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Existence results for coupled systems of impulsive functional stochastic differential inclusions with non-convex case

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Abstract We present some existence results for the impulsive functional differential inclusions with multiple delay with fractional Brownian motion. The results are based on the classical fixed point theorem for contraction multi-valued operators provided by Covitz and Nadler fixed point theorem in generalized metric and Banach spaces.

Keywords Fractional Brownian motion · Impulsive functional differential inclusions · Matrix convergent to zero · Generalized Banach space · Fixed point

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1 Introduction

Stochastic differential equations have many applications in science and engineer, and, for this reason, these equations have been receiving much attention over the last decades (see, e.g., [1,3]). Also, in recent years, stochastic differential equations driven by fractional Brownian motions (fBm), have attracted much attention and there are only a few papers published in this field (see, e.g. [1,3,16,32]). Boudaoui et al. [10] discussed the existence of solutions to stochastic impulsive evolution equations with time delays, driven by fBm with Hurst index $H > \frac{1}{2}$.
Neutral differential equations arise in many areas of applied mathematics and, for this reason, these equations have received much attention over the last few decades. Some good literature for ordinary neutral functional differential equations were considered for the first time by Mil'man and Myškis [28] with impulsive effects and multiple delay of active research which culminated with the monograph by Halanay and Wexler [23]. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity). A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al. [4], Graef et al. [21], Laskshmikantham et al. [25]. The reader can also find a detailed and

extensive bibliography in the previously mentioned books (see also Da Prato and Zabczyk [14], Gard [18], Gihman and Skorokhod [19], Sobczyk [34] and Tsokos and Padgett [35]). For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [35] to a closed system with a simplified heat, one organ or capillary bed, and recirculation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Bharucha-Reid [5], Mao [27], Øksendal [30].

More recently, Blouhi et al. [8] derived a set of sufficient conditions for existence results. Impulsive functional differential inclusions with multiple delay with fractional Brownian motion are based on the following classical fixed point theorem for contraction multi-valued operators we prove some existence results based on the nonlinear alternative of the Leary Schauder type (in the convex case) fixed point in generalized metric and Banach spaces. However, there are only few works that have been reported on the study of stochastic differential inclusions with standard and fractional Brownian motions [26]. Recently, Boudaoui et al. [10], Blouhi et al. [9] have discussed the cases in which the right hand side can be either convex or nonconvex-valued. The results are obtained by using two different fixed point theorems for multi-valued mappings, more precisely, the technique is based on a multi-valued version of Perovs fixed point theorem and a new version of a nonlinear alternative of Leray Schauders fixed point theorem in generalized Banach spaces.

More specifically, in this paper we are interested in proving the existence of solutions for a system of stochastic impulsive differential inclusions with multiple delay of the following type:

$$\begin{cases} d(x(t)-h^1(t,x_t,y_t)) \in \Big(\sum_{i=1}^{n_*} x(t-T_i)+G^1(t,x_t,y_t)\Big)dt \\ \qquad\qquad +f^1(t)d^\circ B^{H_1}(t),\ t\in J, t\neq t_k \\ d(y(t)-h^2(t,x_t,y_t)) \in \Big(\sum_{i=1}^{n_*} y(t-T_i)+G^2(t,x_t,y_t)\Big)dt \\ \qquad\qquad +f^2(t)d^\circ B^{H_2}(t),\ t\in J, t\neq t_k \\ x(t_k^+)-x(t_k) = I_k(x(t_k)), \quad t=t_k \quad k=1,2,\ldots,m \\ y(t_k^+)-y(t_k^-) = \overline{I}_k(y(t_k)), \\ x(t) = \phi(t)\in\mathcal{D}_{\mathcal{F}_0}, \quad t\in[-r,0], \\ y(t) = \overline{\phi}(t)\in\mathcal{D}_{\mathcal{F}_0} \quad t\in[-r,0], \end{cases} \tag{1.1}$$

where $n_* \in \{1, 2, ...\}$, $r = \max_{1\leq i\leq n_*} T_i$ and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. The impulse times t_k satisfy $0 = t_0 < t_1 < t_2 < \ldots, t_m < b$. As for y_t we mean the segment solution which is defined in the usual way, that is, if $y(\cdot, \cdot) : [-r, b]\times\Omega \to \mathbb{R}^n$, then for any $t \geq 0$, $y_t(\cdot, \cdot) : [-r, 0] \times \Omega \to \mathbb{R}^n$ is given by:

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in [-r, 0], \ \omega \in \Omega,$$

Here $y_t()$ represents the history of the state from time $t-r$, up to the present time t. Before describing the properties fulfilled by the operators f^i, g^i and $I_k, \overline{I}_k$, we need to introduce some notation and describe some spaces.
Let $\mathcal{D}_{\mathcal{F}_0}$ be the following space defined by

$$\mathcal{D}_{\mathcal{F}_0} = \big\{\phi\colon [-r, 0] \times \Omega \to \mathbb{R}^n, \text{ ,is continuous everywhere except for a finite number of points; } \phi(t_k^-) \text{ and } \phi(t_k^+) \text{ with } \phi(t_k) = \phi(t_k^-)\big\},$$

endowed with the norm

$$||\phi(t)||_{\mathcal{D}_{\mathcal{F}_0}} = \int_{-r}^{0} |\phi(t)|^2 dt$$

Now, for a given $b > 0$, we define

$$\mathcal{D}_{\mathcal{F}_b} = \big\{y\colon [-r, b] \times \Omega \to \mathbb{R}^n, \ y_k \in C(J_k, \mathbb{R}^n) \text{ for } k = 1, \ldots m, \ y_0 \in \mathcal{D}_{\mathcal{F}_0}, \text{ and there exist; } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-), \ k = 1, \cdots, m, \text{ and } \sup_{t\in[0,b]} E(|y(t)|^2) < \infty \int_{-r}^{0} |\phi(t)|^2 dt < \infty\big\},$$

endowed with the norm

$$\|y\|_{\mathcal{D}_{\mathcal{F}_b}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0\leq s\leq b} (\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}},$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = [-r, 0]$.

Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$ where $J := [0, b]$. Let $h^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathbb{R}^n$ and $f^i : J \to \mathbb{R}^n$ are Carathéodory functions, $G^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{P}(\mathbb{R}^{n)}$ for each $i = 1, 2$, and $B^{H_l} = (B^{H_l, j}, \ j = 1, 2 \ldots n)$ are a cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $l = 1, 2$ and $I_k, \overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(k = 1, \ldots, m)$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), \Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k-h)$ stand

for the right and the left limits of the function y at $t = t_k$, respectively.
It is obvious that system (1.1) can be seen as a fixed point problem:

$$\begin{cases} d(z(t)–h(t,,z_t)) = \Big(\sum_{i=1}^{n_*} z(t–T_i)+G(t)\Big)dt+f(t)d^\circ B^H(s),\ t\in[0,b], t\neq t_k, \\ \Delta z(t) \qquad\qquad = I_k^*(z(t_k)), \quad t = t_k \quad k = 1,2,\ldots,m \\ z(t) \qquad\qquad = \widetilde{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0} \quad t \in [-r,0], \end{cases} \tag{1.2}$$

where

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\ G(t) = \begin{bmatrix} G^1(t,x_t,y_t) \\ G^2(t,x_t,y_t) \end{bmatrix}, f(t) = \begin{bmatrix} f^1(t) \\ f^2(t) \end{bmatrix}$$

and

$$\widetilde{\phi}(t) = \begin{bmatrix} \phi(t) \\ \overline{\phi}(t) \end{bmatrix}, h(t,z_t) = \begin{bmatrix} h^1(t,x_t,y_t) \\ h^2(t,x_t,y_t) \end{bmatrix}.$$

Very recently, for $h = 0$, problem (1.1) was studied by Ferhat et al. [17]. In the deterministic and single valued framework, the above system was used to analyze initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [29] and mathematical economics [24] which can be set in the operator form (1.1).
Motivated by the previous works, in the present paper, it is interesting to analyze the influence of impulsive effects, stochastic noise on a solution of (1.1). Under suitable assumptions, to the best of our knowledge, there are no results concerning coupled systems of impulsive functional stochastic differential inclusions with non-convex case.
The paper is organized as follows. In Section 2 we recall some definitions and facts which will be needed in our analysis. Section 3 we prove some existence results based on a a multi-valued version type of Perov's fixed point theorem [31] and prove another result on the existence of solution in a non-convex case.

2 Preliminaries

In this section, we introduce some notations, and recall some definitions and preliminary facts which are used throughout this paper. Actually we will borrow them from [7]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.

2.1 Some results on stochastic integrals

In this section, we introduce notations, definitions and preliminary facts that are used throughout this article.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).
For a stochastic process $x(.,.) : [0,T] \times \Omega \to \mathbb{R}^n$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t,w)$. SDEs with respect to fBm have been interpreted via various stochastic integrals, such as the Wick integral, the Wiener integral, the Skorokhod integral, and path-wise integrals [11, 19, 33].

Definition 2.1 *The fractional Brownian motion $(B^H(t))$ with Hurst index H is a centered self-similar Gaussian process $B^H = B^H(t), t \in \mathbb{R}^+$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with the properties :*

(i) $B^H(0) = 0$;
(ii) $\mathbb{E}(B^H(t)) = 0, t \in \mathbb{R}^+$;
(ii) $\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}), t, s \in \mathbb{R}^+$

For $H = \frac{1}{2}$, this is the usual Brownian motion.

Definition 2.2 *The process $(F(t), t \geq 0)$ is said to be ϕ-differentiable if for each $t \in \mathbb{R}^+$, $F(t)$ is ϕ-differentiable and $D^\phi F : \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is jointly measurable.*

We recall some type of stochastic integration with respect to the fractional Brownian motion (see for instance, [33]).
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be Borel measurable and $\frac{1}{2} \leq H < 1.\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be given by

$$\phi(t,s) = H(2H-1)|t-s|^{2H-2}, t, s \in \mathbb{R}^+,$$

Then we say that define

$$L^2_\phi = \{f : |f|^2_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(t)f(s)\phi(t,s)dsdt < \infty\}.$$

If we equip L^2_ϕ with the inner product

$$\langle f_1, f_2\rangle_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f_1(t)f_2(s)\phi(t,s)dsdt$$

then $L^2_\phi(\mathbb{R}^+)$ becomes a separable Hilbert space.
Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n)),$$

where $n \geq 1$, $f \in C^\infty_b(\mathbb{R}^n)$ (i.e., f all its partial derivatives are bounded), $\psi_i \in \mathcal{H}$,$\mathcal{H}$ is a Hilbert space [2].

The derivative operator D_t^H of a smooth and cylindrical random variable F is defined as the $\mathcal{H}$-valued random variable:

$$D_t^H F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}((B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n))\psi_i.$$

Introduce the Malliavin ϕ-derivative of F

$$D_t^\phi F = \int_{\mathbb{R}^+} \phi(t,v) D_v^H F dv$$

Refer to [6, pp. 129 ff.] to obtain more details. The Wick product of two random variables is denoted $\diamond$. This product is important in the construction of the stochastic integrals (of Itô type).

Definition 2.3 *Let $\mathcal{L}(0,T)$ be the family of processes on $[0,T]$ such that $F \in \mathcal{L}(0,T)$ if $\mathbb{E}|F|_\phi^2 < \infty$, F is ϕ-differentiable, the trace of $(D_s^\phi F_t, s,t \in [0,T])$ exists and*

$$\mathbb{E}\int_0^T (D_s^\phi F_s)^2 ds < \infty$$

and for each sequence of partitions $(\pi, n \in \mathbb{N})$ of $[0,T]$ such that $|\pi| \to 0$ as $n \to +\infty$

$$\sum_{i=0}^{n-1} \mathbb{E}\Big[\int_{t_i^n}^{t_{i+1}^n} (|D_s^\phi F_{t_i^n} - D_s^\phi F_s|) ds\Big]^2$$

and

$$\mathbb{E}|F^\pi - F|_\phi^2$$

tend to 0 as $n \to +\infty$ where $\pi_n = \{t_0^n, \ldots, t_n^n\}$ and F^π is the simple process induced by π_n.

The stochastic integral of $F \in \mathcal{L}(0,T)$ is constructed from Riemann sums using the Wick product as

$$\sum_{i=0}^{n-1} F_{t_i}^\pi \diamond (B^H(t_{i+1} - B^H(t_i)).$$

Definition 2.4 *Let $u(t)$ be a stochastic process with integrable trajectories.*

(1) The symmetric integral of $u(t)$ with respect to $B^H(t)$ is defined as the limit in probability as ϵ tends to zero of

$$\frac{1}{2\epsilon}\int_0^T u(s)\Big(B^H(s+\epsilon) - B^H(s-\epsilon)\Big) ds$$

provided this limit exists in probability, and is denoted by $\int_0^T u(s) d^\circ B^H(s)$.

