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# Magic conics, their integer points and complementary ellipses

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**Abstract** The aim of this paper is to introduce and study the class of conics provided by the symmetric matrices of the $3 \times 3$ magic squares. This class depends on three real parameters and various relationships between these parameters give special subclasses of conics. Although there are no magic circles we find an ellipse, a parabola and two hyperbolas of magic type. A search of integer points and a complex approach are also included. We study also a pair of complementary ellipses, called Pythagorean.

**Keywords** conic · magic square · integer point · self-complementary ellipse

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## 1 Introduction

After more than two thousand years the conics continue to be a versatile object of mathematics and the very recent book [16] is a concrete and illustrated proof of this fact. A lot of techniques, from analytical to those of projective geometry, are developed to handle these remarkable curves.

The starting points of this note are the articles [10] and [12] where symmetric Pythagorean triple preserving (PTP) matrices and respectively the adjoint representation of Lie group $SU(2)$ are used to generate conics. Hence, we continue this line of research with another class of symmetric matrices of order 3, namely the those produced by the magic $3 \times 3$ squares. These conics depend in an 1-homogeneous manner of three real parameters, denoted $a, b, c$, and we produce classes of magic conics by imposing natural constraints on these parameters. For example, although there are no magic circles since the eccentricity is strictly positive we obtain a parabola, an ellipse and two hyperbolas (one being Luoshu) by imposing different conditions on the given parameters $a$, $b$, $c$.

A special quest in our research concerns with integer (or lattice) points on a given magic conic. For many computations, we use the WolframAlpha soft which produces a lot of information for our particular conics. For the obtained magic parabola we get three sequences of integer points while the magic ellipse has only six such points. Another constant interest is in providing the canonical forms of these magic conics. Other remarkable magic conics are associated to the well-known Luoshu square and zero-sum magic square. An amazing result of our work is that the integer point $(-1, 0)$ belongs to all magic conics and then it provides a *rational parametrization* through stereographic projection.

In the second section we discuss a fixed magic conic $\Gamma$ in terms of its Hermitian coefficients. Also, we continue the algebraic study by discussing the nature of a binary quadratic form $F$ associated to the $\Gamma$. Every such hyperbolic binary quadratic form has a constant involving in the problem of harmonic decomposition (with a factor $F$) of polynomials. We compute this constant for our two magic hyperbolas. Also, we compute a Lorentz inner product (providing the determinant as quadratic form) of pairs of magic conics. Since our magic ellipse is an example of self-complemented ellipse, in the last section we discuss shortly the subject of complementary ellipses. We introduce the pair of Pythagorean complementary ellipses as an example and we study their arithmetic-geometric deformation.

## 2 Magic conics

In the setting of two-dimensional Euclidean space $(\mathbb{R}^2, g_{can} = diag(1, 1))$ let us consider the conic $\Gamma$ implicitly defined by $f \in C^\infty\left(\mathbb{R}^2\right)$ as:
$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ where $f$ is a quadratic function of the form $f(x, y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 > 0$.

The study of $\Gamma$ is based on the symmetric matrices ($e$ means extended):

$$\Gamma := \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} \in Sym(2), \quad \Gamma^e := \begin{pmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{pmatrix} \in Sym(3). \tag{2.1}$$

The algebraic invariants associated to $\Gamma$ are:

$$\begin{cases} I := r_{11} + r_{22} = Tr\Gamma, \quad \delta := \det \Gamma, \quad \Delta := \det \Gamma^e, \\ D := \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2. \end{cases} \tag{2.2}$$

It follows the necessity to search for remarkable symmetric matrices of order three.

