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On the uniqueness of L-function and meromorphic function sharing one or two sets

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Abstract In this paper, we study the uniqueness of an L-function and a meromorphic function when they share one or two sets. We focus on sharing of two sets, one set with finite weight 2 and another set IM. We also have considered the case of sharing only one set with finite weight 2. The results of this paper improve the recent results due to Yuan, Q.-Q; Li, X.-M.; Yi, H.-X. [Lith. Math. J., 58 (2018), no. 2, 249-262].

Keywords Nevanlinna theory · Meromorphic function · L-function · Set sharing · Uniqueness

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1 Introduction, Definitions and Results

In this paper, by meromorphic function we always mean meromorphic function in the complex plane $\mathbb{C}$. Throughout the paper, we adopt the standard notations of Nevanlinna value distribution theory such as $T(r,f)$ as the characteristic function, $N(r,f)$ as the counting function, $N(r,a;f) = N(r,\frac{1}{f-a})$, $m(r,f)$ as the proximity function and $S(r,f)$ as any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly on a set of finite logarithmic measure not necessarily the same at each occurance, as explained in [4,15, 16]. When a pole is counted only once irrespective of its multiplicity, then the counting function is called reduced counting function and is denoted by $\overline{N}(r,f)$. A zero of $f-a$ is called an $a-$point of f. $\overline{N}(r,a;f)$ stands for the reduced counting function of the $a-$points of f. The order of f, denoted by

$\rho(f)$, is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Let f and g be two nonconstant meromorphic functions, and a be a finite value (complex number). We say that f and g share a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with same multiplicities. Similarly, we say that f and g share a IM (ignoring multipliticities) if $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

In the beginning of this century a notion of weighted sharing has been introduced by Lahiri [6,7], which is defined as follows.

Definition 1.1. *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all $a-$points of f where an $a-$point of multiplicity m is counted m times if $m \le k$ and k+1 times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

For a meromorphic function f and a set $S \subset \mathbb{C}\cup\{\infty\}$, we define $E_f(S)$ $(\overline{E}_f(S))$ to be the set of all zeros of $f-a$, where $a \in S$, together with their multiplicities (ignoring their multiplicities). We say that two functions f and g share the set S CM (IM) if $E_f(S) = E_g(S)$ $(\overline{E}_f(S) = \overline{E}_g(S))$. Clearly, two functions f and g share a set S with weight k if $E_k(a;f) = E_k(a;g)$ for all $a \in S$ and in notation, we say f and g share (S,k).

In 1989, Selberg [13] introduced a new class of Dirichlet series, called Selberg class function which eventually has become an important field of research in Analytic Number Theory. In this paper the term "L-function" will mean a Selberg class function with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype. The Selberg class S of L-functions is defined as the set of all Dirichlet series $\mathcal{L} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ that satisfy the following axioms [13]:

1. Ramanujan hypothesis: $a(n) \ll n^{\epsilon}$ for each $\epsilon > 0$;
2. Analytic continuation: there is a nonnegative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order;
3. Functional equation: $\mathcal{L}$ satisfies a functional equation of the type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\bar{s})}$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j; the usual gamma function Γ and complex numbers ν_j, ω with $Re(\nu_j) \geq 0$ and $|\omega| = 1$;

4. Euler product hypothesis: $\mathcal{L}$ can be written over primes as follows:

$$\mathcal{L}(s) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ such that $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined by $d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j$, where λ_j and K are respectively the positive real number and the positive integer defined as in axiom (3) above. The value distribution of the L-function $\mathcal{L}$ concerns the distribution of zeros of $\mathcal{L}$ and more generally the roots of the equation $\mathcal{L}(s) = c$ for some $c \in \mathbb{C} \cup \{\infty\}$. Recently, a lot of research work on this topic has been done across the globe (see [2], [5], [9], [11], [14]). The notion of sharing a value or set of values by an L-function is similar to those of a meromorphic function described in this section.

In 2007, Steuding [14] proved the following theorem.

Theorem A. *If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ with $a(1) = 1$ share a complex value c $(\neq \infty)$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Remark 1.1. *In 2016, Hu and Li [5] have shown that Theorem 1 is not true when $c = 1$ by giving a counter example.*

Though L-functions possess analytic continuation to meromorphic functions, in 2010, Li has shown (see [9]) that Theorem A is not true if one L-function is replaced by a meromorphic function. The following example clarifies this.

