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A new characterization of symplectics groups PSP$(8, q)$

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Abstract In this paper, we prove that the symplectics groups $PSP(8, q)$, where q be odd prime number can be uniquely determined by its order and the second largest elements order.

Keywords Elements order · Largest order of elements · Second largest elements order · Prime graph · Symplectics groups

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1 Introduction

All groups considered in this article are finite. In the way, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the set of order of elements of G is denoted by $\pi_e(G)$. Also, the largest order of elements and the second largest elements order of G is denoted by $k_1(G)$ and $k_2(G)$, respectively. Moreover, we denote a Sylow p-subgroup of G by G_p. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we always assume that $2 \in \pi_1$.
The discussion topic of this article is group characterization. There are different kinds of group characterization. For example, the group characterization by the set of the order of elements, prime graph, the set of the number of elements with the same order, etc. One of the methods, is group characterization by using order of the group and largest elements order or the second largest elements order. In other words, we say the group G is characterizable by using order of the group and the largest elements order or the second largest elements order if there is the group H, so that $k_i(G) = k_i(H)$, where $i = 1, 2$ and $|G| = |H|$, then $G \cong H$. In the way,

for example the researchers in ([7], [5], [8], [2], [10], [3], [6]) proved that the groups $PSL_2(q)$ with $q < 125$, sporadic simple groups, $PSL_3(q)$, the simple K_4-groups with type $L_2(p)$ and $PSU_3(q)$, for some q, $PGL_2(q)$, $PSU_3(3^n)$, where $3^{2n} - 3^n + 1$ is a prime number and the Suzuki groups $Sz(q)$, where $q - 1$ or $q \pm \sqrt{2q} + 1$ is prime number by using the order of the group and the largest elements order are characterizable. In this article, we prove that the symplectics groups $PSP(8, q)$, where q, a odd prime number can be uniquely determined by its order and the second largest elements order. In fact, we prove the following main theorem:

Main Theorem. Let G be a group with $|G| = |PSP(8, q)|$ and $k_2(G) = k_2(PSP(8, q))$, where q is an odd prime number. Then $G \cong PSP(8, q)$.

2 Notation and Preliminaries

In this section, we give some useful lemmas which will be used in the proof of the main theorem.

Definition 2.1 *[11] A group G is called a Frobenius group if there is a non-trivial proper subgroup H such that $H \cap H^x = 1$, for each $x \in G - H$. In the way, $N = G - \cup_{x \in G}(H - 1)^x$ be Frobenius kernel and H is Frobenius complement.*

Lemma 2.2 *[4] Let G be a Frobenius group of even order with kernel K and complement H. Then*

i *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
ii *$|H|$ divides $|K| - 1$;*
iii *K is nilpotent.*

Definition 2.3 *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.4 *[1] Let G be a 2-Frobenius group of even order. Then*

i *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
ii *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.5 *[12] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

i *G is a Frobenius group;*
ii *G is a 2-Frobenius group;*
iii *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$ and also the odd order components of G are the odd order components of K/H.*

3 Proof of the Main Theorem

In this section, we prove the main theorem. First, we know that the only components of the symplectics group $PSP(8,q)$ is $\frac{q^4+1}{2}$. From now on, we denote the symplectics groups $PSP(8,q)$ and $\frac{q^4+1}{2}$, by C and p respectively. We recall that G is a group with $|G| = |C| = \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$ and $k_2(G) = k_2(C) = \frac{q^4+1}{2}$. To prove the main theorem, we need to prove the following Lemmas.

Lemma 3.1 *p is an isolated vertex of $\Gamma(G)$.*

Proof. We prove that p is an isolated vertex of $\Gamma(G)$. In the contradiction, we assume that p is not an isolated vertex, so there is prime number $t \in \pi(G) - p$, so that $tp \in \pi_e(G)$. As a result, we have $tp \geq 2(\frac{q^4+1}{2}) > \frac{q^4+1}{2}$, as a result $k_2(G) > \frac{q^4+1}{2}$, which is a contradiction, hence $t(G) \geq 2$. So Lemma 2.5 implies that G satisfies one of the following Lemmas:

□

Lemma 3.2 *G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.2, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/p$ and $|K| = p$, or (ii) $|H| = p$ and $|K| = |G|/p$. Assume that $|H| = |G|/p$ and $|K| = p$. Then Lemma 2.2 implies that $|G|/p$ divides $p-1$, which is impossible. So the case $|H| = p$ and $|K| = |G|/p$ can be occurred. Lemma 2.2 implies that p divides $\frac{|G|}{p} - 1$, in the other words $\frac{q^4+1}{2} \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)-1}{q^4+1} - 1$. Therefore $q^4+1 \mid 2q^{32} - 2q^{30} - 4q^{28} + 2q^{26} + 4q^{24} + 2q^{22} - 4q^{20} - 2q^{18} + 2q^{16} - 2$. As a result $q^4+1 \mid (q^4+1)(2q^{28} - 2q^{26} - 6q^{24} + 4q^{22} + 10q^{20} - 2q^{18} - 14q^{16} + 16q^{12} - 16q^8 + 16q^4 - 16) + 14$. So q^4+1 must be divided 14, that is a contradiction. Therefore, we deduce that G is not a Frobenius group.

□

Lemma 3.3 *G is not a 2-Frobenius group.*

Proof. Let G be a 2-Frobenius group. Then by Lemma 2.4, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|Aut(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{p\}$ and $|K/H| = p$. Hence $|Aut(K/H)| =$

$p-1$. As $|G/K|$ divides $|Aut(K/H)|$, we deduce that $|G/K||\frac{q^4-1}{2}$. Now since $(\frac{q^4-1}{2}, \frac{q^4+1}{2}) = 1$, where t be prime divisor of $\frac{q^4+1}{2}$ so $t \mid |H|$. Now $K/H \rtimes H_t$ be a Frobenius group by kernel H_t, so $\frac{q^4+1}{2} \mid |H_t|-1$. As a result $\frac{q^4+1}{2} \mid \frac{q^4-1}{2}$, which is a contradiction. So G is not 2-Frobenius group.

□

Lemma 3.4 *G is isomorphic to C.*

Proof. By Lemma 3.2 and Lemma 3.3, then we have that case 3 of lemma 2.5 is satisfied, as G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, so $p \in \pi(K/H)$ and $t(K/H) \geq 2$. As a result, according to the classification of the finite simple groups we know that the possibilities for K/H are alternating group A_m, $m \geq 5$, 26 sporadic groups, the simple groups Lie type. First we have the following isomorphic:
Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = r'$,$r'+1$, $r'+2$. Then by [12] $\pi(A_m) = r', r'-2$. So, we consider $\frac{q^4+1}{2} = r'$, then $r'+1 = \frac{q^4+3}{2}$. Now, we know that $r'+1 \mid |A_m| \mid |G|$, but $\frac{q^4+1}{2} \nmid |G|$, which this is a contradiction. Now, if $\frac{q^4+1}{2} = r'-2$, then $r' = \frac{q^4+5}{2}$. On the other hand, we know that $r' \mid |A_m| \mid |G|$, but $\frac{q^4+5}{2} \nmid |G|$, which is a contradiction.
Step 2. If K/H be isomorphic sporadic groups, then by [9], $k_2((S) = \{8, 10, 12, 15, 21, 24, 28, 42, 45, 35, 36, 66, 28, 17, 26, 44, 110\}$, where S is the sporadic simple groups. Now, we consider $\frac{q^4+1}{2} = 8$, then $q^4 - 15 = 0$, which is a contradiction. For the other value, we have a contradiction.
Step 3. In this case, we consider K/H is isomorphic to a group of Lie-type.
3.1. If $K/H \cong^3 D_4(q')$, then by [9], $k_2({}^3D_4(q')) = q'(q'^3+1)$ and $q'^4-q'^2+1$. Also we know $|{}^3D_4(q')| \mid |G|$. Now, we consider $\frac{q^4+1}{2} = q'(q'^3+1)$, then $q^4 = 2q'^4 + 2q' - 1$, we know that $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by considering $q^4 = 2q'^4 + 2q' - 1$, we can see easily $|{}^3D_4(q')| \nmid |G|$, where is a contradiction. Now if $\frac{q^4+1}{2} = q'^4-q'^2+1$, then we deduce $q^4 = 2q'^4 - 2q'^2 + 1$. Since $|{}^3D_4(q')| \nmid |G|$, where is a contradiction.
3.2. $K/H \ncong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For this purpose, first we consider the following items.
(i) If $K/H \cong E_6(q')$, then by [9] $k_2(E_6(q') = \frac{(q'+1)(q'^5-1)}{(3,q'-1)}$ and $\frac{(q'^2+q'+1)(q'^4-q'^2+1)}{(3,q'-1)}$. On the other hand, $|E_6(q')| = q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1) \mid |G|$. Hence, we consider $\frac{q^4+1}{2} = \frac{(q'+1)(q'^5-1)}{(3,q'-1)}$. First, if $(3, q'-1) = 1$, then $\frac{q^4+1}{2} = (q'+1)(q'^5-1)$. It follows that $q^4 = 2q'^6+2q'^5-2q'-3$. Now, by note to $q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$,