Let us recall that a *magic square* is an $n \times n$ square grid $A$ of nonnegative real numbers such that the entries along any row, column and diagonal, all add up to the same value, denoted $\mathcal{C}(A)$. Fix $n = 3$ and look at the matrix version of these squares; it turns out any $3 \times 3$ magic square is a linear combination, with nonnegative coefficients, of only three of the following *basic* square matrices ([13, p. 419]):

$$A_0 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, A_0^t = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}. \tag{2.3}$$

Remark that $A_1$ and $A_2$ are already symmetric and:

$$A_0 + A_0^t = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = A_1 + A_2.$$

Allowing arbitrary real coefficients $a$, $b$, $c$ we arrive at the general symmetric $3 \times 3$ magic matrix:

$$\begin{cases} X := \frac{a}{2}(A_0 + A_0^t) + bA_1 + cA_2 = \begin{pmatrix} a+2c & a+2b & a+b+c \\ a+2b & a+b+c & a+2c \\ a+b+c & a+2c & a+2b \end{pmatrix}, \\ \mathcal{C}(X(a,b,c)) = 3(a+b+c). \end{cases} \tag{2.4}$$

This magic (which is also Hankel and left circulant) matrix $X$ yields the following class of conics:

**Definition 2.1** *A magic conic is a conic depending on $(a, b, c) \in \mathbb{R}^3$ in the form:*

$$\Gamma(a,b,c) : (a+2c)x^2+2(a+2b)xy+(a+b+c)y^2+2(a+b+c)x+2(a+2c)y+ \\ +(a+2b) = 0. \tag{2.5}$$

A straightforward computation yields:

**Proposition 2.2** *A magic conic has:*

$$r_{11}^2+r_{12}^2+r_{22}^2 = 3a^2+5b^2+5c^2+6a(b+c)+2bc = 3(a+b+c)^2+(b-c)^2. \tag{2.6}$$

*The invariants of $\Gamma(a,b,c)$ are:*

$$\delta(a,b,c) = 2c^2 - 4b^2 - 3ab + 3ac + 2bc = (c-b)(2c+4b+3a) = \\ = (c-b)[2(c-b) + 3(a+2b)], \tag{2.7}$$

$$I(a,b,c) = 2a+b+3c = 3(c-b)+2(a+2b), D(a,b,c) = -3(c-b)^2 \leq 0, \tag{2.8}$$

$$\Delta(a,b,c) = 5abc - 9b^3 + 3c^3 - 6a^2b + 6a^2c - 13ab^2 + 8ac^2 + 3b^2c + 3bc^2 = \\ = (c-b)(3c^2 + 9b^2 + 6a^2 + 13ab + 6bc + 8ca). \tag{2.9}$$

*$\Gamma$ is invariant to homogeneous scalings $(a,b,c) \to (\lambda a, \lambda b, \lambda c)$, $\lambda \neq 0$ and its eccentricity $e_\Gamma$ is:*

$$e_\Gamma^2 = \frac{2\sqrt{(c-b)^2 + 4(a+2b)^2}}{\sqrt{(c-b)^2 + 4(a+2b)^2} - sign(\Delta)(2a+b+3c)} \tag{2.10}$$

*where $sign(\Delta)$ is the signum of $\Delta(a,b,c)$. There are no magic circles since the strict positivity of $r_{11}^2 + r_{12}^2 + r_{22}^2$ implies the strict positivity of $e_\Gamma$.*

**Remark 2.1** i) In [17] a triple product $\{\cdots\}$ is considered on $Sym(3)$:

$$\{xyz\} := xyz + zyx - Tr(xy)z.$$

Our matrix $X(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ satisfies: $\{XXX\} = (18-9)X = 9X$ since its usual powers are $[X(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})]^k = 3^{k-1}X(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Also $\{A_1A_1A_1\} = X(2,2,11) = X(4,1,10)$ and $\{A_2A_2A_2\} = X(2,5,-1)$.
ii) The open problem (conform [7]) of possible $3\times3$ magic square containing 9 distinct square numbers connects the subjects of magic squares with the Pythagorean right triangles. The symmetric Pythagorean triple preserving matrices (PTPM on short) are considered in [10]:

$$Y(r,s,w) := \begin{pmatrix} r^2-2s^2+w^2 & 2s(r-w) & r^2-w^2 \\ 2s(r-w) & 2(rw+s^2) & 2s(r+w) \\ r^2-w^2 & 2s(r+w) & r^2+2s^2+w^2 \end{pmatrix}$$