Example 1.1. *For an entire function g, let $\mathcal{L} = \zeta$ and $f = \zeta e^g$, where ζ denotes the Riemann zeta function. Then f and $\mathcal{L}$ share 0 CM but $\mathcal{L} \neq f$.*

However, for sharing two distinct values, Li [9] proved the following result for a meromorphic function and an L-function.

Theorem B. *Let f be a meromorphic function in $\mathbb{C}$ having finitely many poles and let a, b be two distinct finite complex values. If f and an L-function $\mathcal{L}$ share a CM and b IM, then $f = \mathcal{L}$.*

In [3], Gross asked the following questions:

Question 1.1. *Do there exist two finite sets S_1 and S_2 such that any two nonconstant entire functions f and g must be identical if $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$?*

Question 1.2. *Is there a finite set S such that any two nonconstant entire functions f and g must be identical if $E_f(S) = E_g(S)$?*

In [17] and [18], Yi has given an affirmative answer to the above questions by proving the following results.

Theorem C. *Let $S_1 = \{a + b, a + b\omega, \cdots, a + b\omega^{n-1}\}$ and $S_2 = \{c\}$ be two subsets of $\mathbb{C}$, where $b \neq 0$ and $\omega = exp(\frac{2\pi i}{n})$ with $n > 4$, and let $c \neq a$ be such that $(c-a)^{2n} \neq b^{2n}$. If two nonconstant entire functions f and g satisfy $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f = g$.*

Theorem D. *Let ω_j $(j = 1, 2, \cdots, n)$ be n distinct roots of the equation $P(\omega) = \omega^n + a\omega^m + b = 0$, where n, m are relatively prime positive integers with $n \geq 15$ and $n > m \geq 5$ and a, b are nonzero finite constants satisfying*

$$\frac{b^{n-m}}{a^n} \neq \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}.$$

Suppose that $S = \{c + d\omega_1, c + d\omega_2, \cdots, c + d\omega_n\}$ where d $(\neq 0)$ and c are constants. If two nonconstant entire functions f and g satisfy $E_f(S) = E_g(S)$, then $f = g$.

In connection to these results a lot of works have been done later. In [21], Yuan-Li-Yi raised the following question.

Question 1.3. *What can be said about the uniqueness of a meromorphic function f and an L-function $\mathcal{L}$ when they share one or two finite sets?*

In the same paper the authors have given the answer to the above question by proving the following theorems.

Theorem E. *Let $S = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}$ be the set of distinct roots of the equation $P(\omega) = \omega^n + a\omega^m + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq n$ and n, m are relatively prime positive integers with $n \geq 5$, $n > m$ and a, b, c are nonzero finite constants where $c \neq \gamma_j$ for $1 \leq j \leq l$. Let f be meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function. If f and $\mathcal{L}$ share S CM and c IM, then $f = \mathcal{L}$.*

Theorem F. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and $\mathcal{L}$ be a nonconstant L-function. Let $P(\omega) = \omega^n + a\omega^m + b$, where n and m are relatively prime positive integers with $n > 2m + 4$, and a, b are two nonzero constants. If f and $\mathcal{L}$ share $S = \{\omega : P(\omega) = 0\}$ CM, then $f = \mathcal{L}$.*

Regarding the above two results one may ask the following question.

Question 1.4. *If we replace the CM sharing with weighted sharing, how far the conclusions of Theorems E and F hold?*

Studying for a possible answer to the above question we find the following two results.

Theorem 1.1. *Let $S = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}$ be the set of distinct roots of the equation $P(\omega) = \omega^n + a\omega^m + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq n$ and n, m are relatively prime positive integers with $n > 2k + 4$ or $n > 2m + 4$, where $k = n - m \geq 1$ and a, b, c are nonzero finite constants where $c \neq \gamma_j$ for $1 \leq j \leq l$. Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function. If f and $\mathcal{L}$ share $(S, 2)$ and c IM, then $f = \mathcal{L}$.*

Theorem 1.2. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and $\mathcal{L}$ be a nonconstant L-function. Let S be the set of distinct roots of the equation $P(\omega) = \omega^n + a\omega^m + b = 0$, where n, m are relatively prime positive integers with $n > 2m + 4$, and a, b are nonzero finite constants. If f and $\mathcal{L}$ share $(S, 2)$, then $f = \mathcal{L}$.*

We now state some necessary definitions.

