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On dual Horadam quaternions

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Abstract In this paper, we define the dual Horadam quaternions and investigate some of their properties, we describe the relations between dual Horadam quaternions and the other dual quaternios (dual Fibonacci, dual Lucas, dual Pell, dual Pell-Lucas, dual Jacobsthal and dual Jacobsthal-Lucas quaternions). Also, we give Binet's formula, generating function and Catalan's identity of dual Horadam quaternions.

Keywords Quaternions · Generating functions · Fibonacci and Lucas numbers and generalizations

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1 Introduction

The quaternions were defined by W. R. Hamilton in 1843. Quaternions are used in many fields such as computer graphics, computer vision, animation and eurospace. The quaternions are a number system which extends to the complex numbers. Any f quaternion number consists of four real coefficients and the set of all quaternions is defined by

$$Q = \{f = f_0 + f_1 i + f_2 j + f_3 k | f_0, f_1, f_2, f_3 \in \mathbb{R}\} \tag{1.1}$$

where i, j and k are the standard orthonormal basis in $\mathbb{R}^3$. The quaternion multiplication is defined by the following rules;

$$i^2 = j^2 = k^2 = -1$$
$$ij = -jk = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

This demonstrates that the multiplication of quaternions is not commutative. If two quaternions are $f = f_0 + f_1 i + f_2 j + f_3 k$ and $f^{'} = f_0^{'} + f_1^{'} i + f_2^{'} j + f_3^{'} k$, then the addition and subtraction of them is

$$f \pm f^{'} = (f_0 \pm f_0^{'}) + (f_1 \pm f_1^{'}) i + (f_2 \pm f_2^{'}) j + (f_3 \pm f_3^{'}) k$$

and the multiplication is

$$ff' = f_0f_0' - (f_1f_1' + f_2f_2' + f_3f_3') + i(f_0f_1' + f_1f_0' + f_2f_3' - f_3f_2') + j(f_0f_2' + f_2f_0' + f_3f_1' - f_1f_3') + k(f_0f_3' + f_3f_0' + f_1f_2' - f_2f_1')$$

The conjugate of the quaternion f is denoted by $\overline{f}$ and defined as

$$\overline{f} = f_0 - f_1i - f_2j - f_3k$$

Clifford extented the real numbers to the dual numbers in 1873. A dual number d is the form

$$d = a + \varepsilon a^* \tag{1.2}$$

where $a, a^* \in \mathbb{R}$, ε is the dual unit, $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. Dual numbers form two dimensional commutative algebra over the real numbers. Let $\widetilde{Q}$ be the set of dual quaternions. A dual quaternion $\widetilde{q}$ can be defined in a similar way to the dual numbers, as follows

$$\widetilde{q} = q + \varepsilon q^* \tag{1.3}$$

where q and q^* are quaternions and ε is the dual unit. If $q = a_0 + a_1i + a_2j + a_3k$ and $q^* = a_0' + a_1'i + a_2'j + a_3'k$ then the dual quaternion $\widetilde{q}$ can be written as

$$\widetilde{q} = q + \varepsilon q^* = (a_0 + \varepsilon a_0') + (a_1 + \varepsilon a_1')i + (a_2 + \varepsilon a_2')j + (a_3 + \varepsilon a_3')k$$

Now, the dual quaternion $\widetilde{q}$ consist of a scaler part and a vector part given by

$$S_{\widetilde{q}} = a_0 + \varepsilon a_0' = S_q + \varepsilon S_{q^*}$$

$$V_{\widetilde{q}} = (a_1 + \varepsilon a_1')i + (a_2 + \varepsilon a_2')j + (a_3 + \varepsilon a_3')k = V_q + \varepsilon V_q^*$$

respectively.

If $\widetilde{q} = q + \varepsilon q^*$ and $\widetilde{p} = p + \varepsilon p^*$ are two dual quaternions then the addition and subtraction is given by

$$\widetilde{p} \pm \widetilde{q} = p \pm q + \varepsilon(p^* \pm q^*)$$

and multiplication is given by

$$\widetilde{p}\widetilde{q} = pq + \varepsilon(pq^* + p^*q)$$

where $p = a_0 + a_1 i + a_2 j + a_3 k$, $p^* = a_0{}' + a_1{}' i + a_2{}' j + a_3{}' k$, $q = b_0 + b_1 i + b_2 j + b_3 k$ and $q^* = b_0{}' + b_1{}' i + b_2{}' j + b_3{}' k$.
The conjugate of $\widetilde{p} = p + \varepsilon p^*$ is defined by

$$\begin{aligned}\overline{\widetilde{p}} &= \overline{p} + \varepsilon\overline{p^*} \\ &= a_0 - a_1 i - a_2 j - a_3 k + \varepsilon(a_0{}' - a_1{}' i - a_2{}' j - a_3{}' k) \\ &= (a_0 + \varepsilon a_0{}') - i(a_1 + \varepsilon a_1{}') - j(a_2 + \varepsilon a_2{}') - k(a_3 + \varepsilon a_3{}')\end{aligned}$$