The sum on first two columns is the same if and only if: $r^2 = rs + rw + sw + 2s^2$ while the sum on last two columns is the same if and only if: $r^2 = rs + rw - sw$ and then these two conditions yield: $s(s+w) = 0$. We have the PTPMs $Y(r,0,r) = 2r^2I_3$ and:

$$Y(r,r,-r) = 4r^2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y(r,-r,r) = 4r^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and hence a PTPM can not be magic but only semi-magic. Hence we have the PTPM-semi-magical conics: 1) the equilateral hyperbola $xy = -\frac{1}{2}$; 2) the double horizontal line $y = 1$; 3) the void conic $x^2 + y^2 + 1 = 0$.
iii) The point $M_1(-1,0)$ belongs to all magic conics since $r_{11} + r_{00} - 2r_{10} = 0$. The stereographic projection from $M_1$ yields the following *rational parametrization* for a magic conic:

$$\Gamma(a,b,c) : \begin{cases} x(t) = \frac{2(b-c)(2t-1)}{(a+b+c)t^2+2(a+2b)t+(a+2c)} - 1 \\ y(t) = \frac{2(b-c)t(2t-1)}{(a+b+c)t^2+2(a+2b)t+(a+2c)}, \quad t \in \mathbb{R}. \end{cases} \tag{2.11}$$

*Example 2.1* A degenerate case is $b = c$ for which:

$$\Gamma(a,b,b) : (a+2b)(x+y+1)^2 = 0 \tag{2.12}$$

and hence for $a + 2b \neq 0$ we have the double line:

$$l(a \neq -2b, b, b) : x + y + 1 = 0. \tag{2.13}$$

In the projective form $x + y + z = 0$ this line is naturally associated to a $SU(2)$-Barning hyperbola in [12]. Recall also that the general conic $\Gamma$ is called *symmetric* if $r_{11} = r_{22}$. For magical conics this condition is exactly the equality $b = c$.

A second special case is provided by the magic parabola $c = -\frac{3}{2}a - 2b$ (for $\delta = 0$) given by:

$$\Gamma(a,b) : -\frac{1}{2}(a+2b)(4x^2 - 4xy + y^2 + 2x + 8y - 2) = 0. \tag{2.14}$$

Then $a \neq -2b$ implies the equation of the magic parabola:

$$P : 4x^2 - 4xy + y^2 + 2x + 8y - 2 = 0. \tag{2.15}$$

The change of coordinates:

$$x = \frac{1}{\sqrt{5}}(x' - 2y'), \quad y = \frac{1}{\sqrt{5}}(2x' + y') \tag{2.16}$$

gives the canonical form:

$$P : \left(y' + \frac{2}{5\sqrt{5}}\right)^2 = -\frac{18}{5\sqrt{5}}\left(x' - \frac{3}{5\sqrt{5}}\right) \tag{2.17}$$

and hence the vertex is $V(x = \frac{7}{25}, y = \frac{4}{25})$ and the directrix is $d : y = -\frac{x}{2} + \frac{3}{4}$. With WolframAlpha we get three pairs of integer points:

$$\begin{cases} (x_n, y_n) = (1 - 18n^2, -2(18n^2 - 9n + 1)) \\ (X_n, Y_n) = (-18n^2 + 12n - 1, -6(6n^2 - 7n + 2)) \\ (\alpha_n, \beta_n) = (-18n^2 + 24n - 7, -6(6n^2 - 11n + 5)), \quad n \in \mathbb{Z}. \end{cases} \tag{2.18}$$

For example: $(x_1, y_1) = (-17, -20)$, $(X_1, Y_1) = (-7, -6)$ and $(\alpha_1, \beta_1) = (-1, 0) = M_1$. The rational parametrization of $P$ is:

$$P : x(t) = \frac{-t^2 - 8t + 2}{(t-2)^2}, \quad y(t) = \frac{6t(2t-1)}{(t-2)^2}, \quad t \in \mathbb{R} \setminus \{2\} \tag{2.19}$$

and $M_1(t = \frac{1}{2})$. Recall also that for 3 collinear points $a, b, m$ we can associate their simple ratio $(a, b; m) := \frac{m-a}{b-m}$. For our three sequences (2.18) we have:

$$(x_n, X_n; \alpha_n) = \frac{4(3n-1)}{3(-2n+1)}, \quad (y_n, Y_n; \beta_n) = \frac{2(-12n+7)}{3(4n-3)}$$

and then $\lim_{n\to\infty}(x_n, X_n; \alpha_n) = \lim_{n\to\infty}(y_n, Y_n; \beta_n) = -2$.