Definition 1.2. [6] *For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $\overline{N}(r, a; f| = k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k. In particular, $\overline{N}(r, a; f| = 1)$ or $N(r, a; f| = 1)$ is the counting function of the simple a-points of f.*

Definition 1.3. [6] *For a positive integer k, we denote by $N(r, a; f| \leq k)$ $(N(r, a; f| \geq k))$ the counting function of those a-points of f whose multiplicities are not greater (less) than k, where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f| \leq k)$ and $\overline{N}(r, a; f| \geq k)$ are the corresponding reduced counting functions.*

Definition 1.4. [7] *We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.*

Definition 1.5. [20] *Let f and g be two nonconstant meromorphic functions sharing a value $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f of order p and an a-point of g of order q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the counting function of those common a-points of f and g where $p > q$ $(q > p)$, each a-point being counted once only.*

Definition 1.6. [6,7] *Let f and g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those common a-points of f and g whose multiplicities are not the same. Clearly, $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.*

2 Lemmas

We now present some lemmas which will be needed to prove our results. Henceforth, we denote H as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions.

Lemma 2.1. [22] *Let p (>0) and q be two relatively prime integers, and let a be a finite complex number satisfying $a^p=1$. Then the expressions ω^p-1 and ω^q-a have exactly one common zero.*

Lemma 2.2. (See Lemma 2.7, [10]) *Let $P(\omega)=\omega^n+a\omega^m+b$, where n and m are positive integers such that $n>m$, a and b are finite complex values. Then the following cases occur:*

1. *The algebraic equation $P(\omega)=0$ has no root of multiplicity ≥ 3;*
2. *If*
$$\frac{b^{n-m}}{a^n}\neq\frac{(-1)^n m^m(n-m)^{n-m}}{n^n}, \tag{2.1}$$
then the algebraic equation $P(\omega)=0$ has n distinct roots;
3. *If n and m are relatively prime and*
$$\frac{b^{n-m}}{a^n}=\frac{(-1)^n m^m(n-m)^{n-m}}{n^n}, \tag{2.2}$$
then the algebraic equation $P(\omega)=0$ has $n-1$ distinct roots, i.e., exactly one root is of multiplicity 2.

Lemma 2.3. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let S be the set given as in Theorem 1.1. If f and a nonconstant L-function $\mathcal{L}$ share S IM, then $\rho(f)=\rho(\mathcal{L})=1$.*

Proof. The proof of the lemma follows by using Lemma 2.2 and by similar arguments as in the first part of the proof of Theorem 5 in [21]. □

Lemma 2.4. [12] *Let f be a nonconstant meromophic function and let $F=\sum_{i=0}^{p}a_if^i/\sum_{j=0}^{q}b_jf^j$ be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$, where $a_p\neq 0$ and $b_q\neq 0$. Then*

$$T(r,F)=dT(r,f)+O(1),$$

where $d=\max\{p,q\}$.

Lemma 2.5. [19] *If two nonconstant meromorphic functions F and G share 1 IM and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;F) \leq N(r,H) + S(r,F) + S(r,G).$$

Taking $m = 2$ in Lemma 2.2 of [1] we can obtain the following lemma.

Lemma 2.6. *Let F and G be two nonconstant meromorphic functions sharing $(1,2)$. Then*

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)\leq\frac{1}{2}[N(r,1;F)+N(r,1;G)]+N_E^{1)}(r,1;F)-\frac{3}{2}\overline{N}_*(r,1;F,G).$$

Lemma 2.7. *Let F and G be two nonconstant meromorphic functions sharing $(1,2)$. If $H \not\equiv 0$, then*

$$\begin{aligned}N(r,H) \leq{}& \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}(r,\infty;F|\geq 2)+\overline{N}(r,\infty;G|\geq 2)\\ &+ \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + O(\log r).\end{aligned}$$

where $\overline{N}_0(r,0;F')$ denotes the reduced counting function of those zeros of F' which are not zeros of $F(F-1)$. $\overline{N}_0(r,0;G')$ is defined similarly.

Proof. Since simple zeros and poles of F and G can not be poles of H, the lemma follows from the proof of Lemma 4 [8].