In [5] Nurkan and Güven defined the dual Fibonacci quaternion and the dual Lucas quaternion and they investigated some of their properties. Szynal-Liana and Włoch in [7] introduced Jacobsthal quaternions and Jacobsthal-Lucas quaternions and they described their properties. Horadam [3] studied the Fibonacci and Lucas quaternions. In [6], Swamy gave the relations of generalized Fibonacci quaternions. In 2016, Yüce and Aydin [8] studied the generalized dual Fibonacci quaternions. Also in [9], Yüce and Aydin redefined the dual Fibonacci quaternions, they investigated the relations between the dual Fibonacci and the dual Lucas quaternions.
In this paper, we introduce dual Horadam quaternions and prove some of their properties. Also, we give Binet's formula, generating function, Catalan identity, Cassini identity and d'Ocagne identity of dual Horadam quaternions.

2 Horadam Sequence

In [4] Horadam introduced the following sequence

$$\{w_n\} = \{w_n(a, b; p, q) : w_n = pw_{n-1} + qw_{n-2}, w_0 = a, w_1 = b, n \geq 2\} \quad (2.1)$$

where w_n is called as the nth Horadam number.
Let λ and γ denote the roots of the characteristic equation $x^2 - px - q = 0$ of the recurrence relation (2.1). Then $\lambda = \frac{p+\sqrt{p^2+4q}}{2}$ and $\gamma = \frac{p-\sqrt{p^2+4q}}{2}$
Note that:

$$\lambda + \gamma = p, \quad \lambda\gamma = -q \quad \text{and} \quad \lambda - \gamma = \sqrt{p^2 + 4q}. \quad (2.2)$$

The Binet formula for Horadam sequence is given by

$$w_n = \frac{N\lambda^n - K\gamma^n}{\lambda - \gamma} \quad \text{where} \quad N = b - a\gamma \quad \text{and} \quad K = b - a\lambda. \quad (2.3)$$

By the recurrence relation and Binet formula for Horadam sequence, we have

$$w_n = \frac{1}{\lambda - \gamma}\Big\{b(\lambda^n - \gamma^n) + aq(\lambda^{n-1} - \gamma^{n-1})\Big\}$$

which is another Binet formula for Horadam sequence.

Remark 2.1 Consider Eq (2.1);
• If $p = 1$, $q = 1$, $a = 0$ and $b = 1$, the Fibonacci sequence is obtained,
• If $p = 1$, $q = 1$, $a = 2$ and $b = 1$, the Lucas sequence is obtained,
• If $p = 2$, $q = 1$, $a = 0$ and $b = 1$, the Pell sequence is obtained,
• If $p = 1$, $q = 2$, $a = 0$ and $b = 1$, the Jacobsthal sequence is obtained,
• If $p = 1$, $q = 2$, $a = 2$ and $b = 1$, the Jacobsthal-Lucas sequence is obtained,
• If $p = 2$, $q = 1$, $a = 2$ and $b = 2$, the Pell-Lucas sequence is obtained.

Hence, the Horadam sequence is the generalization of sequences above.

3 Dual Horadam Quaternions

In [3], Horadam defined complex Fibonacci numbers as follows

$$C_n = F_n + iF_{n+1}, \quad i^2 = -1$$

where F_n is the nth Fibonacci number. Halici, in [1], introduced the nth complex Fibonacci quaternion by

$$R_n = \mathfrak{F}_{\mathfrak{n}} + i\mathfrak{F}_{\mathfrak{n}+\mathfrak{1}}, \quad i^2 = -1$$

where $\mathfrak{F}_{\mathfrak{n}} = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$ is the nth complex Fibonacci quaternion. The nth dual Horadam number is defined by

$$\widetilde{w_n} = w_n + \varepsilon w_{n+1}$$

where ϵ is the dual unit and $\varepsilon \neq 0$, $\varepsilon^2 = 0$. Here w_n is the nth Horadam number.
The nth dual Fibonacci quaternion is defined by

$$\widetilde{\mathfrak{F}_{\mathfrak{n}}} = \mathfrak{F}_{\mathfrak{n}} + \varepsilon\mathfrak{F}_{\mathfrak{n}+\mathfrak{1}}$$

Here $\mathfrak{F}_{\mathfrak{n}} = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$ is the nth Fibonacci quaternion, i, j, k are standard orthonormal basis in $\mathbb{R}^3$ which satisfy the following properties

$$i^2 = j^2 = k^2 = -1$$
$$ij = k = -jk, \quad jk = i = -kj, \quad ki = j = -ik.$$

With the same idea we can define dual Horadam quaternions; The nth dual Horadam quaternion is defined as

$$\widetilde{H_n} = H_n + \varepsilon H_{n+1} \tag{3.1}$$

Here,

$$H_n = w_n + iw_{n+1} + jw_{n+2} + kw_{n+3} \tag{3.2}$$

is the *nth* Horadam quaternion, w_n is the *nth* Horadam number i, j, k are standard orthonormal basis in $\mathbb{R}^3$ which have the following properties

$$i^2 = j^2 = k^2 = -1$$
$$ij = k = -ji, \quad jk = i = -kj =, \quad ki = j = -ik.$$