(2) The forward integral of $x(t)$ with respect to B^H is defined as the limit in probability as ϵ tends to zero of

$$\frac{1}{\epsilon}\int_0^T u(s)\Big(\frac{B^H(s+\epsilon)-B^H(s)}{\epsilon}\Big)ds$$

provided this limit exists in probability, and is denoted by $\int_0^T u(s)d^-B^H(s)$

(3) The backward integral of $u(t)$ with respect to B^H is defined as the limit in probability as ϵ tends to zero of

$$\frac{1}{\epsilon}\int_0^T u(s)\Big(\frac{B^H(s-\epsilon)-B^H(s)}{\epsilon}\Big)ds$$

provided this limit exists in probability, and is denoted by $\int_0^T u(s)d^+B^H(s)$

Theorem 2.5 *(See e.g. [6], Theorem 5.5.1) Let $\mathcal{L}_\phi(0,T)$ of integrands is defined as the family of stochastic processes $u(t)$ on $[0,T]$ such that $u(t)\in\mathcal{L}_\phi(0,T)$ if $E|u(t)|^2_\phi<\infty$. Assume that $u(t)$ is a stochastic process in $L(0,T)$ and satisfies*

$$\int_{[0,T]}\int_{[0,T]}|D^H_s u(t)||t-s|^{2H-2}dsdt<\infty.$$

Then the symmetric integral exists and the following relation holds:

$$\int_0^T u(s)d^\circ B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T(D^\phi_s u(t))ds, \tag{2.1}$$

where $\diamond$ denotes the Wick product, $\frac{1}{2}<H<1$

Remark 2.1 (See e.g. [6], Theorem 6.2.7) If $u(t)\in\mathcal{L}_\phi(0,T)$, the definition of the forward and backward integrals with respect to fBm is as follows:

$$\int_0^T u(s)d^-B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T(D^\phi_s u(t))ds \tag{2.2}$$

and

$$\int_0^T u(s)d^+B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T(D^\phi_s u(t))ds \tag{2.3}$$

The detailed proof of Lemma 2.6 can be found in the authors previous work [37].

Lemma 2.6 *Suppose that $Z(s)$ is a stochastic process in $\mathcal{L}_\phi(0,T)$, and $B^H(t)(H>\frac{1}{2})$ is a fractional Brownian motion. For any $0<T<\infty$, there exists a constant $C(H,T)$, such that the following inequality holds*

$$\mathbb{E}\Big(\int_0^T Z(s)d^\circ B^H(s)\Big)^2\le 2C(H,T)\mathbb{E}\Big(\Big(\int_0^T|Z(s)|^2ds\Big)+4CT^2 \tag{2.4}$$

where $C(H,T)=HT^{2H-1}$.

2.2 Some results on fixed point theorems and set-valued analysis

For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.7 *A generalized metric space (X,d), where* $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}$, *is complete if* (X, d_i) *is a complete metric space for every* $i = 1, \ldots, n$.

Definition 2.8 *A square matrix of real numbers* M *is said to be convergent to zero if its spectral radius* $\rho(M)$ *is strictly less than* 1. *In other words, this means that all the eigenvalues of* M *are in the open unit disc. (i.e.* $|\lambda| < 1$, *for every* $\lambda \in \mathbb{C}$ *with* $det(M - \lambda I) = 0$, *where* I *denote the unit matrix of* $\mathcal{M}_{n\times n}(\mathbb{R})$*).*

Definition 2.9 *A non-singular matrix* $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *is said to have the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Lemma 2.10 *(See e.g.[36], Theorem 3.15.) Let* M *be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) M *is convergent towards zero;*
(ii) the matrix $I - M$ *is non-singular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $|\lambda| < 1$ *for every* $\lambda \in \mathbb{C}$ *with* $det(M - \lambda I) = 0$
(iv) $(I - M)$ *is non-singular and* $(I - M)^{-1}$ *has nonnegative elements.*

Some examples of matrices convergent to zero can be seen in [7].

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X) : y \text{ closed }\},$$
$$\mathcal{P}_b(X) = \{y \in \mathcal{P}(X) : y \text{ bounded }\},$$
$$\mathcal{P}_c(X) = \{y \in \mathcal{P}(X) : y \text{ convex }\},$$
$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X) : y \text{ compact }\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Let (X, d) be a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \dots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \dots, n$ are metrics on X, $H_d(A, B) = \ max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$, where $d(A, b) = \inf_{a \in A} d(a, b), d(a, B) = \inf_{b \in B} d(a, b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multi-valued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y \in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.

If the multi-valued map F is completely continuous with nonempty compact valued, then F is *u.s.c.* if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \longrightarrow \mathcal{P}_{cp,c}(X)$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 2.11 *The set-valued map $F : J \times X \times X \to \mathcal{P}(X \times X)$ is said to be L^2-Carathéodory if*

(i) $t \mapsto F(t, v)$ is measurable for each $v \in X \times X$;
(ii) $v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$;
(iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, v)\|^2 := \sup_{f \in F(t,v)} \|f\|^2 \leq h_q(t), \quad \textit{for all } \|v\|^2 \leq q \textit{ and for a.e. } t \in J.$$

and we say that a multi-valued operator G has a fixed point if there exists $x \in X$ such that $x \in G(X)$.
Our proofs are based on the following classical fixed point theorem for contraction multi-valued operators proved by Covitz and Nadler in 1970 [13] (see also Ouahab [22], Theorem 8.33).

Theorem 2.12 *(See e.g. [22], Theorem 8.33) Let (X, d) be a generalized complete metric space, and let $F : X \to \mathcal{P}_{cl}(X)$ be a multi-valued map. Assume that there exist $A, B, C \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x, y) + Bd(y, F(x)) + Cd(x, F(x)) \tag{2.5}$$

where $A + C$ converge to zero. Then there exist $x \in X$ such that $x \in F(x)$.

3 Existence result

Let us start by defining what we mean by a solution of problem (1.1).
$AC^i(J,\mathbb{R}^n)$ is the space of functions $y: J \to \mathbb{R}^n$ times differentiable whose ith derivative, $y^{(i)}$, is absolutely continuous.