so by assumption $q^4 = 2q'^6 + 2q'^5 - 2q' - 3$, we can see easily $|E_6(q')| \nmid |G|$, where is a contradiction. Now if $(3, q' - 1) = 3$, then $\frac{q^4+1}{2} = \frac{(q'+1)(q'^5-1)}{3}$, similarly $|E_6(q')| \nmid |G|$, where is a contradiction.

(ii) If $K/H \cong F_4(q')$, then by [9] $k_2(F_4(q') = (q'^3 - 1)(q' + 1)$, $q'^4 + 1$. First we assume $\frac{q^4+1}{2} = q'^4 + 1$, then we deduce $q^4 = 2q'^4 + 1$. On the other hand, $q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, since $q^4 = 2q'^4 + 1$, we deduce $|F_4(q')| \nmid |G|$, where is a contradiction. Another case is a contradiction, similarly.

For $K/H \ncong E_7(q')$, $E_8(q')$, we have a contradiction, similarly.

3.3. If $K/H \cong^2 E_6(q')$, then by [9] $k_2({}^2E_6(q') = \frac{q'(q'^5+1)}{(3,q'+1)}$ and $\frac{q'^6-1}{(3,q'+1)}$ and also $|{}^2E_6(q')| = \frac{q'^{36}(q'^{12}-1)(q'^9+1)(q'^8-1)(q'^6-1)(q'^5+1)(q'^2-1)}{(3,q'+1)} \mid |G|$. First if $(3, q'+1) = 1$, then we consider $\frac{q^4+1}{2} = q'(q'^5+1)$. Hence $q^4 = 2q'^6+2q'-1$. On the other hand, we know that, $q'^{36}(q'^{12} - 1)(q'^9 + 1)(q'^8 - 1)(q'^6 - 1)(q'^5 + 1)(q'^2 - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by the assumption $q^4 = 2q'^6 + 2q' - 1$, we see that $|{}^2E_6(q')| \nmid |G|$, where is a contradiction. Similarly, if $(3, q' + 1) = 3$, then we get a contradiction.

3.4. If $K/H \cong^2 G_2(q')$, where $m \geq 1$, $q' = 3^{2m+1}$, then by [9], $k_2({}^2G_2(q'))) = q' - 1$. Now, we consider $\frac{q^4+1}{2} = q' - 1$, so $q^4 = 2q' - 2$. On the other hand, we know $q'^3(q'^3 + 1)(q' - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by considering $q^4 = 2q' - 2$, we see $|{}^2G_2(q')| \nmid |G|$, where is a contradiction.

3.5. If $K/H \cong^2 F_4(q')$, where $q' = 2^{2m+1} > 2$ then, by [9], $k_2({}^2F_4(q') = 2^{4m+2} - 2^{3m+2} + 2^{2m+1} - 2^{m+1} + 1$. So, we consider $\frac{q^4+1}{2} = 2^{4m+2} - 2^{3m+2} + 2^{2m+1} - 2^{m+1} + 1$. As a result $q^4 = 2^{4m+3} - 2^{3m+3} + 2^{2m+2} - 2^{m+2} + 1$. On the other hand, we know $q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by considering $q^4 = 2^{4m+3}-2^{3m+3}+2^{2m+2}-2^{m+2} + 1$, we can see easily $|{}^2F_4(q')| \nmid |G|$, where is a contradiction.