The dual Horadam quaternion $\widetilde{H_n}$ can be represented as

$$\widetilde{H_n} = (w_n + \varepsilon w_{n+1}) + (w_{n+1} + \varepsilon w_{n+2})i + (w_{n+2} + \varepsilon w_{n+3})j + k(w_{n+3} + \varepsilon w_{n+4}). \tag{3.3}$$

By using dual Horadam numbers, we have

$$\widetilde{H_n} = \widetilde{w_n} + i\widetilde{w_{n+1}} + j\widetilde{w_{n+2}} + k\widetilde{w_{n+3}} \tag{3.4}$$

The scaler and the vector part of the dual Horadam quaternion $\widetilde{H_n}$ are given by

$$S_{\widetilde{H_n}} = w_n + \varepsilon w_{n+1},$$

$$V_{\widetilde{H_n}} = (w_{n+1} + \varepsilon w_{n+2})i + (w_{n+2} + \varepsilon w_{n+3})j + (w_{n+3} + \varepsilon w_{n+4})k$$

respectively. Thus, the dual Horadam quaternion $\widetilde{H_n}$ is given by $\widetilde{H_n} = S_{\widetilde{H_n}} + V_{\widetilde{H_n}}$.

Let $\widetilde{H_n} = H_n + \varepsilon H_{n+1}$ and $\widetilde{R_n} = R_n + \varepsilon R_{n+1}$ be two dual Horadam quaternions. Then the addition and subtraction of them is defined by

$$\widetilde{H_n} \pm \widetilde{R_n} = (H_n \pm R_n) + \varepsilon(H_{n+1} \pm R_{n+1}), \tag{3.5}$$

the multiplication of them is given by

$$\widetilde{H_n}\widetilde{R_n} = (H_n R_n) + \varepsilon(H_n R_{n+1} + H_{n+1} R_n) \tag{3.6}$$

where $H_n = w_n + iw_{n+1} + jw_{n+2} + kw_{n+3}$, $H_{n+1} = w_{n+1} + iw_{n+2} + jw_{n+3} + kw_{n+4}$, $R_n = w_n{}' + iw_{n+1}{}' + jw_{n+2}{}' + kw_{n+3}{}'$ and $R_{n+1} = w_{n+1}{}' + iw_{n+2}{}' + jw_{n+3}{}' + kw_{n+4}{}'$ are Horadam quaternions.

The conjugate of the dual Horadam quaternion $\widetilde{H_n}$ is given by

$$\begin{aligned}\overline{\widetilde{H_n}} &= \widetilde{w_n} - i\widetilde{w_{n+1}} - j\widetilde{w_{n+2}} - k\widetilde{w_{n+3}} \\ &= (w_n + \varepsilon w_{n+1}) - i(w_{n+1} + \varepsilon w_{n+2}) - j(w_{n+2} + \varepsilon w_{n+3}) \\ &\quad - k(w_{n+3} + \varepsilon w_{n+4}) \\ &= S_{\widetilde{H_n}} - V_{\widetilde{H_n}}.\end{aligned}$$

The norm of $\widetilde{H_n}$ is given by

$$\begin{aligned}N_{\widetilde{H_n}} = \widetilde{H_n}\overline{\widetilde{H_n}} &= (w_n + \varepsilon w_{n+1})^2 + (w_{n+1} + \varepsilon w_{n+2})^2 + (w_{n+2} + \varepsilon w_{n+3})^2 \\ &\quad + (w_{n+3} + \varepsilon w_{n+4})^2.\end{aligned}$$

Theorem 3.1 *Let $\widetilde{H_n}$ be a dual Horadam quaternion. For $n \geq 1$ the following relations hold*

(i) $\widetilde{H_{n+2}} = p\widetilde{H_{n+1}} + q\widetilde{H_n}$,
(ii) $\widetilde{H_n} + \overline{\widetilde{H_n}} = 2\widetilde{w_n}$,
(iii) $\widetilde{w_n} - i\widetilde{w_{n+1}} - j\widetilde{w_{n+2}} - k\widetilde{w_{n+3}} = \overline{H_n} + \varepsilon\overline{H_{n+1}} = \overline{\widetilde{H_n}}$,
(iv) $\widetilde{H_n}\overline{\widetilde{H_n}} = H_n\overline{H_n} + 2\varepsilon\sum_{l=0}^{3} w_{n+l}w_{n+1+l}$.