Lemma 3.1 *Let $f^i, h^i : \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathbb{R}^n$ be a continuous function and $g^i : J \to \mathbb{R}^n$, $i = 1, 2$. Let $I_k, \overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ for each $k = 1, \ldots, m$ and let $x, y \in \mathcal{D}_{\mathcal{F}_b} \cap \cup_{k=1}^{k=m} AC^1(J_k, \mathbb{R}^N)$ be a classical solution of the problem*

$$\begin{cases} d(x(t)–h^1(x_t,y_t)) = \Big(\sum_{i=1}^{n_*} x(t-T_i)+g^1(x_t,y_t)\Big)dt+f^1(t)d^\circ B^{H_1}(t), \ t\in J, t\neq t_k \\ d(y(t–h^2(x_t,y_t)) = \Big(\sum_{i=1}^{n_*} y(t-T_i)+g^2(x_t,y_t)\Big)dt+f^2(t)d^\circ B^{H_2}(t), \ t\in J, t\neq t_k \\ \Delta x(t) = I_k(x(t_k)), \quad t = t_k \quad k = 1,2,\ldots,m \\ \Delta y(t) = \overline{I}_k(y(t_k)), \\ x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ y(t) = \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0} \quad t \in [-r, 0], \end{cases} \tag{3.1}$$

where $r = \max\limits_{1\le i\le n_*} T_i$ if and only if z is a solution of impulsive integral functional differential equation

$$\begin{cases} x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ x(t) = \phi(0) + h^1(x_t, y_t) - h^1(\phi, \bar{\phi}) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds \\ \qquad + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) \\ \qquad + \int_0^t g^1(x_s, y_s)ds + \sum\limits_{0\le t_k\le t} I_k(x(t_k)), \quad for, \ t \in J, a.e.\, w \in \Omega, \end{cases} \tag{3.2}$$

and

$$\begin{cases} y(t) = \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ y(t) = \bar{\phi}(0) + h^2(x_t, y_t) - h^2(\phi, \bar{\phi}) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \bar{\phi}(s)ds \\ \qquad + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \int_0^t f^2(s)d^\circ B^{H_2}(s) \\ \qquad + \int_0^t g^2(x_s, y_s)ds + \sum\limits_{0\le t_k\le t} \bar{I}_k(x(t_k)), \quad for, \ t \in J, a.e.\, w \in \Omega, \end{cases} \tag{3.3}$$

Proof. Let (x, y) be a possible solution of the problem (3.1). Then $z|_{[-r,t_1]} = (x|_{[-r,t_1]}, y|_{[-r,t_1]})$ is a solution to

$$d(x(t)-h^1(x_t,y_t))=\Big(\sum_{i=1}^{n_*}x(t-T_i)+g^1(x_t,y_t)\Big)dt+f^1(t)d^\circ B^{H_1}(t),\ t\in J:=[0,b].$$

Assume that $t_k < t \le t_{k+1}$, $k = 1, \dots, m$. By integration of above inequality yields

$$\begin{aligned}
x(t_1^-)-x(0)-(h^1(x_{t_1^-},y_{t_1^-})-h^1(\phi,\bar{\phi}))&=\sum_{i=1}^{n_*}\int_0^{t_1}x(s-T_i)ds+\int_0^{t_1}f^1(s)d^\circ B^{H_1}(s)\\
&\quad+\int_0^{t_1}g^1(x_s,y_s)ds\\
x(t_1^-)-x(0)-(h^1(x_{t_1^-},y_{t_1^-})-h^1(\phi,\bar{\phi}))&=\sum_{i=1}^{n_*}\int_{-T_i}^{t_1-T_i}x(s)ds+\int_0^{t_1}f^1(s)d^\circ B^{H_1}(s)\\
&\quad+\int_0^{t_1}g^1(x_s,y_s)ds\\
x(t_2^-)-x(t_1^-)-(h^1(x_{t_2^-},y_{t_2^-})-h^1(x_{t_1^+},y_{t_1^+}))&=I_1(x(t_1))+\sum_{i=1}^{n_*}\int_{t_1-T_i}^{t_2-T_i}x(s)ds\\
&\quad+\int_{t_1}^{t_2}f^1(x_s,y_s)d^\circ B^{H_1}(s)+\int_{t_1}^{t_2}g^1(x_s,y_s)ds.
\end{aligned}$$

More generally, for $t_{k-1} < t < t_k$

$$\begin{aligned}
x(t_k^-)-x(t_{k-1}^-)-(h^1(x_{t_k^-},y_{t_k^-})-h^1(x_{t_{k-1}^+},y_{t_{k-1}^+}))&=I_k(x(t_k))+\sum_{i=1}^{n_*}\int_{t_{k-1}-T_i}^{t_k-T_i}x(s)ds\\
&\quad+\int_{t_{k-1}}^{t_k}f^1(s)d^\circ B^{H_1}(s)+\int_{t_{k-1}}^{t_k}g^1(x_s,y_s)ds\\
x(t)-x(t_k^-)-(h^1(x_t,y_t)-h^1(t_k^+,x_{t_k^+},y_{t_k^+}))&=I_k(x(t_k))+\sum_{i=1}^{n_*}\int_{t_k-T_i}^{t-T_i}x(s)ds\\
&\quad+\int_{t_k}^{t}f^1(x_s,y_s)d^\circ B^{H_1}(s)+\int_{t_k}^{t}g^1(x_s,y_s)ds.
\end{aligned}$$

Adding these together, we get

$$x(t) = x(0) + h^1(x_t, y_t) - h^1(\phi, \bar{\phi})) + \sum_{0 \le t_k \le t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)ds$$

$$x(t) = \phi(0) + h^1(x_t, y_t) - h^1(\phi, \bar{\phi})) + \sum_{0 \le t_k \le t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} x(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)ds.$$

If x satisfies the integral equation (3.3), then x is solution of the problem (3.1). Let $t \in [0, b] \backslash \{t_1, \ldots, t_m\}$ and

$$x(t) = \phi(0) + h^1(x_t, y_t) - h^1(\phi, \bar{\phi})) + \sum_{0 \le t_k \le t} I_k(x(t_k))$$

$$+ \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)ds.$$

Similarly

$$y(t) = \overline{\phi}(0) + h^2(x_t, y_t) - h^2(\phi, \bar{\phi})) + \sum_{0 \le t_k \le t} \overline{I}_k(y(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \overline{\phi}(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds$$

$$+ \int_0^t f^2(s)d^\circ B^{H_2}(s) + \int_0^t g^2(x_s, y_s)ds.$$

□

3.1 The nonconvex case

Now we first define the concept of the solution to our problem.

