Shen's square metric and Kropina metric have same Douglas tensor if and only if both are Douglas metrics and find some relations to characterize projective change between them on a manifold. Moreover, we consider this projective change when Z-Shen's square metric satisfies some curvature properties.

Keywords Projective change · Finsler metric · Z-Shen's square metric · Kropina metric · S-curvature

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1 Introduction

The projective change of Finsler metrics is an delightful topic to study in Finsler geometry. The projective relation is said to be trivial if the corresponding sprays are equal. Two Finsler metrics F and $\tilde{F}$ on a manifold M are said to be projectively related if every geodesic of the first is also a geodesic of the second and vice versa. The projective changes between two Finsler spaces have been studied by many authors [[13], [20], [10], [11], [9], [4], [1] and [14]].

In 1929, L. Berwald constructed an example of a projectively flat Finsler metric of constant flag curvature $K = 0$, on unit ball B^n given by

$$F(x,y) = \frac{\left(\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}, (x,y) \in T\mathbb{R}^n.$$

Here $|\cdot|$ and $\langle , \rangle$ denote the standard Euclidean norm and inner product, respectively on $\mathbb{R}^n$.

Z. Shen introduced a (α,β)-metric called square metric given by $F = \frac{(\alpha+\beta)^2}{\alpha}$. It is interesting to note that Berwald metric is a special case of Z. Shen's square metric. Z. Shen and G. Civi Yildirim [17] classified this special (α,β)-metric under the restriction of projectively flatness and constant flag curvature. Recently in [21], L. Zhou proved that a square metric with constant flag curvature must be projectively flat and hence classified all projectively flat square (α,β)-metrics.

Kropina metric was first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina [5]. Kropina metrics are non-regular Finsler metrics. Kropina metrics are simplest non-trivial Finsler metrics with many interesting application in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [18]. Also, they have interesting applications in relativistic field theory, control theory, evolution and developmental biology.

A crucial result related with the projective change was given by Rapcsák's paper [13]. He proved a necessary and sufficient condition of projective change. In 1984, H. Park and Y. Lee [11] studied projective changes between a Finsler space with (α,β)-metric and the associated Riemannian metric. S. Bácsó and M. Matsumoto [1] in 1994, considered the projective changes among Finsler spaces with (α,β)-metric. In [16] projective change between two Randers metrics is studied and it is obtained that two Randers metrics are projectively related if and only if the corresponding Riemannian metrics are projectively related and they have the same Douglas tensor. In 2009, N.W. Cui and Y. Bing Shen [4] studied projective change between Z. Shen square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ and a Randers metric and in 2011, M. Zohrehvand and M. M. Rezaii [22] studied projectively related Matsumoto metric and a Randers metric.
In this paper, we study projective change between Z-Shen's square metric and Kropina metric.

2 Preliminaries

Let M be an n-dimensional smooth manifold. Denote T_xM, the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote elements of TM by (x, y), where $x = (x^i)$ be a point of M and $y \in T_xM$ called supporting element. We denote $TM_0 = TM \setminus \{0\}$.

Definition: A Finsler metric on M is a function $F : TM \to [0,\infty)$ with the following properties:
(i) F is smooth on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and
(iii) the Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

For a given Finsler metric $F = F(x, y)$, the geodesic of F are characterized locally by a following system of second order differential equations [2]

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the spray coefficients of F, given by

$$G^i = \frac{1}{4}g^{il}\left\{\left[F^2\right]_{x^m y^l} y^m - \left[F^2\right]_{x^l}\right\}.$$

In Riemannian geometry, two Riemannian metrics α and $\tilde{\alpha}$ are projectively related if and only if their spray coefficients are given by [4]

$$G^i_\alpha = G^i_{\tilde{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function.
A Finsler metric F on M is said to be projective to $\tilde{F}$ if there exists a scalar function $P(x, y)$ on $TM \setminus \{0\}$, homogeneous of degree one in y, such that

$$G^i = \tilde{G}^i + P(x, y)y^i, \tag{2.2}$$

where G^i and $\tilde{G}^i$ are the spray coefficients of F and $\tilde{F}$ respectively.
A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