□

Lemma 2.8. *Let $F = \frac{f^n+af^m}{-b}$ and $G = \frac{g^n+ag^m}{-b}$, where f and g are any two meromorphic functions having finitely many poles in $\mathbb{C}$, n, m are positive integers such that $n > m \geq 1$ and a, b are nonzero finite constants. Let $\omega_1, \omega_2, \cdots, \omega_k$ be the roots of the equation $\omega^k + a = 0$, where $k = n - m$. If $H \not\equiv 0$ and F, G share $(1,2)$, then*

$$\begin{aligned}\frac{n}{2}[T(r,f)+T(r,g)]\leq{}&\sum_{j=1}^{k}\{N_2(r,\omega_j;f)+N_2(r,\omega_j;g)\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &-\frac{1}{2}\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).\end{aligned}$$

Proof. Since f and g have only finitely many poles in $\mathbb{C}$, we have

$$\overline{N}(r,F) = O(\log r) \quad \text{and} \quad \overline{N}(r,G) = O(\log r). \tag{2.3}$$

Using (2.3), from Nevanlinna's second fundamental theorem we get,

$$T(r,F) \leq \overline{N}(r,1;F) + \overline{N}(r,0;F) - N_0(r,0,F') + S(r,F)$$

and

$$T(r,G)\le \overline{N}(r,1;G)+\overline{N}(r,0;G)-N_0(r,0,G')+S(r,G).$$

Therefore,

$$\begin{aligned}T(r,F)+T(r,G)\le{}&\overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\&-N_0(r,0,F')-N_0(r,0,G')+S(r,F)+S(r,G).\end{aligned}\tag{2.4}$$

Using (2.3), from Lemmas 2.5, 2.6 and 2.7 we have

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\le{}&\frac{1}{2}[N(r,1;F)+N(r,1;G)]+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)\\&-\frac{1}{2}\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+O(\log r).\end{aligned}\tag{2.5}$$

Combining (2.4) and (2.5) and using Definition 1.4 we get

$$\begin{aligned}T(r,F)+T(r,G)\le{}&\frac{1}{2}[N(r,1;F)+N(r,1;G)]+N_2(r,0;F)+N_2(r,0;G)\\&-\frac{1}{2}\overline{N}_*(r,1;F,G)+O(\log r)\\\le{}&\frac{1}{2}[T(r,F)+T(r,G)]+\{N_2(r,0;f^m(f^k+a))+N_2(r,0;g^m(g^k+a))\}\\&-\frac{1}{2}\overline{N}_*(r,1;F,G)+O(\log r).\end{aligned}$$

Therefore, by Lemma 2.4 we get

$$\begin{aligned}\frac{n}{2}[T(r,f)+T(r,g)]\le{}&\sum_{j=1}^{k}\{N_2(r,\omega_j;f)+N_2(r,\omega_j;g)\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\&-\frac{1}{2}\overline{N}_*(r,1;F,G)+O(\log r).\end{aligned}$$

This completes the proof.

□

Lemma 2.9. *Let* $F_1=\frac{-f^n}{af^m+b}$ *and* $G_1=\frac{-g^n}{ag^m+b}$*, where* f *and* g *are any two meromorphic functions having finitely many poles in* $\mathbb{C}$*,* n*,* m *are positive integers such that* $n>m\ge 1$ *and* a*,* b *are nonzero finite constants and* H

be defined in terms of F_1 *and* G_1*. Let* $\nu_1,\ \nu_2,\ \cdots,\nu_m$ *be the roots of the equation* $\omega^m+\frac{b}{a}=0$*. If* $H\not\equiv 0$ *and* F_1*,* G_1 *share* $(1,2)$*, then*

$$\frac{n}{2}[T(r,f)+T(r,g)]\leq\sum_{j=1}^{m}\{N_2(r,\nu_j;f)+N_2(r,\nu_j;g)\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}$$
$$-\frac{1}{2}\overline{N}_*(r,1;F_1,G_1)+S(r,f)+S(r,g).$$

Proof. The proof of the lemma is similar to that of Lemma 2.8.

□

The following two lemmas can be proved by using Lemma 2.1 (as and when required) and proceeding in a similar manner as in the proof of Theorems 6 and 5 (see [21]) respectively. Note that, the inequalities (3.5) and (3.6) in [21] are valid also when f and $\mathcal{L}$ share $(S,2)$ and the equation (3.10) in [21] is true also when $P(f)$ and $P(\mathcal{L})$ share 0 CM.