□

*Example 2.2* In duality with the case $(b - c = 0, a + 2b \neq 0)$ we study now the case $(b - c \neq 0, a + 2b = 0)$ which means the magic ellipse:

$$E(-2b, b, c) : 2x^2 + y^2 + 2x + 4y = 0 \tag{2.20}$$

with foci $F_1(-0.5,-3.5)$, $F_2=(-0.5,-0.5)$ and eccentricity $e_6=\frac{1}{\sqrt{2}}=0.707....$ The canonical form of this ellipse is:

$$E: \frac{(x+\frac{1}{2})^2}{\frac{9}{4}}+\frac{(y+2)^2}{\frac{9}{2}}-1=0. \tag{2.21}$$

We adopt the notation $E_6$ for this ellipse since it has 6 integer points:

$$E_6 \ni M_1, M_2(-2,-2), M_3(-1,-4), M_4(0,-4), M_5(0,0)=O, M_6(1,-2). \tag{2.22}$$

We point out also that in [16, p. 371] is giving another ellipse with 6 integer points:

$$\begin{cases} \tilde{E}_6: x^2-xy+y^2-1=0, \\ \tilde{M}_1=M_1, \tilde{M}_2(-1,-1), \tilde{M}_3(1,1), \tilde{M}_4(1,0), \tilde{M}_5(0,-1), \tilde{M}_6(0,1), \end{cases} \tag{2.23}$$

with eccentricity $\tilde{e}_6=0.816>e_6$. But $\tilde{E}_6$ is smallest from the length of semimajor axis point of view: $1.414<2.121=\frac{3}{\sqrt{2}}$ of $E_6$. The rational parametrization of $E_6$ is:

$$E_6: x(t)=-\frac{t(t+4)}{t^2+2}, \quad y(t)=\frac{2t(1-2t)}{t^2+2}, \quad t\in\mathbb{R} \tag{2.24}$$

and $M_1(t=\frac{1}{2})$, $M_2(t=2)$, $M_3(t\to+\infty)$, $M_4(t=-4)$, $M_5(t=0)$ and $M_6(t=-1)$.

In the book [14] is presented a consequence of the pigeonhole principle that 5 is the minimal number of integer points $M_i$ which assures that at least one of their midpoints is also integer. For our magic ellipse $E_6$ we have that the midpoints of $[M_1M_3]$, $[M_1M_6]$, $[M_2M_4]$, $[M_2M_5]$ and $[M_4M_5]$ are integers while for $\tilde{E}_6$ the same fact hold only for $[M_1\tilde{M}_4]$, $[\tilde{M}_2\tilde{M}_3]$ and $[\tilde{M}_5\tilde{M}_6]$.

In [8] there were founded infinitely many ellipses and hyperbolas on which there exist six points with integer co-ordinates $P_j(x_j;y_j); j=1,...,6$ such that the products $p_j=x_jy_j$ are in arithmetic progression. The magic ellipse $E_6$ and $\tilde{E}$ are not such ellipses since $p_1=0, p_2=4=p_3, p_4=0=p_5, p_6=-2$ respectively $\tilde{p}_1=p_1=0, \tilde{p}_2=1=\tilde{p}_3, \tilde{p}_4=0=\tilde{p}_5=\tilde{p}_6$.

We finish this example with the remark that the equation of $\tilde{E}_6$ is an instance (namely $z=1$) of the equation $x^2-xy+y^2=z^2$ whose solutions provide Eisenstein triangles $x,y,z$ having the angle $\angle Z=60^0$.

□

**Remark 2.2** In [21, p. 360] is defined a function, called *canonical conic function*, on the set of ellipses with the same eccentricity $e$ as:

$$C_{ellipse}(e):=\frac{1}{8(1-e^2)^2}\left[\pi\sqrt{1-e^2}-2e+2e^3-2\sqrt{1-e^2}\arcsin e\right]. \tag{2.25}$$

For our ellipse $E_6$ we obtain:

$$C_{ellipse}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi - 2}{4\sqrt{2}} = 0.2018069..., \tag{2.26}$$

which is a transcendental number.