In 1972, M. Matsumoto [8] introduced (α, β)-metrics. By definition, an (α, β)-metric is a Finsler metric expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form satisfying $\|\beta_x\| < b_0$, $\forall x \in M$. It is well known that $F = \alpha\phi(s)$ is a regular (α, β)-metric iff the function $\phi(s)$ is a positive C^∞ function with $|s| < b_0$ satisfying [2]

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0. \tag{2.3}$$

In this case the metric tensor induced by F is positive definite.
Let

$$r_{ij} := \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \quad s_{ij} := \frac{1}{2}\left(b_{i|j} - b_{j|i}\right)$$

where $b_{i|j}$ means the coefficients of the covariant derivative of β with respect to α. Clearly β is closed if and only if $s_{ij} = 0$.
An (α, β)-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Furthermore, we denote

$$r^i_j := a^{ik}r_{kj}, \quad s^i_j := a^{ik}s_{kj},$$

$$r_{00} := r_{ij}y^iy^j, \quad r_{i0} := r_{ij}y^j,$$
$$s_i := b_js^j_i, \quad s_0 := s_iy^i,$$
$$r := r_{ij}b^ib^j, \quad s_{i0} := s_{ij}y^j.$$

The relation between the spray coefficients G^i of F and geodesic coefficients G^i_α of α are given by [15]

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\left\{\Psi b^i + \Theta\alpha^{-1}y^i\right\}, \tag{2.4}$$

where

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\left((\phi - s\phi') + (b^2 - s^2)\phi''\right)},$$
$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

The tensor $D := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called Douglas tensor where,

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right). \tag{2.5}$$

A Finsler metric is called Douglas metric if the Douglas tensor vanishes ([4], [19]). The Douglas tensor is projective invariant [12]. Since the spray coefficients of a Riemannian metric are quadratic forms. It can be easily seen that for Riemannian metric Douglas tensor vanishes, which shows that Douglas tensor is a non-Riemannian quantity. The fundamental fact is that all Berwald metrics must be Douglas metrics.
Here we use quantities with a tilde to denote the corresponding quantities of the metric $\tilde{F}$. Now, we enumerate the Douglas tensor of an (α, β)-metric. Let

$$\tilde{G}^i = G^i_\alpha + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.6}$$

Then (2.4) becomes

$$G^i = \tilde{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, if sprays G^i and $\tilde{G}^i$ are projective equivalent then, they have same Douglas tensor. Denote

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.7}$$

Differentiating equation (2.7) with respect to y^m, we have

$$T^m_{y^m} = Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \tag{2.8}$$

The equation (2.6) can be rewritten as $\tilde{G}^i = G^i_\alpha + T^i$. By (2.5), we have

$$\begin{aligned}\tilde{D}^i_{jkl} &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m} y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right).\end{aligned}$$

The Douglas tensor of an (α,β)-metric is given by

$$\tilde{D}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right). \tag{2.9}$$

Thus, if F and $\tilde{F}$ have the same Douglas tensor, i.e. $D^i_{jkl} = D^{\tilde{i}}_{jkl}$. From (2.5) and (2.9), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - T^{\tilde{m}}_{y^m}) y^i\right] = 0. \tag{2.10}$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$, such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - T^{\tilde{m}}_{y^m}) y^i = H^i_{00}, \tag{2.11}$$

where $H^i_{00} := H^i_{jk}(x) y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations (2.7) and (2.8) respectively.

3 Projective change of the two metrics

In this section, we consider on the projectively related two (α,β)-metrics, namely Z. Shen square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ and Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$.