Lemma 2.10. *Let* $P(\omega)$*,* S*,* f*,* $\mathcal{L}$ *be defined as in Theorem 1.1, and let* n*,* m *be relatively prime positive integers with* $n\geq 5$ *and* $n>m$*. If* f *and* $\mathcal{L}$ *share* $(S,2)$ *and* c *IM, and* $P(f)$*,* $P(\mathcal{L})$ *share* 0 *CM, then* $f=\mathcal{L}$*.*

Lemma 2.11. *Let* $P(\omega)$*,* S*,* f*,* $\mathcal{L}$ *be defined as in Theorem 1.2, and let* n*,* m *be relatively prime positive integers with* $n>2m+4$*. If* f *and* $\mathcal{L}$ *share* $(S,2)$ *and* $P(f)$*,* $P(\mathcal{L})$ *share* 0 *CM, then* $f=\mathcal{L}$*.*

3 Proof of the Theorem

Proof of Theorem 1.1. Let us define $F=\frac{f^n+af^m}{-b}$ and $G=\frac{\mathcal{L}^n+a\mathcal{L}^m}{-b}$, where $k=n-m>0$. Since f and $\mathcal{L}$ share $(S,2)$, F and G share $(1,2)$. By Lemma 2.3, it is obvious that $S(r,f)=S(r,\mathcal{L})=O(\log r)$. Also since f has finitely many poles and possible poles of $\mathcal{L}$ may occur only at $s=1$, then we have $N(r,f)=O(\log r)$ and $N(r,\mathcal{L})=O(\log r)$. Let $\omega_1,\omega_2,\cdots,\omega_k$ be the roots of the equation $\omega^k+a=0$.
If possible, let us assume that $H\not\equiv 0$. Then by Lemma 2.8 we get

$$\frac{n}{2}[T(r,f)+T(r,\mathcal{L})]\leq\sum_{j=1}^{k}\{N_2(r,\omega_j;f)+N_2(r,\omega_j;\mathcal{L})\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L})\}$$
$$-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,\mathcal{L})$$
$$\leq(k+2)[T(r,f)+T(r,\mathcal{L})]+S(r,f)+S(r,\mathcal{L}).$$

Therefore,

$$\left\{\frac{n}{2}-(k+2)\right\}[T(r,f)+T(r,\mathcal{L})]\le S(r,f)+S(r,\mathcal{L}).$$

Thus $n\le 2k+4$, which contradicts our assumption that $n>2k+4$.

Hence $H\equiv 0$. That is

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}.$$

Integrating twice we get

$$\frac{1}{F-1}=\frac{k_1}{G-1}+k_2, \tag{3.1}$$

where $k_1(\neq 0)$ and k_2 are constants. From (3.1) it is clear that F and G share 1 CM. Therefore $P(f)$ and $P(\mathcal{L})$ share 0 CM.

Let F_1 and G_1 be defined as in Lemma 2.9. Since f and $\mathcal{L}$ share $(S,2)$, F_1 and G_1 share $(1,2)$. Therefore, if $\nu_1,\ \nu_2,\ \cdots,\nu_m$ be the roots of the equation $\omega^m+\frac{b}{a}=0$, then for H (involving F_1 and G_1) $\not\equiv 0$, using Lemma 2.9 we get,

$$\begin{aligned}\frac{n}{2}[T(r,f)+T(r,\mathcal{L})]\le&\sum_{j=1}^{m}\{N_2(r,\nu_j;f)+N_2(r,\nu_j;\mathcal{L})\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L})\}\\&-\frac{1}{2}\overline{N}_*(r,1;F_1,G_1)+S(r,f)+S(r,\mathcal{L})\\\le&(m+2)[T(r,f)+T(r,\mathcal{L})]+S(r,f)+S(r,\mathcal{L}).\end{aligned}$$

This is again a contradiction for $n>2m+4$. Thus, $H\equiv 0$. Then F_1 and G_1 share 1 CM and hence $P(f)$ and $P(\mathcal{L})$ share 0 CM. Also, since f and $\mathcal{L}$ share c IM, by Lemma 2.10, we have $f=\mathcal{L}$. This proves Theorem 1.1.

□

Proof of Theorem 1.2. Let F_1 and G_1 be defined as in Lemma 2.9. If H (involving F_1 and G_1) $\not\equiv 0$, then by similar reasoning as in Theorem 1.1 we will arrive at a contradiction for $n>2m+4$. Thus, we will have $H\equiv 0$. So F_1 and G_1 share 1 CM, i.e., $P(f)$ and $P(\mathcal{L})$ share 0 CM. Now by Lemma 2.11, we have $f=\mathcal{L}$. This proves Theorem 1.2.

□

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