Proof. (i) By using the equation (3.1), we have

$$\begin{aligned} p\widetilde{H_{n+1}} + q\widetilde{H_n} &= p(H_{n+1} + \varepsilon H_{n+2}) + q(H_n + \varepsilon H_{n+1}) \\ &= pH_{n+1} + qH_n + \varepsilon(pH_{n+2} + qH_{n+1}) \end{aligned}$$

From the Eqs. (3.2) and (2.1), we obtain

$$\begin{aligned} &pH_{n+1} + qH_n + \varepsilon(pH_{n+2} + qH_{n+1}) \\ &= pw_{n+1} + pw_{n+2}i + pw_{n+3}j + pw_{n+4}k + qw_n + qw_{n+1}i + qw_{n+2}j \\ &+ qw_{n+3}k + \varepsilon\big[pw_{n+2} + pw_{n+3}i + pw_{n+4}j + pw_{n+5}k + qw_{n+1} + qw_{n+2}i \\ &+ qw_{n+3}j + qw_{n+4}k\big] \\ &= (pw_{n+1} + qw_n) + (pw_{n+2} + qw_{n+1})i + (pw_{n+3} + qw_{n+2})j \\ &+ (pw_{n+4} + qw_{n+3})k + \varepsilon[pw_{n+2} + qw_{n+1} + (pw_{n+3} + qw_{n+2})i \\ &+ (pw_{n+4} + qw_{n+3})j + (pw_{n+5} + qw_{n+4})k] \\ &= w_{n+2} + w_{n+3}i + w_{n+4}j + w_{n+5}k + \varepsilon(w_{n+3} + w_{n+4}i + w_{n+5}j + w_{n+6}k) \\ &= H_{n+2} + \varepsilon H_{n+3} \\ &= \widetilde{H_{n+2}}. \end{aligned}$$

(ii) From the definition of conjugate of dual Horadam quaternion and Eq. (3.4), we obtain the result.
(iii) By the definitions of dual Horadam numbers and dual Horadam quaternions, we have the result.
(iv) By using the definition of dual Horadam quaternions and the definition of conjugate of dual Horadam quaternions, we get the result.

□

An easy computation gives the proof of the following proposition.

Proposition 3.2 *Let $e_1 = i$, $e_2 = j$ and $e_3 = k$. For $u \geq 0$, the following identities hold:*

$$(i) \qquad \widetilde{H_u}^2 = 2{w_u}^2 - H_u\overline{H_u} + 2\big\{\varepsilon(w_u w_{u+1} - \sum_{l=1}^{3} w_{u+l}w_{u+1+l}) + \sum_{l=1}^{3}[w_u\widetilde{w_{u+l}} + w_{u+1}w_{u+l}\varepsilon]e_l\big\},$$

$$(ii) \qquad \overline{\widetilde{H_u}}^2 = 2{w_u}^2 - H_u\overline{H_u} + 2\big\{\varepsilon(w_u w_{u+1} - \sum_{l=1}^{3} w_{u+l}w_{u+1+l}) - \sum_{l=1}^{3}[w_u\widetilde{w_{u+l}} + w_{u+1}w_{u+l}\varepsilon]e_l\big\},$$

$$(iii) \qquad \widetilde{H_u}^2 + \overline{\widetilde{H_u}}^2 = 4{w_u}^2 - 2H_u\overline{H_u} + 4w_u w_{u+1}\varepsilon - 4\varepsilon\sum_{l=1}^{3} w_{u+l}w_{u+1+l},$$

$$(iv) \qquad \widetilde{H_u}^2 - \overline{\widetilde{H_u}}^2 = 4\sum_{l=1}^{3}[w_u\widetilde{w_{u+l}} + w_{u+1}w_{u+l}\varepsilon]e_l.$$

Throughout this paper $\widetilde{\mathfrak{F}_{\mathrm{m}}}$, $\widetilde{\mathfrak{L}_{\mathrm{m}}}$, $\widetilde{\mathfrak{J}_{\mathrm{m}}}$, $\widetilde{\mathfrak{j}_{\mathrm{m}}}$, $\widetilde{\mathfrak{P}_{\mathrm{m}}}$ and $\widetilde{\mathfrak{p}_{\mathrm{m}}}$ denote dual Fibonacci, dual Lucas, dual Jacosthal, dual Jacobsthal-Lucas, dual Pell and dual Pell-Lucas quaternions, respectively.

Now, we will express the following theorem which is about Binet formula;

Theorem 3.3 *Let $\widetilde{H_n}$ be a dual Horadam quaternion. For $n \geq 0$, the Binet formula for this quaternion is given by*

$$\widetilde{H_n} = \frac{1}{\lambda - \gamma}[N\underline{\lambda}\lambda^n(1 + \lambda\varepsilon) - K\underline{\gamma}\gamma^n(1 + \gamma\varepsilon)] \tag{3.7}$$

where $\underline{\lambda} = 1 + \lambda i + \lambda^2 j + \lambda^3 k$, $\underline{\gamma} = 1 + \gamma i + \gamma^2 j + \gamma^3 k$, $N = b - a\gamma$ and $K = b - a\lambda$