For Z. Shen square metric $\frac{(\alpha+\beta)^2}{\alpha}$, one can prove by (2.3), that F is a regular Finsler metric if and only if $\|\beta_x\| < 1$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$$\begin{aligned} Q &:= \frac{2}{1-s}, \\ \Theta &:= \frac{1-2s}{1+2b^2-3s^2}, \\ \Psi &:= \frac{1}{1+2b^2-3s^2}. \end{aligned} \tag{3.1}$$

For Kropina-metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$, one can see that F is not a regular (α,β)-metric, but the relation $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0$ is still true for

$|s| > 0$.
The geodesic coefficients are given by (2.4) with

$$\tilde{Q} := -\frac{1}{2s}, \quad \tilde{\Theta} = -\frac{s}{\tilde{b}^2}, \quad \tilde{\Psi} := \frac{1}{2\tilde{b}^2}. \tag{3.2}$$

For simplicity, we assume in this paper that $\lambda := \frac{1}{(n+1)}$.

Lemma 3.1 *[7] Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M. Then*
(1) $(n \geq 3)$ Kropina metric F with $b^2 \neq 0$ is a Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}\left(b_i s_k - b_k s_i\right). \tag{3.3}$$

(2) $(n = 2)$ Kropina metric F is a Douglas metric.

Since the Douglas tensor is a projective invariant, we have

Proposition 3.2 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a Z. Shen square metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have same Douglas tensor if and only if both are Douglas metrics.*

Proof. The sufficiency is obvious, we just need to prove the necessity. Suppose that F and $\tilde{F}$ have the same Douglas tensor on an n-dimensional manifold M when $n \geq 3$. Then $D^i_{jkl} = \tilde{D}^i_{jkl}$, which implies that (2.11) holds. Putting equation (3.1) and (3.2) in equation (2.11). Hence, we obtain

$$\frac{A^i\alpha^6+B^i\alpha^5+C^i\alpha^4+D^i\alpha^3+E^i\alpha^2+F^i\alpha+H^i}{I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N}+\frac{\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i}{2\tilde{b}^2\tilde{\beta}}=H^i_{00}, \tag{3.4}$$

where
$A^i := 2\left(1+2b^2\right)\left[(1+2b^2)s^i_0 - 2b^i s_0\right]$,
$B^i := \left(1+2b^2\right)\left[b^i r_{00} - 2\lambda y^i\left(r_0+s_0\right)\right]$,
$C^i := -\beta\left[\left(1+2b^2\right)\left(b^i r_{00}+12\beta s^i_0-2\lambda y^i r_0\right)-\left(12\beta b^i+2\lambda y^i+28\lambda y^i b^2\right)s_0\right]$,
$D^i := -\beta\left[6\lambda y^i\left(b^2 r_{00}-\beta r_0-\beta s_0\right)+3b^i r_{00}\beta\right]$,
$E^i := 3\beta^2\left[6\beta^2 s^i_0+\beta b^i r_{00}+2\lambda y^i\left(b^2 r_{00}-\beta r_0-5\beta s_0\right)\right]$,
$F^i := 6\lambda y^i\beta^3 r_{00}$,
$H^i := -6\lambda y^i\beta^4 r_{00}$,

$I := -\left(1+2b^2\right)^2, \quad J := \beta\left(1+2b^2\right)^2,$
$W := 6\beta^2\left(1+2b^2\right), \quad L := -6\beta^3\left(1+2b^2\right),$
$X = -9\beta^4, \quad N = 9\beta^5,$
and
$\tilde{A}^i := \tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s}_0,$
$\tilde{B}^i := \tilde{\beta}\left[2\lambda y^i\left(\tilde{r}_0+\tilde{s}_0\right) - \tilde{b}^i\tilde{r}_{00}\right].$
Here $\lambda := \frac{1}{n+1}$. Further, (3.4) is equivalent to

$$\begin{aligned}&\left(A^i\alpha^6+B^i\alpha^5+C^i\alpha^4+D^i\alpha^3+E^i\alpha^2+F^i\alpha+H^i\right)\left(2\tilde{b}^2\tilde{\beta}\right)\\&+\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\left(I\alpha^5+J\alpha^4+W\alpha^3+L\alpha^2+X\alpha+N\right)\\&=H^i_{00}\left(2\tilde{b}^2\tilde{\beta}\right)\left(I\alpha^5+J\alpha^4+W\alpha^3+L\alpha^2+X\alpha+N\right).\end{aligned} \tag{3.5}$$