In [11] a quaternion-inspired (but non-internal) product is considered on the set $(0, 1]$:

$$u_1 \odot_c u_2 = \sqrt{\frac{(1 - (u_1 u_2)^2}{u_1^2 + u_2^2}}. \tag{2.27}$$

The $\odot_c$-square of eccentricity $e_6 = \frac{1}{\sqrt{2}}$ is $\frac{\sqrt{3}}{2} \in (0, 1]$.

□

*Example 2.3* The well-known Luoshu magic matrix is [23, p. 132]:

$$L = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \to L + L^t = \begin{pmatrix} 8 & 12 & 10 \\ 12 & 10 & 8 \\ 10 & 8 & 12 \end{pmatrix} = X(4, 4, 2) = 2X(2, 2, 1). \tag{2.28}$$

We have the Luoshu conic:

$$\Gamma(2, 2, 1) := Lc : 4x^2 + 12xy + 5y^2 + 10x + 8y + 6 = 0 \tag{2.29}$$

which is a hyperbola with eccentricity $e_{Lc} = 1.069...$, $\delta(Lc) = -16$ and having the integer points:

$$Lc \ni M_1, \quad M_7(-7, 2), \quad M_8(1, -2) \tag{2.30}$$

as well as the rational parametrization:

$$Lc : x(t) = -\frac{5t^2 + 8t + 6}{5t^2 + 12t + 4}, y(t) = \frac{2t(2t - 1)}{5t^2 + 12t + 4}, t \in \mathbb{R} \setminus \{-\frac{2}{5}, -2\} \tag{2.31}$$

and $M_1(t = \frac{1}{2})$, $M_7(t = -\frac{1}{3})$ and $M_8(t = -1)$.

We remark here that the triple $(2, 2, 1)$ connects the Luoshu magic matrix and the classical (or Babylonian) right-triangle with legs $(3, 4, 5)$. Indeed, as is pointed out in [6], if the triple $(a, b, c)$ belongs to the cone:

$$Cone : 2bc = a^2 \tag{2.32}$$

then $x = b + a, y = c + a, z = a + b + c$ is a Pythagorean triple. We have that $(2, 2, 1) \in Cone$ and the associated Pythagorean triple is exactly $(4, 3, 5)$.

□

*Example 2.4* In [23] a magic square matrix $N$ with $\mathcal{C}(N) = 0$ is used in the parametrization of magic squares. Hence we consider the magic conic:

$$\Gamma(\mathcal{C} = 0) : x^2 - 2xy + 2y - 1 = 0 = (x - 1)(x - 2y + 1) \tag{2.33}$$

which is a pair of lines, concurrent exactly in the point $\tilde{M}_3$. The universal point $M_1$ belongs to the second line: $x - 2y + 1 = 0$.

□

## 3 A complex approach to magic conics

The aim of this section is to study the magic conic $\Gamma$ by using the complex structure of the plane. More precisely, with the usual notation $z = x+iy \in \mathbb{C}$ we derive the complex expression of a general conic $\Gamma$:

$$\Gamma : F(z,\bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{3.1}$$

with:

$$A = \frac{r_{11}-r_{22}}{4} - \frac{r_{12}}{2}i \in \mathbb{C}, 2B = r_{11}+r_{22} = I \in \mathbb{R}, C = r_{10}-r_{20}i \in \mathbb{C}. \tag{3.2}$$

It follows that the usual rotation performed to eliminate the mixed term $xy$ has the meaning to reduce/rotate $A$ in the real line while the translation which eliminates the term $y$ has a similar meaning with respect to $C$. The inverse relationship between $f$ and $F$ is:

$$r_{11} = B + 2\Re A, r_{22} = B - 2\Re A, r_{12} = -2\Im A, r_{10} = \Re C, r_{20} = -\Im C \tag{3.3}$$

with $\Re$ and $\Im$ respectively the real and the imaginary part. $\Gamma$ is a symmetric conic if and only if $\Re A = 0$ i.e. is a pure imaginary number.