Proof. In [2], Halici gave the Binet formula for Horadam quaternion by

$$H_n = \frac{1}{\lambda - \gamma}[N\underline{\lambda}\lambda^n - K\underline{\gamma}\gamma^n] \tag{3.8}$$

where $\underline{\lambda} = 1 + \lambda i + \lambda^2 j + \lambda^3 k$, $\underline{\gamma} = 1 + \gamma i + \gamma^2 j + \gamma^3 k$, $N = b - a\gamma$ and $K = b - a\lambda$.
So it can be written

$$H_{n+1} = \frac{1}{\lambda - \gamma}[N\underline{\lambda}\lambda^{n+1} - K\underline{\gamma}\gamma^{n+1}] \tag{3.9}$$

Thus, by using the Eqs. (3.1), (3.8) and (3.9), we obtain

$$\begin{aligned}\widetilde{H_n} &= H_n + \varepsilon H_{n+1} \\ &= \frac{1}{\lambda-\gamma}[N\underline{\lambda}\lambda^n - K\underline{\gamma}\gamma^n] + \frac{\varepsilon}{\lambda-\gamma}[N\underline{\lambda}\lambda^{n+1} - K\underline{\gamma}\gamma^{n+1}] \\ &= \frac{1}{\lambda-\gamma}[N\underline{\lambda}\lambda^n(1+\lambda\varepsilon) - K\underline{\gamma}\gamma^n(1+\gamma\varepsilon)].\end{aligned}$$

□

In the Eq. (3.7), if we put $a = 0, b = 1, p = 1$ and $q = 1$, we obtain the Binet formula for dual Fibonacci quaternion as follows:

$$\widetilde{\mathfrak{F}_{\mathfrak{n}}} = \frac{1}{\lambda-\gamma}[\underline{\lambda}\lambda^n(1+\lambda\varepsilon) - \underline{\gamma}\gamma^n(1+\gamma\varepsilon)]$$

which is given in [5]. Here $\lambda = \frac{1+\sqrt{5}}{2}$ and $\gamma = \frac{1-\sqrt{5}}{2}$.

In the Binet formula for dual Horadam quaternion, if we take $a = 2, b = 1, p = 1$ and $q = 1$, we get the Binet formula for dual Lucas quaternion $\widetilde{\mathfrak{L}_{\mathfrak{n}}} = \underline{\lambda}\lambda^n(1+\lambda\varepsilon) + \underline{\gamma}\gamma^n(1+\gamma\varepsilon)$ which is given in [5]. Here $\lambda = \frac{1+\sqrt{5}}{2}$ and $\gamma = \frac{1-\sqrt{5}}{2}$.

In the following table we give the Binet formula for dual Fibonacci, dual Lucas, dual Jacobsthal, dual Jacobsthal-Lucas, dual Pell and dual Pell-Lucas quaternions.

Remark 3.1 The results of Theorem 3.3 are given in the following table (Table 1):

Table 1 Binet formulas for dual quaternions.

Du. Qu	$w_m(a,b;p,q)$	N	K	λ	γ	Binet Formulas
Du. Fi	$w_m(0,1;1,1)$	1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\widetilde{\mathfrak{F}_{\mathrm{m}}} = \frac{\underline{\lambda}\lambda^m(1+\lambda\varepsilon)-\underline{\gamma}\gamma^m(1+\gamma\varepsilon)}{\sqrt{5}}$
Du. Lu	$w_m(2,1;1,1)$	$\sqrt{5}$	$-\sqrt{5}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\widetilde{\mathfrak{L}_{\mathrm{m}}} = \underline{\lambda}\lambda^m(1+\lambda\varepsilon) + \underline{\gamma}\gamma^m(1+\gamma\varepsilon)$
Du. Jac	$w_m(0,1;1,2)$	1	1	2	-1	$\widetilde{\mathfrak{J}_{\mathrm{m}}} = \frac{\underline{\lambda}\lambda^m(1+\lambda\varepsilon)-\underline{\gamma}\gamma^m(1+\gamma\varepsilon)}{3}$
Du J L	$w_m(2,1;1,2)$	3	-3	2	-1	$\widetilde{\mathfrak{j}_{\mathrm{m}}} = \underline{\lambda}\lambda^m(1+\lambda\varepsilon) + \underline{\gamma}\gamma^m(1+\gamma\varepsilon)$
Du Pe	$w_m(0,1;2,1)$	1	1	$1+\sqrt{2}$	$1-\sqrt{2}$	$\widetilde{\mathfrak{P}_{\mathrm{m}}} = \frac{\underline{\lambda}\lambda^m(1+\lambda\varepsilon)-\underline{\gamma}\gamma^m(1+\gamma\varepsilon)}{2\sqrt{2}}$
Du P L	$w_m(2,2;2,1)$	$2\sqrt{2}$	$-2\sqrt{2}$	$1+\sqrt{2}$	$1-\sqrt{2}$	$\widetilde{\mathfrak{p}_{\mathrm{m}}} = \underline{\lambda}\lambda^m(1+\lambda\varepsilon) + \underline{\gamma}\gamma^m(1+\gamma\varepsilon)$

Theorem 3.4 *The generating function of dual Horadam quaternion is*

$$h(t) = \frac{\widetilde{H_0} + (\widetilde{H_1} - p\widetilde{H_0})t}{1 - pt - qt^2}. \tag{3.10}$$