3.6. If $K/H \cong^2 B_2(q')$, where $q' = 2^{2m+1}$, $m \geq 1$ then by [9], $k_2({}^2B_2(q')) = q' - 1$, also $|{}^2B_2(q')| \mid |G|$. For this purpose, we consider $\frac{q^4+1}{2} = q' - 1$. As a result $q^4 = 2q' - 3$. On the other hand, we know $q'^2(q'^2 + 1)(q' - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by the assumption $q^4 = 2q' - 3$, we deduce $|{}^2B_2(q')| \nmid |G|$, where is a contradiction.

3.7. If $K/H \cong G_2(q')$, then by [9], $k_2(G_2(q')) = q'^2 + q'$ and $q'^2 - 1$ and also $|G_2(q')| = q'^6(q'^6 - 1)(q'^2 - 1) \mid |G|$. For this purpose, we consider $\frac{q^4+1}{2} = q'^2 + q'$. As a result, $q^4 = 2q'^2 + 2q' - 1$. We know that $q'^6(q'^6 - 1)(q'^2 - 1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now, by the attention to $q^4 = 2q'^2 + 2q' - 1$, we see $|G_2(q')| \nmid |G|$, which is a contradiction. Now if $\frac{q^4+1}{2} = q'^2 - 1$, then we deduce $q^4 = 2q'^2 + 3$, similarly $|G_2(q')| \nmid |G|$, which is a contradiction.

3.8. If $K/H \cong^2 A_n(q')$, where $n > 1$. Then by [9], $k_2({}^2A_n(q')) = \frac{q'^{2n}-1}{(n+1,q'+1)}$,

also we know that $|^2A_n(q')| \mid |G|$. Hence $\frac{1}{(n+1,q'+1)}q'^{n(n+1)/2}\prod_{i=2}^{n+1}(q'^i - (-1)^i \mid |G|$. For this purpose, we consider $\frac{q^4+1}{2} = \frac{q'^{2n}-1}{(n+1,q'+1)} < q'^{2n} - 1$. As a result, $q^4 < 2q'^{2n} - 3$. We know that $q'^{2n} \mid |G|$, but $\frac{q^4+3}{2} \nmid |G|$, which is a contradiction.

3.9. If $K/H \cong B_n(q')$, where $n \geq 4$, then by [9], $k_2(B_n(q') = q'^n - q'$. Now, we consider $\frac{q^4+1}{2} = q'^n - q'$, so $q^4 = 2q'^n - 2q' - 1$. On the other hand, we know $\frac{q'^{n^2}\prod_{i=1}^{n}(q'^{2i}-1)}{(2,q'-1)} \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now since that $q^4 = 2q'^n - 2q' - 1$, so we see $|B_n(q')| \nmid |G|$, where is a contradiction. Similarly, for $K/H \ncong D_n(q')$, we have the proof, similarly.

3.10. If $K/H \cong L_{n+1}(q')$, where $n \geq 1$. Then by [9], $k_2(L_{n+1}(q')) = \frac{q'^{n+1}-1}{(n+1,q'-1)}$. Now, we consider $\frac{q^4+1}{2} = \frac{q'^{n+1}-1}{(n+1,q'-1)} < q'^{n+1} - 1$. As a result, $q^4 < 2q'^{n+1} - 3$. On the other hand, $q'^{n+1} \mid |G|$, but $\frac{q^4+3}{2} \nmid |G|$, we get a contradiction. Similarly, for $n > 1$, $K/H \ncong L_{n+1}(q')$.

3.11. Hence $K/H \cong C$. It follows that $|K/H| = |C|$. On the other hand, we know that $H \trianglelefteq K \trianglelefteq G$, where p is an isolated vertex of $\Gamma(G)$. Also, by the assumption, we have $k_2(K/H) \mid k_2(G)$. Hence $\frac{q^4+1}{2} = \frac{q'^4+1}{2}$. As a result $q = q'$. Now, since $|K/H| = |C|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong C$.

□

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