The linear invariant $I$ and the quadratic invariant $\delta$ are respectively the trace and the determinant of the Hermitian matrix:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix} \tag{3.4}$$

which is a special one, the entries of the main diagonal being equal; hence their set is the three-dimensional subspace $Sym(2)$ of the four-dimensional real linear space $H(2)$ of $2 \times 2$ Hermitian matrices.

For our magic conic (2.5) we have the new coefficients, which we call *Hermitian*:

$$\begin{cases} A(a,b,c) = \frac{c-b}{4} - \frac{a+2b}{2}i, \quad B(a,b,c) = a + \frac{b+3c}{2}, \\ C(a,b,c) = a+b+c-(a+2c)i, \end{cases} \tag{3.5}$$

which are (real) 1-homogeneous functions of $(a,b,c)$ and satisfy the relation:

$$B = \Re(C + 2A). \tag{3.6}$$

The square of the eccentricity is:

$$e^2 = \frac{4|A|}{2|A| - Bsign(\Delta)} \tag{3.7}$$

where $|z|$ is the modulus of the complex number $z$. For example, the Luoshu Hermitian coefficients are:

$$A(Lc) = -\frac{1}{4} - 3i, \quad B(Lc) = \frac{9}{2}, \quad C(Lc) = 5 - 4i. \tag{3.8}$$

*Example 3.1* $C$ from (3.5) is a real number if and only if $a = -2c$ and thus we get the magic hyperbola:

$$H(-2c, b \neq c, c) : 4xy + y^2 + 2x + 2 = 0 \tag{3.9}$$

with eccentricity $e_H = \frac{\sqrt{17-\sqrt{17}}}{2\sqrt{2}} = 1.268.....$ We can denote $H_3$ since it contains 3 integer points:

$$H_3 \ni M_1, \quad M_8(1, -2), \quad M_9(-1, 4). \tag{3.10}$$

Its center is $C(\frac{1}{4}, -\frac{1}{2})$ and has $\delta(H_3) = -4$, $\Delta(H_3) = -9$. Its rational parametrization is:

$$H_3 : x(t) = -\frac{t^2 + 2}{t(t + 4)}, \quad y(t) = \frac{2t(2t - 1)}{t(t + 4)}, \quad t \in \mathbb{R} \setminus \{0, -4\} \tag{3.11}$$

and $M_1(t = \frac{1}{2})$, $M_9(t \to +\infty)$ and $M_8(t = -1)$. In conclusion, the Luoshu hyperbola and $H_3$ have two common integer points: $M_1$ and $M_8$. We remark here that the hyperbola $H_3$ have the integer points with arithmetic progression of products since $p_1 = 0, p_8 = -2, p_9 = -4$ but there are only three and not six as discussed in example 2.2.

The canonical conic function on the set of hyperbolas with the same eccentricity $e$ is ([21, p. 360]):

$$C_{hyperbola}(e) := \frac{1}{4(e^2 - 1)^2}\left[e^3 - e - \sqrt{e^2 - 1}\ln(e + \sqrt{e^2 - 1})\right] \tag{3.12}$$

and then:

$$\sqrt{2}(9 - \sqrt{17})^2 C_{hyperbola}(e_{H_3}) =$$

$$= (9 - \sqrt{17})\sqrt{17 - \sqrt{17}} - 4(\sqrt{34} - \sqrt{2})\ln\frac{\sqrt{34 - 2\sqrt{17}} + \sqrt{17} - 1}{4}. \tag{3.13}$$

□

*Example 3.2* We associate binary quadratic forms to our magic conics $P$, $E_6$ and $H_3$:

$$f_P(x, y) = (2x - y)^2, f_{E_6}(x, y) = 2x^2 + y^2, f_{H_3}(x, y) = 4xy + y^2. \tag{3.14}$$