Proof. The generating function for the dual Horadam quaternion is

$$h(t) = \sum_{n=0}^{\infty} \widetilde{H_n} t^n \tag{3.11}$$

Now, we multiply both sides of Eq. (3.11) with $-pt$ and $-qt^2$ respectively. Then, we obtain
$(1 - pt - pt^2)h(t) = \widetilde{H_0} + (\widetilde{H_1} - p\widetilde{H_0})t + (\widetilde{H_2} - p\widetilde{H_1} - q\widetilde{H_0})t^2 + (\widetilde{H_3} - p\widetilde{H_2} - q\widetilde{H_1})t^3 + ...$
Employing Theorem 3.1 (i), we get $h(t) = \frac{\widetilde{H_0} + (\widetilde{H_1} - p\widetilde{H_0})t}{1 - pt - qt^2}$.

□

Theorem 3.5 *Let $\widetilde{H_n}$ be the nth term of the dual Horadam quaternion sequence $(\widetilde{H_n})$, for $n \geq 1$ the Cassini identity for $\widetilde{H_n}$ is given as*

$$\widetilde{H_{n-1}}\widetilde{H_{n+1}} - \widetilde{H_n}^2 = \frac{NK(-q)^{n-1}(1 + p\varepsilon)\left[\underline{\lambda}\underline{\gamma}\gamma - \underline{\gamma}\underline{\lambda}\lambda\right]}{\lambda - \gamma} \tag{3.12}$$

Proof. Using the Binet formula for dual Horadam quaternions, we get

$$\begin{aligned}
&\widetilde{H_{n-1}}\widetilde{H_{n+1}} - \widetilde{H_n}^2 \\
&= \frac{1}{(\lambda - \gamma)^2}\left\{N\underline{\lambda}\lambda^{n-1}(1 + \varepsilon\lambda) - K\underline{\gamma}\gamma^{n-1}(1 + \varepsilon\gamma)\right\} \\
&\times \left\{N\underline{\lambda}\lambda^{n+1}(1 + \varepsilon\lambda) - K\underline{\gamma}\gamma^{n+1}(1 + \varepsilon\gamma)\right\} \\
&- \left\{\frac{1}{\lambda - \gamma}\left\{N\underline{\lambda}\lambda^{n}(1 + \varepsilon\lambda) - K\underline{\gamma}\gamma^{n}(1 + \varepsilon\gamma)\right\}\right\}^2 \\
&= \frac{NK\lambda^{n-1}\gamma^{n-1}}{(\lambda - \gamma)^2}\Big\{ - \underline{\lambda}\underline{\gamma}\gamma^2(1 + \varepsilon\lambda)(1 + \varepsilon\gamma) - \underline{\gamma}\underline{\lambda}\lambda^2(1 + \varepsilon\gamma)(1 + \varepsilon\lambda) \\
&+ \underline{\lambda}\underline{\gamma}\lambda\gamma(1 + \varepsilon\lambda)(1 + \varepsilon\gamma) + \underline{\gamma}\underline{\lambda}\lambda\gamma(1 + \varepsilon\gamma)(1 + \varepsilon\lambda)\Big\} \\
&= \frac{NK(-q)^{n-1}(1 + p\varepsilon)\left[\underline{\lambda}\underline{\gamma}\gamma - \underline{\gamma}\underline{\lambda}\lambda\right]}{\lambda - \gamma}.
\end{aligned}$$

□

Theorem 3.6 *The exponential generating function for the dual Horadam quaternion is*

$$\sum_{s=0}^{\infty} \frac{\widetilde{H_s}}{s!} y^s = \frac{N\underline{\lambda}(1 + \varepsilon\lambda)e^{\lambda s} - K\underline{\gamma}(1 + \varepsilon\gamma)e^{\gamma s}}{\lambda - \gamma}. \tag{3.13}$$

Proof. Using the Binet formula for dual Horadam quaternions, we have

$$\begin{aligned}\sum_{s=0}^{\infty}\frac{\widetilde{H_s}}{s!}y^s &= \sum_{s=0}^{\infty}\Big[\frac{N\underline{\lambda}\lambda^s(1+\varepsilon\lambda)-K\underline{\gamma}\gamma^s(1+\varepsilon\gamma)}{\lambda-\gamma}\Big]\frac{y^s}{s!}\\ &= \frac{N\underline{\lambda}(1+\varepsilon\lambda)}{\lambda-\gamma}\sum_{s=0}^{\infty}\lambda^s\frac{y^s}{s!}-\frac{K\underline{\gamma}(1+\varepsilon\gamma)}{\lambda-\gamma}\sum_{s=0}^{\infty}\gamma^s\frac{y^s}{s!}\\ &= \frac{N\underline{\lambda}(1+\varepsilon\lambda)e^{\lambda y}-K\underline{\gamma}(1+\varepsilon\gamma)e^{\gamma y}}{\lambda-\gamma}.\end{aligned}$$