Replacing y^i by $-y^i$ in (3.5) yields

$$\begin{aligned}&\left(-A^i\alpha^6+B^i\alpha^5-C^i\alpha^4+D^i\alpha^3-E^i\alpha^2+F^i\alpha-H^i\right)\left(-2\tilde{b}^2\tilde{\beta}\right)\\&-\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\left(I\alpha^5-J\alpha^4+W\alpha^3-L\alpha^2+X\alpha-N\right)\\&=H^i_{00}\left(-2\tilde{b}^2\tilde{\beta}\right)\left(I\alpha^5-J\alpha^4+W\alpha^3-L\alpha^2+X\alpha-N\right).\end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), yields

$$\begin{aligned}&\left(A^i\alpha^6+C^i\alpha^4+E^i\alpha^2+H^i\right)\left(2\tilde{b}^2\tilde{\beta}\right)+\left(J\alpha^4+L\alpha^2+N\right)\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\\&\qquad=H^i_{00}\left(2\tilde{b}^2\tilde{\beta}\right)\left(J\alpha^4+L\alpha^2+N\right).\end{aligned} \tag{3.7}$$

Again, subtracting (3.6) from (3.5), yields

$$\begin{aligned}&\left(B^i\alpha^5+D^i\alpha^3+F^i\alpha\right)\left(2\tilde{b}^2\tilde{\beta}\right)+\left(I\alpha^5+W\alpha^3+X\alpha\right)\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\\&\qquad=H^i_{00}\left(2\tilde{b}^2\tilde{\beta}\right)\left(I\alpha^5+W\alpha^3+X\alpha\right).\end{aligned}$$

That is,

$$\begin{aligned}&\left(B^i\alpha^4+D^i\alpha^2+F^i\right)\left(2\tilde{b}^2\tilde{\beta}\right)+\left(I\alpha^4+W\alpha^2+X\right)\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\\&\qquad=H^i_{00}\left(2\tilde{b}^2\tilde{\beta}\right)\left(I\alpha^4+W\alpha^2+X\right).\end{aligned} \tag{3.8}$$

From equation (3.8), we can see that $\left(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i\right)\left(I\alpha^4+W\alpha^2+X\right)$ can be divided by $\tilde{\beta}$. Again, $\tilde{A}^i\tilde{\alpha}^2\left(I\alpha^4+W\alpha^2+X\right)$ also can be divided by $\tilde{\beta}$. Since $\beta=\mu\tilde{\beta}$, then $\tilde{A}^i\tilde{\alpha}^2 I\alpha^4$ can be divided by $\tilde{\beta}$. Because $\tilde{\beta}$ is prime with respect to α and $\tilde{\alpha}$, therefore $\tilde{A}^i:=\tilde{b}^2\tilde{s}^i_0-\tilde{b}^i\tilde{s}_0$ can be divided by $\tilde{\beta}$. Hence, there is a scalar function $\varphi^i(x)$ such that

$$\tilde{b}^2\tilde{s}^i_0-\tilde{b}^i\tilde{s}_0=\tilde{\beta}\varphi^i. \tag{3.9}$$

Contracting (3.9) by $\tilde{y}_i:=\tilde{a}_{ij}y^j$, we obtained $\varphi^i(\tilde{y}_i)=-\tilde{s}_0$. Since $\tilde{y}_i$ is arbitrary vector, we get $\varphi^i(x)=-\tilde{s}^i$. Then we have

$$\tilde{s}_{ij}=\frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i\right], \tag{3.10}$$

provided, $\tilde{b^2}\neq 0$.
Thus, by Lemma (3.1), $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric. Since F and $\tilde{F}$ have the same Douglas tensor, both of them are Douglas metrics.
When $n=2$, $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric by Lemma (3.1). Thus F and $\tilde{F}$ having the same Douglas tensor means that they are all Douglas metrics. This completes the proof of proposition (3.2).
Now, we are in the position to state the following Theorem.