Their complex variant is:

$$\begin{cases} F_P(z, \bar{z}) = \frac{1}{4}(3z^2 + 10z\bar{z} + 3\bar{z}^2) + i(z^2 - \bar{z}^2), \\ F_{E_6}(z, \bar{z}) = \frac{1}{4}(z^2 + 6z\bar{z} + \bar{z}^2), \quad F_{H_3}(z, \bar{z}) = -\frac{1}{4}(z - \bar{z})^2 + i(\bar{z}^2 - z^2) \end{cases} \tag{3.15}$$

and hence:

$$\begin{cases} A(P) = \frac{3}{4} + i \in \mathbb{C}, \quad B(P) = \frac{5}{2}, \\ A(E_6) = \frac{1}{4} \in \mathbb{R}, \quad B(E_6) = \frac{3}{2}, \\ A(H_3) = -\frac{1}{4} - i \in \mathbb{C}, B(H_3) = \frac{1}{2}. \end{cases} \tag{3.16}$$

The natural numbers properly represented by the quadratic form $f_{E_6}$ are studied in Exercise 2.5.4. from page 85 and Corollary 7.1.2. from page 200 of the book [18].

$\square$

**Remark 3.1** From $\delta = \det \Gamma = \det \Gamma^c = B^2 - 4|A|^2$ it follows that the determinant is a binary quadratic form on $Sym(2)$ associated to the Lorentzian inner product:

$$< \Gamma^c, \tilde{\Gamma}^c >= B\tilde{B} - 4(\Re A \Re \tilde{A} + \Im A \Im \tilde{A}) = \frac{1}{2}(r_{11}\tilde{r}_{22} + \tilde{r}_{11}r_{22}) - r_{12}\tilde{r}_{12}.$$

But also we can consider a bracket on the space of binary quadratic forms:

$$\{f_1, f_2\} = B_1 B_2 - 4|A_1||A_2| \tag{3.17}$$

which is only commutative and positive homogeneous: $\{\lambda f_1, f_2\} = \lambda\{f_1, f_2\}$, for $\lambda \geq 0$. It is not a bilinear form but gives $\{f, f\} = \det f$. For example, if $f_{S^1} = x^2 + y^2 = z\bar{z}$ is the circle bilinear form then the subspace of forms $\tilde{f}$ satisfying $\{f_{S^1}, \tilde{f}\} = 0$ is given by the traceless forms $\tilde{f} = r_{11}(x^2 - y^2) + 2r_{12}xy$. We have for our binary quadratic forms:

$$\{f_P, f_{E_6}\} = \frac{35}{16}, \quad \{f_{E_6}, f_{H_3}\} = \frac{7}{4}, \quad \{f_{H_3}, f_P\} = \frac{345}{64}. \tag{3.18}$$

The null cone of this bracket, which is not a linear subspace, is provided by the Hermitian matrices:

$$\Gamma^c = \Gamma^c(A) : B = 2|A|$$

and then we arrive at the map: $Null : \mathbb{C} = \mathbb{R}^2 \to H(2) = \mathbb{R}^4$:

$$Null(z) = \begin{pmatrix} |z| & \bar{z} \\ z & |z| \end{pmatrix}. \tag{3.19}$$

The canonical basis of $H(2)$ is $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ with $\sigma_i, i = 1, 2, 3$ the Pauli matrices and hence:

$$Null(z) = |z|I_2 + (\Re z)\sigma_1 + (\Im z)\sigma_2. \tag{3.20}$$

In other words, we have the map $Null : \mathbb{R}^2 \to \mathbb{R}^3 \subset \mathbb{R}^4$:

$$Null(x, y) = (\sqrt{x^2 + y^2}, x, y, 0) \tag{3.21}$$

and its image is the elliptic cone:

$$(x^1)^2 = (x^2)^2 + (x^3)^2. \tag{3.22}$$

Consider the linear map $Inv_{13} : \mathbb{R}^4 \to \mathbb{R}^4$, $(x^1, x^2, x^3, x^4) \to (x^3, x^2, x^1, x^4)$, the map $Proj_1 : \mathbb{R}^4 \setminus \{x^1 = 0\} \to \mathbb{R}^4$, $(x^1, x^2, x^3, x^4) \to \left(1, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \frac{x^4}{x^1}\right)$ and the composition $\frac{1}{2} Inv_{13} \circ Proj_1 \circ Null : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^4$:

$$\frac{1}{2} Inv_{13} \circ Proj_1 \circ Null(x, y) = \frac{1}{2}\left(\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 1, 0\right) \in S^3.$$