□

Theorem 3.7 *For $u\in\mathbb{Z}^+$, $v\in\mathbb{N}$, the generating function of the dual Horadam quaternion $\widetilde{H_{u+v}}$ is*

$$\sum_{v=0}^{\infty}\widetilde{H_{v+u}}y^v=\frac{\widetilde{H_u}+qy\widetilde{H_{u-1}}}{1-py-qy^2}. \tag{3.14}$$

Proof. From (3.7) and (2.2), we have

$$\begin{aligned}\sum_{v=0}^{\infty}\widetilde{H_{v+u}}y^v &= \sum_{v=0}^{\infty}\Big[\frac{N\underline{\lambda}\lambda^{u+v}(1+\varepsilon\lambda)-K\underline{\gamma}\gamma^{u+v}(1+\varepsilon\gamma)}{\lambda-\gamma}\Big]y^v\\ &= \frac{N(1+\varepsilon\lambda)\underline{\lambda}\lambda^u}{\lambda-\gamma}\sum_{v=0}^{\infty}(\lambda y)^v-\frac{K(1+\varepsilon\gamma)\underline{\gamma}\gamma^u}{\lambda-\gamma}\sum_{v=0}^{\infty}(\gamma y)^v\\ &= \frac{N(1+\varepsilon\lambda)\underline{\lambda}\lambda^u(1-\gamma y)-K\underline{\gamma}(1+\varepsilon\gamma)\gamma^u(1-\lambda y)}{(\lambda-\gamma)(1-\gamma y-\lambda y+\lambda\gamma y^2)}\\ &= \frac{1}{1-py-qy^2}\Big\{\frac{N(1+\varepsilon\lambda)\underline{\lambda}\lambda^u-K(1+\varepsilon\gamma)\underline{\gamma}\gamma^u}{\lambda-\gamma}\\ &+qy\Big[\frac{N(1+\varepsilon\lambda)\underline{\lambda}\lambda^{u-1}-K(1+\varepsilon\gamma)\underline{\gamma}\gamma^{u-1}}{\lambda-\gamma}\Big]\Big\}\\ &= \frac{\widetilde{H_u}+qy\widetilde{H_{u-1}}}{1-py-qy^2}.\end{aligned}$$

□

We next give a Theorem for the dual Horadam quaternions. The proof of theorem can be done by using Binet formula for $\widetilde{H_u}$.

Theorem 3.8 *For $u\geq 0$, the following identities hold:*

(i) $\sum_{l=0}^{u}\widetilde{H_l}=\frac{1}{1-p-q}\Big(\widetilde{H_0}-\widetilde{H_{u+1}}-q\widetilde{H_u}-\big[\frac{N\underline{\lambda}(1+\lambda\varepsilon)\gamma-K\underline{\gamma}(1+\gamma\varepsilon)\lambda}{\lambda-\gamma}\big]\Big)$,

(ii) $\sum_{l=0}^{u}\widetilde{H_{2l}}=\frac{1}{1-p^2-2q+q^2}\Big(\widetilde{H_0}-\widetilde{H_{2u+2}}+q^2\widetilde{H_{2u}}-[\frac{N\underline{\lambda}(1+\lambda\varepsilon)\gamma^2-K\underline{\gamma}(1+\gamma\varepsilon)\lambda^2}{\lambda-\gamma}]\Big)$,

(iii) $\sum_{l=1}^{u}\widetilde{H_{2l-1}}=\frac{1}{1-p^2-2q+q^2}\Big(\widetilde{H_1}-\widetilde{H_{2u+1}}+q^2\widetilde{H_{2u-1}}+q[\frac{N\underline{\lambda}(1+\lambda\varepsilon)\gamma-K\underline{\gamma}(1+\gamma\varepsilon)\lambda}{\lambda-\gamma}]\Big)$.

Theorem 3.9 *"Catalan's identity" For $u,t\in\mathbb{Z}^+$ such that $t\le u$. Then*

$$\widetilde{H_{u-t}}\widetilde{H_{u+t}}-\widetilde{H_u}^2=\frac{NK(1+\varepsilon p)(-q)^{u-t}(\lambda^t-\gamma^t)[\underline{\lambda}\underline{\gamma}\gamma^t-\underline{\gamma}\underline{\lambda}\lambda^t]}{p^2+4q}. \quad (3.15)$$