Definition 3.2 *Given $(\phi, \bar{\phi}) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$, a $\mathbb{R}^n-$valued stochastic process $u(t) = \{(x(t), y(t)), t \in [-r, b]\}$ is called a solution of the problem (1.1) if $(x(t), y(t))$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $u(t) = (\phi(t), \bar{\phi}(t))$ on $[-r, 0]$, $\Delta u|_{t=t_k} = (I_k(x(t_k^-)), \bar{I}_k(y(t_k^-)))$, $k = 1, 2, \cdots, m$, the restriction of $x(\cdot, \cdot)$ to $[0, b) - \{t_1, t_2, \cdots, t_m\}$ is continuous, (x, y) and $u(t) \in \mathbb{R}^n \times \mathbb{R}^n$*

has càdlàg paths on $t \in [0,b]$ a.s., for every $0 \leq s < t \leq b$, there exist $g^i \in S_{G^i,u}$ for each $i = 1,2$ such that the following integral equation holds

$$x(t) = \phi(0) + h^1(x_t, y_t) - h^1(\phi, \bar{\phi})) + \sum_{0 \leq t_k \leq t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)ds., \ t \in J. \tag{3.4}$$

and

$$y(t) = \bar{\phi}(0) + h^2(x_t, y_t) - h^2(\phi, \bar{\phi})) + \sum_{0 \leq t_k \leq t} \bar{I}_k(y(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \bar{\phi}(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \int_0^t f^2(s)d^\circ B^{H_2}(s) + \int_0^t g^2(x_s, y_s)ds., \ t \in J. \tag{3.5}$$

where $S_{G^i,u}$ is known as the selectors set from G^i and given by

$$g^i \in S_{G^i,y} = \{f \in L^2([0,T], \mathbb{R}^n) : g^i(t) \in G^i(t,y) \quad for\ a.e.t \in [0,b]\}.$$

Now we present a second result for the problem (1.1) with a nonconvex valued right-hand side. Our considerations are based on a fixed point theorem for contraction multi-valued operators given by Covitz and Nadler in 1970 (see also Deimling [15], Theorem 11.1). In this section, we will provide a multi-valued version of Perov's fixed point theorem in [31].

Definition 3.3 *Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x), N(y)) \leq Md(x,y) \text{ for all } \ x, y \in X.$$

Theorem 3.4 *[31] Let (X,d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ a contractive multi-valued operator with Lipschitz matrix M. Then N has at least one fixed point.*

Let us introduce the following hypotheses:

- (H_1) $G^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow \mathcal{P}_{cp}(\mathbb{R}^n); (t,y) \longrightarrow G^i(t,x,y)$ is measurable for each $(x,y) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$.
- (H_2) There exist functions $a_i,\ b_i \in L^1([0,b], \mathbb{R}^+)$ such that

$$\begin{cases} EH^2_{d_1}(G^1(t,x,y), G^1(t,\overline{x},\overline{y})) \leq a_1(t)\|x - \overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b_1(t)\|y - \overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ EH^2_{d_2}(G^2(t,x,y), G^2(t,\overline{x},\overline{y})) \leq a_2(t)\|x - \overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b_2(t)\|y - \overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \end{cases}$$

with

$$d_i(0, G^i(t,0,0) \leq a_i(t)$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathcal{D}_{\mathcal{F}_0}$ for each $i = 1,2$.

(H_3) There exist constants α_i, β_i for each $i = 1, 2$ such that

$$\begin{cases} E|h^1(t,x,y) - h^1(t,\overline{x},\overline{y})\|^2 \leq \alpha_1\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \beta_1\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ E|h^2(t,x,y) - h^2(t,\overline{x},\overline{y})|^2 \leq \alpha_2\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \beta_2\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{cases}$$

The function $f^i : J \to \mathbb{R}^n$ satisfies

$$\int_0^b \|f^i(t)\|^2 dt < \infty, \quad \text{for all} \quad t \in J$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathcal{D}_{\mathcal{F}_0}$ and $t \in J$.

(H_4) There exist constants $d_k \geq 0$ and $\overline{d}_k \geq 0,\ k = 1, \ldots, m$ such that

$$|I_k(x) - I_k(\overline{x})|^2_X \leq d_k|x-\overline{x}|^2$$

and

$$|\overline{I}_k(y) - \overline{I}_k(\overline{y})|^2 \leq \overline{d}_k|y-\overline{y}|^2$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}^n$.

Theorem 3.5 *Assume that (H_1)-(H_4) are satisfied and the matrix*

$$M = \begin{pmatrix} \sqrt{\frac{1}{\tau} + 4m\sum_{k=1}^m d_k + 4\alpha_1} & \sqrt{\frac{1}{\tau} + 4\beta_1} \\ \sqrt{\frac{1}{\tau} + 4\alpha_2} & \sqrt{\frac{1}{\tau} + 4\beta_2 + 4m\sum_{k=1}^m \overline{d}_k} \end{pmatrix}.$$

where τ is sufficiently large. If M converges to zero. Then the problem (1.1) *has unique solution on $[-r, b]$.*

Proof. In order to transform the problem (1.1) into a fixed point problem, let the multi-valued operator $N : \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} \to \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)),\ (x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$$

where

$$N(x,y) = \{(P_1, P_2) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}\}$$

given by

$$P_1(t) = \begin{cases} \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^1(t, x_t, y_t) - h^1(0, \phi, \bar{\phi}) + \sum\limits_{0 \leq t_k \leq t} I_k(x(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} x(s)ds + \int_0^t f^1(s)d^\circ B^{H_1}(s) + \int_0^t g^1(s)ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.6}$$

and

$$\bar{P}_2(t) = \begin{cases} \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^2(t, x_t, y_t) - h^2(0, \phi, \bar{\phi}) + \sum\limits_{0 \le t_k \le t} \overline{I}_k(y(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \overline{\phi}(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} y(s) ds + \int_0^t f^2(s) d^\circ B^{H_2}(s) + \int_0^t g^2(s) ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.7}$$

where

$$g^i \in S_{G^i,u} = \{g^i \in L^2(J, X) : g^i(t) \in G^i(t, x, y) \quad \text{for a.e} \quad t \in J\}.$$

We shall show that N satisfies the assumptions of Theorem 3.4. Note that (H_1) implies that G^i for each $i = 1, 2$ has at most linear growth, i.e.

$$\mathbb{E}|G^i(t, x, y)|_X^2 \le a_i(t)\|x\|_{\mathcal{D}_{\mathcal{F}_0}}^2 + b_i(t)\|y\|_{\mathcal{D}_{\mathcal{F}_0}}^2$$

for a.e. $t \in J$ and all $x, y \in X$.