Theorem 3.3 *Let* $F=\frac{(\alpha+\beta)^2}{\alpha}$ *and* $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *be two* (α,β)*-metrics on a manifold* M *with dimension* $n>2$*, where* α *and* $\tilde{\alpha}$ *are two Riemannian metrics,* $\beta=b_iy^i$ *and* $\tilde{\beta}$ *are two non-zero collinear* 1*- forms. Then* F *is projectively related to* $\tilde{F}$ *if and only if the following relations hold:*
(i)

$$G^i_\alpha=\tilde{G}^i_{\tilde{\alpha}}+\frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i+\tilde{r}_{00}\tilde{b}^i\right]-\tau\alpha^2b^i+\theta y^i,$$

(ii)

$$b_{i|j}=\tau\left[(1+2b^2)a_{ij}-3b_ib_j\right],$$

(iii) $\tilde{s}_{ij}=\frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i\right]$,
where $\tau=\tau(x)$ *is some scalar function,* $b_{i|j}$ *denote the coefficients of the covariant derivatives of* β *with respect to* α *and* $\tilde{b}^i:=\tilde{a}^{ij}\tilde{b}_j$*,* $\tilde{b}:=||\tilde{\beta}||_{\tilde{\alpha}}$ *and* $\theta=\theta_iy^i$ *is a* 1*-form on* M*.*

Z-Shen's square metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ is a Douglas metric if and only if $b_{i|j}=\tau[(1+2b^2)\alpha_{ij}-3b_ib_j]$ holds for some scalar $\tau=\tau(x)$ [6]. It is known that Kropina metric $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if $\tilde{s}_{ij}=\frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i\right]$.

Corollary 3.4 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β)- metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero collinear 1- forms. Then F is projectively related to $\tilde{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ have the following relation*

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] - \tau\alpha^2 b^i + \theta y^i, \tag{3.11}$$

where $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j, \tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

Proof of Theorem 3.3

First, we prove the necessity. Since Douglas tensor is an invariant under projective change between two Finsler metrics. If F is projectively related to $\tilde{F}$, then they have the same Douglas tensor. By proposition (3.2), we obtain that F and $\tilde{F}$ are both Douglas metrics. It has been proved in [7] that Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]. \tag{3.12}$$

It is well known that Z-Shen's square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = \tau\left[(1+2b^2)a_{ij} - 3b_ib_j\right], \tag{3.13}$$

for some scalar function $\tau = \tau(x)$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α. In this case, β is closed. Plugging (3.13) and (3.1) into (2.4) yields

$$G^i = G^i_\alpha + \tau(\alpha - 2\beta)y^i + \tau\alpha^2 b^i. \tag{3.14}$$

On the other hand, plugging (3.12) and (3.2) into (2.4) yields

$$\tilde{G}^i = \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}\left[-\tilde{\alpha}^2\tilde{s}^i + (2\tilde{s}_0 y^i - \tilde{r}_{00}\tilde{b}^i) + \frac{2\tilde{r}_{00}\tilde{\beta}y^i}{\tilde{\alpha}^2}\right]. \tag{3.15}$$

Since F is projectively related to $\tilde{F}$ again, there is a scalar function $P := P(x, y)$ on $TM\setminus\{0\}$ such that

$$G^i = \tilde{G}^i + Py^i. \tag{3.16}$$

From (3.14), (3.15) and (3.16), we have

$$\left[P - \tau(\alpha - 2\beta) - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right)\right]y^i = G^i_\alpha - \tilde{G}^i_{\tilde{\alpha}} + \tau\alpha^2 b^i - \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right]. \tag{3.17}$$

The Right-hand side of the above equation is a quadratic form in y. Then there exists a one form $\theta = \theta_i(x)y^i$ on M such that,

$$P - \tau(\alpha - 2\beta) - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right) = \theta. \tag{3.18}$$

Then, we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] - \tau\alpha^2 b^i + \theta y^i. \tag{3.19}$$

From (3.12), (3.13) and (3.19), we complete the proof of the necessity.