Proof. Employing (3.7) and (2.2), we obtain

$$\begin{aligned}
&\widetilde{H_{u-t}}\widetilde{H_{u+t}}-\widetilde{H_u}^2\\
&=\frac{\left[N(1+\varepsilon\lambda)\lambda^{u-t}\underline{\lambda}-K(1+\varepsilon\gamma)\underline{\gamma}\gamma^{u-t}\right]}{(\lambda-\gamma)^2}\\
&\times\left[N(1+\varepsilon\lambda)\lambda^{u+t}\underline{\lambda}-K(1+\varepsilon\gamma)\underline{\gamma}\gamma^{u+t}\right]\\
&-\frac{\left[N(1+\varepsilon\lambda)\lambda^{u}\underline{\lambda}-K(1+\varepsilon\gamma)\underline{\gamma}\gamma^{u}\right]^2}{(\lambda-\gamma)^2}\\
&=\frac{-NK[1+\varepsilon(\lambda+\gamma)]\lambda^{u-t}\gamma^{u+t}\underline{\lambda}\underline{\gamma}-NK[1+\varepsilon(\lambda+\gamma)]\gamma^{u-t}\lambda^{u+t}\underline{\gamma}\underline{\lambda}}{(\lambda-\gamma)^2}\\
&+\frac{NK[1+\varepsilon(\lambda+\gamma)]\underline{\lambda}\underline{\gamma}\lambda^u\gamma^u+NK[1+\varepsilon(\lambda+\gamma)]\underline{\gamma}\underline{\lambda}\lambda^u\gamma^u}{(\lambda-\gamma)^2}\\
&=\frac{NK(1+\varepsilon p)(-q)^{u-t}(\lambda^t-\gamma^t)[\underline{\lambda}\underline{\gamma}\gamma^t-\underline{\gamma}\underline{\lambda}\lambda^t]}{p^2+4q}.
\end{aligned}$$

□

Theorem 3.10 *"d'Ocagne's identity" For $u\in\mathbb{Z}^+$ and $v\in\mathbb{N}$ such that $v>u+1$. Then*

$$\widetilde{H_{u+1}}\widetilde{H_v}-\widetilde{H_u}\widetilde{H_{v+1}}=\frac{NK(1+\varepsilon p)(-q)^u[\underline{\gamma}\underline{\lambda}\lambda^{v-u}-\underline{\lambda}\underline{\gamma}\gamma^{v-u}]}{\lambda-\gamma}. \quad (3.16)$$

Proof. Employing the Binet formula for dual Horadam quaternions and Eq. (2.2), we have

$$\begin{aligned}
&\widetilde{H_{u+1}}\widetilde{H_v} - \widetilde{H_u}\widetilde{H_{v+1}} \\
&= \Big[\frac{N\underline{\lambda}(1+\varepsilon\lambda)\lambda^{u+1} - K\underline{\gamma}\gamma^{u+1}(1+\varepsilon\gamma)}{\lambda-\gamma}\Big]\Big[\frac{N\underline{\lambda}(1+\varepsilon\lambda)\lambda^{v} - K\underline{\gamma}\gamma^{v}(1+\varepsilon\gamma)}{\lambda-\gamma}\Big] \\
&- \Big[\frac{N\underline{\lambda}(1+\varepsilon\lambda)\lambda^{u} - K\underline{\gamma}\gamma^{u}(1+\varepsilon\gamma)}{\lambda-\gamma}\Big]\Big[\frac{N\underline{\lambda}(1+\varepsilon\lambda)\lambda^{v+1} - K\underline{\gamma}\gamma^{v+1}(1+\varepsilon\gamma)}{\lambda-\gamma}\Big] \\
&= \frac{NK(1+\varepsilon p)(-q)^u\big[\lambda(\underline{\gamma}\underline{\lambda}\lambda^{v-u} - \underline{\lambda}\underline{\gamma}\gamma^{v-u}) + \gamma(\underline{\lambda}\underline{\gamma}\gamma^{v-u} - \underline{\gamma}\underline{\lambda}\lambda^{v-u})\big]}{(\lambda-\gamma)^2} \\
&= \frac{NK(1+\varepsilon p)(-q)^u[\underline{\gamma}\underline{\lambda}\lambda^{v-u} - \underline{\lambda}\underline{\gamma}\gamma^{v-u}]}{\lambda-\gamma}.
\end{aligned}$$

□

References

1. Halici, S. – *On complex Fibonacci quaternions*, Adv. Appl. Clifford Algebr., **23** (2013), no. 1, 105-112.
2. Halici, S., Karataş A. – *On a generalization for Fibonacci quaternions*, Chaos Solitons Fractals, **98** (2017), 178-182.
3. Horadam, A.F. – *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly **70** (1963), 289-291.
4. Horodam, A.F. – *Generating functions for powers of a certain generalised sequence of numbers*, Duke Math. J., **32** (1965), 437-446.
5. Nurkan, S.K.; Güven, I.A. – *Dual Fibonacci quaternions*, Adv. Appl. Clifford Algebr., **25** (2015), no. 2, 403-414.
6. Swamy, M.N.S. – *On generalized Fibonacci quaternions*, Fibonacci Quart., **11** (1973), no. 5, 547-549.
7. Szynal-Liana, A.; Włoch, I. – *A note on Jacobsthal quaternions*, Adv. Appl. Clifford Algebr., **26** (2016), no. 1, 441-447.
8. Yüce, S.; Aydin, F.T. – *Generalized dual Fibonacci quaternions*, Appl. Math. E-Notes, **16** (2016), 276-289.
9. Yüce, S.; Aydin, F.T. – *A new aspect of dual Fibonacci quaternions*, Adv. Appl. Clifford Algebr., **26** (2016), no. 2, 873-884 .

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