(a) $N(x, y) \in \mathcal{P}_{cl}(\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b})$ for each $(x, y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$. Let $\{P_{1,n}, \bar{P}_{2,n} : n \in \mathbb{N}\} \subset N_1(x, y) \times N_2(x, y)$ be a sequence converge to $(h, \bar{h})$. Then there exists a sequence $g_n^i \in S_{G_n^i,u}$ such that

$$P_{1,n}(t) = \begin{cases} \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^1(t, x_t, y_t) - h^1(0, \phi, \bar{\phi}) + \sum\limits_{0 \le t_k \le t} I_k(x(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} x(s) ds + \int_0^t f^1(s) d^\circ B^{H_1}(s) + \int_0^t g_n^1(s) ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.8}$$

and

$$\bar{P}_{2,n}(t) = \begin{cases} \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^2(t, x_t, y_t) - h^2(0, \phi, \bar{\phi}) + \sum\limits_{0 \le t_k \le t} \overline{I}_k(y(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \overline{\phi}(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} y(s) ds + \int_0^t f^2(s) d^\circ B^{H_2}(s) + \int_0^t g_n^2(s) ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.9}$$

Since $G^i(., ., .)$ has compact values, let $w(.) \in G^1(., .0)$ and $\bar{w}(.) \in G^2(., .0)$ be a measurable function such that

$$|g^1(t) - w(t)| = d(g^1(t), G^1(t, 0, 0))|$$

and

$$|g^2(t) - \bar{w}(t)| = d(g^2(t), G^2(t,0,0))|$$

From (H_1) and (H_2), we infer that for a.e. $t \in [0,b]$

$$\mathbb{E}|g_n^1(t)|^2 \leq \mathbb{E}|g_n^1(t) - w(t)|^2 + \mathbb{E}|w(t)|^2,$$

$$\leq a_1(t)\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b_1(t)\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}} + a_1 := K_1$$

and

$$\mathbb{E}|g_n^2(t)|^2 \leq \mathbb{E}|g_n^2(t) - \bar{w}(t)|^2 + \mathbb{E}|\bar{w}(t)|^2,$$

$$\leq a_2(t)\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b_2(t)\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}} + a_2 := K_2$$

Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\int_0^t \mathbb{E}|g_n^1(s) - g_n^1(s)|^2 ds \to 0 \quad \text{and thus} \quad P_{1,n}(t) \to P_1(t)$$

and

$$\int_0^t \mathbb{E}|g_n^2(s) - g_n^2(s)|^2 ds \to 0 \quad \text{and thus} \quad \bar{P}_{2,n}(t) \to \bar{P}_2(t)$$

with

$$P_1(t) = \begin{cases} \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r,0], \\ \overline{\phi}(0) + h^1(t, x_t, y_t) - h^1(0, \phi, \bar{\phi}) + \sum\limits_{0 \leq t_k \leq t} I_k(x(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^0 \phi(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} x(s) ds + \int_0^t f^1(s) d^\circ B^{H_1}(s) + \int_0^t g^1(s) ds, \ , \quad t \in [0,b]. \end{cases} \tag{3.10}$$

and

$$P_2(t) = \begin{cases} \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r,0], \\ \overline{\phi}(0) + h^2(t, x_t, y_t) - h^2(0, \phi, \bar{\phi}) + \sum\limits_{0 \leq t_k \leq t} \overline{I}_k(y(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^0 \overline{\phi}(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} y(s) ds + \int_0^t f^2(s) d^\circ B^{H_2}(s) + \int_0^t g^2(s) ds, \ , \quad t \in [0,b]. \end{cases} \tag{3.11}$$

proving that $(P_1, P_2) \in N_1(x,y) \times N_2(x,y)$

(b) There exists $M_{\alpha,\beta} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ convergent matric to zero, such that

$$\mathbb{E}H_d(N(x,y),N(\overline{x},\overline{y})) \leq M_{\alpha,\beta}\begin{pmatrix}\|x-\overline{x}\|_{\mathcal{D}_{\mathcal{F}_0}}\\ \|y-\overline{y}\|_{\mathcal{D}_{\mathcal{F}_0}}\end{pmatrix}, \quad \text{for all} \quad x,y,\overline{x},\overline{y}\in\mathcal{D}_{\mathcal{F}_b}.$$

Let $x,y,\overline{x},\overline{y}\in\mathcal{D}_{\mathcal{F}_b}$ and $(P_1,\bar{P}_1)\in N_1(x,y)\times N_2(x,y)$, $i=1,2$. Then there exists $g^i(\cdot)\in S_{G^i,x,y}$ such that for each $t\in J$. Then there exists $(g_1(t),g_2(t))\in G^1(t,x_t,y_t)\times G^2(t,x_t,y_t)$ such that.

From $(H2)$, tells us that

$$\begin{cases}\mathbb{E}H^2_{d_1}(G^1(t,x,y),G^1(t,\overline{x},\overline{y}))\leq a_1(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_1(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, & \text{a.e.}t\in J,\\ \mathbb{E}H^2_{d_2}(G^2(t,x,y),G^2(t,\overline{x},\overline{y}))\leq a_2(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_2(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, & \text{a.e.}t\in J.\end{cases}$$

Hence there is $(w,\overline{w})\in G^1(t,\overline{x}(t),\overline{y}(t))\times G^2(t,\overline{x}(t),\overline{y}(t))$ such that

$$\mathbb{E}|g^1(t)-w|^2_X\leq a_1(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_1(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t\in J$$

$$\mathbb{E}|g^2(t)-\overline{w}|^2\leq a_2(t)\|x-\overline{x}|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_2(t)\|y-\overline{y}|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t\in J$$