Conversely, plugging (3.13) into (2.4) with (3.1) yields (3.14). Plugging (3.12) into (2.4) with (3.2) yields (3.15). From (3.14) and (3.19) we have

$$G^i = \tilde{G}^i_{\tilde{\alpha}} + [\theta + \tau(\alpha - 2\beta)]\, y^i + \frac{1}{2\tilde{b}^2}\left(\tilde{\alpha^2}\tilde{s}^i + \tilde{r}_{00}\tilde{b^i}\right). \tag{3.20}$$

Putting the value of $\tilde{G}^i_{\tilde{\alpha}}$ from equation (3.15) to equation (3.20), we get

$$G^i = \tilde{G}^i + \left[\theta + \tau(\alpha - 2\beta) + \frac{1}{2\tilde{b}^2}\left(2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right)\right] y^i. \tag{3.21}$$

From equation (3.16), we get

$$P = \theta + \tau(\alpha - 2\beta) + \frac{1}{2\tilde{b}^2}\left(2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right). \tag{3.22}$$

i.e. F is projectively related to $\tilde{F}$. We complete the proof of theorem (3.3).

4 Projective change with curvature properties

Berwald curvature tensor of a Finsler metric F is defined by

$$\mathbf{B} := B^i_{jkl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l,$$

where $B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \left[G^i\right]_{y^j y^k y^l}$ and G^i are the spray coefficients of F.

A Finsler metric F is of isotropic Berwald curvature if

$$B^i_{jkl} = c\left(F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^j y^k y^l}y^i\right),$$

where $c = c(x)$ is a scalar function on M [2].

The mean Berwald curvature tensor is defined by $\mathbf{E} := E_{ij}dx^i \otimes dx^j$, where $E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right) = \frac{1}{2}B^m_{mij}$.

A Finsler metric is said to be of isotropic mean Berwald curvature if

$E_{ij} = \frac{n+1}{2}c(x)F_{y^iy^j}$, where $c = c(x)$ is a scalar function on M [14]. Clearly, the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature.

A Finsler metric F is said to have isotropic S-curvature if $\mathbf{S} = (n+1)c(x)F$ for some scalar function $c(x)$ on M [2].
N.W. Cui proves the following

Lemma 4.1 *[3] For metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$, where ϵ and $k \neq 0$ are constants on an n-dimensional manifold M, the following are equivalent:*
(i) F is of isotropic S-curvature, that is, $\boldsymbol{S} = (n+1)c(x)F$;
(ii) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}c(x)F^{-1}\boldsymbol{h}$;
(iii) F has vanished S-curvature, that is, $\boldsymbol{S} = 0$;
(iv) F is a weak Berwald metric, that is, $\boldsymbol{E} = 0$;
(v) β is a Killing 1-form of constant length with respect to α, that is, $r_{00} = s_0 = 0$, where $c = c(x)$ is a scalar function.

The lemma (4.1) is valid for $F = \frac{(\alpha+\beta)^2}{\alpha}$ when we put $\epsilon = 2$ and $k = 1$.

In present section, we suppose that $F = \frac{(\alpha+\beta)^2}{\alpha}$ has some curvature properties. Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is projectively related to F. For $F = \frac{(\alpha+\beta)^2}{\alpha}$ of isotropic S-curvature, we have the following

Theorem 4.2 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_iy^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S-curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*
(i) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i$,
(ii) β is parallel with respect to α, that is, $b_{i|j} = 0$,
(iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i\right]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_iy^i$ is a 1-form on M.

Proof. From Theorem (3.3) the sufficiency is obvious. For the necessity, again from Theorem (3.3) we have that if F is projectively related to $\tilde{F}$, then

$$b_{i|j} = \tau\left[(1+2b^2)a_{ij} - 3b_ib_j\right],$$

for some scalar function $\tau = \tau(x)$. Contracting the above equation with y^i and y^j yields

$$r_{00} = \tau\left[(1+2b^2)\alpha^2 - 3\beta^2\right]. \tag{4.1}$$

By Lemma (4.1), if F has isotropic S-curvature, then $r_{00} = s_0 = 0$.
Plugging $r_{00} = s_0 = 0$ in equation (4.1), we get

$$(1 + 2b^2)\alpha^2 - 3\beta^2 = 0,$$

provided $\tau \neq 0$. That is

$$(1 + 2b^2)a_{ij} - 3b_i b_j = 0.$$

Contracting the above equation with a^{ij}, gives $n + (2n - 3)b^2 = 0$, which is not possible. Therefore $\tau = 0$, which is not possible as $n \geq 2$, putting in Theorem (3.3), we complete the proof.

We know that the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature. Thus, by Lemma (4.1), assuming that F has isotropic Berwald curvature, we immediately obtain the theorem

Theorem 4.3 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic Berwald curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*
(i) $G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right] + \theta y^i$,
(ii) β is parallel with respect to α, i.e. $b_{i|j} = 0$,
(iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}\left[\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i\right]$,
where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M.

Theorem 4.4 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S-curvature.*

Proof. Suppose F has isotropic Berwald curvature, then F has isotropic mean Berwald curvature. Thus, by Lemma (4.1), F is of isotropic S-curvature. This proves necessary condition for proving sufficiency. Since F and $\tilde{F}$ are projectively equivalent, (2.2) holds. Suppose that F has isotropic S-curvature $\mathbf{S} = (n + 1)c(x)F$. By Lemma (4.1), we have F is of isotropic mean Berwald curvature, that is, $E_{ij} = \frac{n+1}{2} cF_{y^i y^j}$.
Given that $\tilde{F}$ has isotropic Berwald curvature, then

$$\tilde{B}^i_{jkl} = \tilde{c}\left(\tilde{F}_{y^j y^k}\delta^i_l + \tilde{F}_{y^j y^l}\delta^i_k + \tilde{F}_{y^k y^l}\delta^i_j + \tilde{F}_{y^j y^k y^l} y^i\right),$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function on M. Hence, by the definition of the mean Berwald tensor, it follows from (2.2) that $cF_{y^i y^j} = \tilde{c}\tilde{F}_{y^i y^j} + P_{y^i y^j}$, which gives that

$$cF_{y^iy^jy^k} = \tilde{c}\tilde{F}_{y^iy^jy^k} + P_{y^iy^jy^k}.$$

Now we have

$$\begin{aligned}
B^i_{jkl} &= \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \\
&= \tilde{B}^i_{jkl} + \left(P_{y^jy^k}\delta^i_l + P_{y^jy^l}\delta^i_k + P_{y^ky^l}\delta^i_j + P_{y^jy^ky^l}y^i\right) \\
&= \tilde{c}\left(\tilde{F}_{y^jy^k}\delta^i_l + \tilde{F}_{y^jy^l}\delta^i_k + \tilde{F}_{y^ky^l}\delta^i_j + \tilde{F}_{y^jy^ky^l}y^i\right) \\
&+ \left(P_{y^jy^k}\delta^i_l + P_{y^jy^l}\delta^i_k + P_{y^ky^l}\delta^i_j + P_{y^jy^ky^l}y^i\right) \\
&= c\left(F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i\right).
\end{aligned}$$

This implies that F has isotropic Berwald curvature. We complete the proof. By the above methods, we could obtain the theorem

Theorem 4.5 *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and F has isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S-curvature.*

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