Consider the multi-valued maps $U_i:J\to\mathcal{P}(\mathbb{R}^n)$, $i=1,2$ defined by

$$U_1(t)=\{w\in G^1(t,\overline{x}(t),\overline{y}(t)):E|g^1(t)-w|^2\leq a_1(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}$$

$$+b_1(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \quad \text{a.e} \quad t\in J\}$$

$$U_2(t)=\{\overline{w}\in G^2(t,\overline{x}(t),\overline{y}(t)):E|g^2(t)-\overline{w}|^2\leq a_2(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}$$

$$+b_2(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \quad \text{a.e} \quad t\in[0,b]\}$$

that is $U_1=\overline{B}(g^1(t),a_1(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_1(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}})$ and $U_2=\overline{B}(g^2(t),a_2(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_2(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}})$. Since $g^i,a_i,b_i,x,y,\overline{x},\overline{y}$ are measurable for each $i=1,2$, Theorem $III.4.1$ in [12], tells us that the closed ball U_i is measurable. In addition $(H1)$ and $(H2)$ imply that for each $(x,y)\in\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}$and $G^i(t,x(t),y(t))$ is measurable. Finally the set $V_i(.)=U_i(.)\cap G^i(.,\overline{x}(.),\overline{y}(.))$ is nonempty. Therefore the intersection multi-valued operator V_i is measurable with nonempty, closed values (see [20]), there exists a function $\overline{g}^i(t)$ which is a measurable selection for $V_i(.)$. Thus

$$\overline{g}^i(t)\in G^i(t,\overline{x}(t),\overline{y}(t)) \quad \text{for a.e.}t\in J.$$

Hence

$$\mathbb{E}|g^1(t)-\overline{g}^1(t)|^2_X\leq a_1(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_1(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad \text{for a.e.}t\in J.$$

and

$$\mathbb{E}|g^2(t)-\overline{g}^2(t)|^2_X\leq a_2(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+b_2(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad \text{for a.e.}t\in J.$$

So

$$\bar{P}_1(t) = \begin{cases} \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^1(t, \bar{x}_t, \bar{y}_t) - h^1(0, \phi, \bar{\phi}) + \sum\limits_{0 \le t_k \le t} I_k(\bar{x}(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} \bar{x}(s) ds + \int_0^t f^1(s) d^\circ B^{H_1}(s) + \int_0^t \bar{g}^1(s) ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.12}$$

and

$$\bar{P}_2(t) = \begin{cases} \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad t \in [-r, 0], \\ \overline{\phi}(0) + h^2(t, \bar{x}_t, \bar{y}_t) - h^2(0, \phi, \bar{\phi}) + \sum\limits_{0 \le t_k \le t} \bar{I}_k(\bar{y}(t_k)) + \sum\limits_{i=1}^{n_*} \int_{-T_i}^{0} \bar{\phi}(s) ds \\ + \sum\limits_{i=1}^{n_*} \int_0^{t-T_i} \bar{y}(s) ds + \int_0^t f^2(s) d^\circ B^{H_2}(s) + \int_0^t \bar{g}^2(s) ds, \ , \quad t \in [0, b]. \end{cases} \tag{3.13}$$

This implies that

$$\begin{aligned} \mathbb{E}|P_1(t) - \overline{P}_1(t)|^2 &= \mathbb{E}\Big| h^1(t, x_t, y_t) + \sum_{0 \le t_k \le t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s) ds \\ &\quad + \int_0^t f^1(s) d^\circ B^{H_1}(s) + \int_0^t g^1(s) ds - h^1(t, \bar{x}_t, \bar{y}_t) \\ &- \sum_{0 \le t_k \le t} I_k(\bar{x}(t_k)) - \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s) ds - \sum_{i=1}^{n_*} \int_0^{t-T_i} \bar{x}(s) ds - \int_0^t f^1(s) d^\circ B^{H_1}(s) - \int_0^t \bar{g}^1(s) ds \Big|^2 \\ &\le 4\mathbb{E}\Big| h^1(t, x_t, y_t) - h^1(t, \bar{x}_t, \bar{y}_t) \Big|^2 \\ &\quad + 4m \sum_{k=1}^{m} \mathbb{E}\left| I_k(x(t_k)) - I_k(\bar{x}(t_k)) \right|^2 \\ &+ 4 \sum_{i=1}^{n_*} \int_0^{t-T_i} \mathbb{E}|x(s) - \bar{x}(s)|^2 ds + 4E \left| \int_0^t g^1(s) - \bar{g}^1(s) \right|^2. \end{aligned}$$

From Lemma 2.6 and thanks to (H_1)-(H_4), we obtain that for any $t \in [0, b]$

$$\begin{aligned}
&\mathbb{E}|P_1(t) - \overline{P}_1(t)|^2 \\
&\leq 4\alpha_1 \|x_t - \overline{x}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 5\beta_1 \|y_t - \overline{y}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&+4n_* \int_0^t E|x(s) - \bar{x}(s)|^2 ds + 4\int_0^t \Big(a_1(s)\|x_s - \overline{x}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b_1(s)\|y_s - \overline{y}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} \Big) ds \\
&\quad +4m \sum_{k=1}^{m} d_k |x(t_k) - \overline{x}(t_k)|^2
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}|P_1(t) - \overline{P}_1(t)|^2 \leq 4\alpha_1 \mathbb{E}|x(t) - \overline{x}(t)|^2 + 5\beta_1 \mathbb{E}|y(t) - \overline{y}(t)|^2 \\
&+4b \int_0^t a_1(s)\mathbb{E}|x(s) - \overline{x}(s)|^2 + 4b \int_0^t b_1(s)\mathbb{E}|y(s) - \overline{y}(s)|^2 ds \\
&\quad +4m \sum_{k=1}^{m} d_k \mathbb{E}|x(t_k) - \overline{x}(t_k)|^2 + 4n_* b \int_0^t \mathbb{E}\left|x(s) - \overline{x}(s)\right|^2 ds
\end{aligned}$$

and therefore, since $(x(s), y(s)) = (\overline{x}(s), \overline{y}(s))$ over the interval $[-r, 0]$, by taking supremum in the above inequality

$$\begin{aligned}
&\sup_{h\in[0,t]} \mathbb{E}|N_1(x(h), y(h)) - N_1(\overline{x}(h), \overline{y}(h))|^2 \\
&\leq \int_0^t \alpha(s) e^{\tau\widehat{\alpha}(s)} e^{-\tau\widehat{\alpha}(s)} \sup_{\theta\in[0,s]} \mathbb{E}|x(\theta) - \overline{x}(\theta)|^2 ds \\
&\qquad + \int_0^t \alpha(s) e^{\tau\widehat{\alpha}(s)} e^{-\tau\widehat{\alpha}(s)} \sup_{\theta\in[0,s]} \mathbb{E}|y(\theta) - \overline{y}(\theta)|^2 ds \\
&+4m \sum_{k=1}^{m} d_k e^{\tau\widehat{\alpha}(t)} e^{-\tau\widehat{\alpha}(t)} \mathbb{E}|x(t_k) - \overline{x}(t_k)|^2 \\
&\qquad +4\alpha_1 e^{\tau\widehat{\alpha}(t)} e^{-\tau\widehat{\alpha}(t)} \mathbb{E}|x(t) - \overline{x}(t)|^2 + 4\beta_1 e^{\tau\widehat{\alpha}(t)} e^{-\tau\widehat{\alpha}(t)} \mathbb{E}|y(t) - \overline{y}(t)|^2 \\
&\leq \int_0^t \alpha(s) e^{\tau\widehat{\alpha}(s)} ds \|x - \overline{x}\|_*^2 + \int_0^t \alpha(s) e^{\tau\widehat{\alpha}(s)} ds \|y - \overline{y}\|_*^2 \\
&\qquad +4m \sum_{k=1}^{m} d_k e^{\tau\widehat{\alpha}(s)} e^{-\tau\widehat{\alpha}(s)} \mathbb{E}|x(t_k) - \overline{x}(t_k)|^2 \\
&\qquad +4\alpha_1 e^{\tau\widehat{\alpha}(t)} e^{-\tau\widehat{\alpha}(t)} \mathbb{E}|x(t) - \overline{x}(t)|^2 + 4\alpha_1 e^{\tau\widehat{\alpha}(t)} e^{-\tau\widehat{\alpha}(t)} \mathbb{E}|y(t) - \overline{y}(t)|^2 \\
&\leq \frac{1}{\tau} \int_0^t (e^{\tau\widehat{\alpha}(s)})' ds \|x - \overline{x}\|_*^2 + \frac{1}{\tau} \int_0^t (e^{\tau\widehat{\alpha}(s)})' ds \|y - \overline{y}\|_*^2 \\
&\qquad +4m \sum_{k=1}^{m} d_k e^{\tau\widehat{\alpha}(t)} \|x - \overline{x}\|_*^2 + 4\alpha_1 e^{\tau\widehat{\alpha}(t)} \|x - \overline{x}\|_*^2 + 4\beta_1 e^{\tau\widehat{\alpha}(t)} \|y - \overline{y}\|_*^2 \\
&\leq \Big(\frac{1}{\tau} + 4m \sum_{k=1}^{m} d_k + 4\alpha_1\Big) e^{\tau\widehat{\alpha}(t)} \|x - \overline{x}\|_*^2 + \Big(\frac{1}{\tau} + 4\beta_1\Big) e^{\tau\widehat{\alpha}(t)} \|y - \overline{y}\|_*^2.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
e^{-\tau\widehat{\alpha}(t)} \sup_{h\in[0,t]} \mathbb{E}|N_1(x(h), y(h)) - N_1(\overline{x}(h), \overline{y}(h))|^2 \leq &\Big(\tfrac{1}{\tau} + 4m \sum_{k=1}^{m} d_k + 4\alpha_1\Big) \|x - \overline{x}\|_*^2 \\
&+ \Big(\tfrac{1}{\tau} + 4\beta_1\Big) \|y - \overline{y}\|_*^2,
\end{aligned}$$

where $\|x\|_*^2$ is the Bielecki-type norm on $\mathcal{D}_{\mathcal{F}_b}$ defined by

$$\|x\|_*^2 = \sup_{h\in[0,t]} \mathbb{E}|x(h, .)|^2 e^{-\tau\widehat{\alpha}(t)}.$$

where

$$\alpha(s) = \begin{cases} 0, & t \in [-r, 0], \\ \max\{4n_* b + 4b\alpha_1(s), 4b\beta_1(s)\}, & t \in [0, b] \end{cases}$$

and

$$\widehat{\alpha}(t) = \int_0^t \alpha(s)ds, \qquad t \in [-r, b].$$

Using the fact that for all $a, b \geq 0$ we have $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we conclude that

$$\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_* \leq \sqrt{\frac{1}{\tau} + 4m\sum_{k=1}^{m} d_k + 4\alpha_1}\|x-\overline{x}\|_* + \sqrt{\frac{1}{\tau} + 4\beta_1}\|y-\overline{y}\|_*. \tag{3.14}$$

Similar computations for N_1 yield

$$\|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \leq \sqrt{\frac{1}{\tau} + 4\alpha_2}\|x-\overline{x}\|_* + \sqrt{\frac{1}{\tau} + 4\beta_2 + 4m\sum_{k=1}^{m} \bar{d}_k}\|y-\overline{y}\|_*. \tag{3.15}$$

Now, (3.14), (3.15) can be put together and be rewritten as

$$\begin{aligned}\|N(x,y) - N(\overline{x},\overline{y})\|_* &= \begin{pmatrix} \|N_1((x,y) - N_1(\overline{x},\overline{y})\|_* \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \end{pmatrix} \\ &\leq \begin{pmatrix} \sqrt{\frac{1}{\tau} + 4m\sum_{k=1}^{m} d_k + 4\alpha_1} & \sqrt{\frac{1}{\tau} + 4\beta_1} \\ \sqrt{\frac{1}{\tau} + 4\alpha_2} & \sqrt{\frac{1}{\tau} + 4\beta_2 + 4m\sum_{k=1}^{m} \bar{d}_k} \end{pmatrix} \\ &\quad \begin{pmatrix} \|x-\overline{x}\|_* \\ \|y-\overline{y}\|_* \end{pmatrix}.\end{aligned}$$

Hence

$$\|N(x,y) - N(\overline{x},\overline{y})\|_* \leq M \begin{pmatrix} \|x-\overline{x}\|_* \\ \|y-\overline{y}\|_* \end{pmatrix},$$

where

$$M = \begin{pmatrix} \sqrt{\frac{1}{\tau} + 4m\sum_{k=1}^{m} d_k + 4\alpha_1} & \sqrt{\frac{1}{\tau} + 4\beta_1} \\ \sqrt{\frac{1}{\tau} + 4\alpha_2} & \sqrt{\frac{1}{\tau} + 4\beta_2 + 4m\sum_{k=1}^{m} \bar{d}_k} \end{pmatrix}.$$

We choose a suitable τ is sufficiently large such that the matrix

$$\|M\| < 1.$$

Then M is nonnegative, $I - M$ is nonsingular and

$$(I-M)^{-1} = 1 + M + M^2 + \dots$$

From Lemma 2.10, we obtain that M converges to zero . Thanks to Theorem 3.4, we can ensure that N has a fixed point (x, y), which is a solution to (1.1).

□

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