We recall the Hopf bundle $\pi : S^3(1) \to S^2(\frac{1}{2}) \subset \mathbb{R} \oplus \mathbb{C}$ where, by using the complex numbers:

$$\pi(z, w) = \left(\frac{1}{2}(|w|^2 - |z|^2), z\bar{w}\right).$$

Finally we get the map from $\mathbb{C}^*$ to $S^1\left(\frac{1}{2}\right)$:

$$\pi \circ \frac{1}{2} Inv_{13} \circ Proj_1 \circ Null(x, y) = \left(0, \frac{i\bar{z}}{2|z|}\right) \in \{0\} \oplus S^1\left(\frac{1}{2}\right). \tag{3.23}$$

**Remark 3.2** Recall after [10, p. 91] that the binary quadratic form:

$$F(x, y) = ux^2 + 2vxy + wy^2$$

has *the discriminant* $\Delta(F) := 4(v^2 - uw)$ and is called *indefinite* if $\Delta(F) > 0$. Also, an indefinite quadratic form is called *reduced* if:

$$|\sqrt{\Delta(F)} - 2|u|| < 2v < \sqrt{\Delta(F)}. \tag{3.24}$$

We have:

$$\Delta(P) = 0, \quad \Delta(E_6) = -8, \quad \Delta(H_3) = 16 \tag{3.25}$$

and then only $H_3$ is indefinite but is not reduced since all parts of inequality (3.24) are 4. But for $\Delta = 16$ we get another binary quadratic form, called *principal*:

$$F^{H_3}(x, y) := x^2 - \frac{\Delta}{4}y^2 = x^2 - 4y^2 \tag{3.26}$$

and recall that the negative Pell equation $x^2 - 4y^2 = -1$ has no integer solutions; the simplest proof of this fact was offered us by the reviewer and is based on the decomposition $(x+2y)(x-2y) = -1$ implying the impossible integer solutions of the system $(x = 0, 4y^2 = 1)$. The same results hold for the Luoshu quadratic form $4x^2 + 12xy + 5y^2$ with $\Delta(Lc) = 4 \cdot 16 = 64$.

With the discussion from the previous remark and relations (2.3) it follows that the quadratic part of any parabola is provided by a complex number $A$ through the formula:

$$\begin{cases} F_A(x, y) = (|A| + \Re A)x^2 - 2(\Im A)xy + (|A| - \Re A)y^2 = (\alpha x + \beta y)^2, \\ A = u + iv \in \mathbb{C} \\ \sqrt{2}\alpha = \sqrt{\sqrt{u^2+v^2} - v} + \sqrt{\sqrt{u^2+v^2} + v}, \\ \sqrt{2}\beta = \sqrt{\sqrt{u^2+v^2} - v} - \sqrt{\sqrt{u^2+v^2} + v}. \end{cases} \tag{3.27}$$

For example, if $\alpha \in \mathbb{R}$ then:

$$F_{e^{-\alpha i}}(x,y) = 2\left(\cos\frac{\alpha}{2}\cdot x + \sin\frac{\alpha}{2}\cdot y\right)^2.$$

**Remark 3.3** For a hyperbolic quadratic binary form $F$ in [3, p. 26] is introduced a constant as test for possible harmonic decomposition. Hence let us call it *the hyperbolic harmonic decomposition* constant of $F$ and denote $HHD(F)$:

$$HHD(F) := \frac{u+w+2i\sqrt{v^2-uw}}{u+w-2i\sqrt{v^2-uw}} \tag{3.28}$$

and the harmonic decomposition holds with a factor $F$ if and only if $HHD(F)$ is not a complex root of unity. By introducing the non-zero complex number $hhd(F) = u+w+2i\sqrt{v^2-uw}$ we have:

$$HHD(F) = \left(\frac{hhd(F)}{|hhd(F)|}\right)^2. \tag{3.29}$$

Remark that the right-hand side is the square of a number from $S^1$ and hence there exists $t = t(F)$ such that $\frac{hhd(F)}{|hhd(F)|} = e^{it}$ and hence $HHD(F) = e^{2ti}$. For our magic hyperbolas we get:

$$\begin{cases} hhd(Lc) = 9+8i, \quad HHD(Lc) = \frac{1}{145}(17+144i), \\ hhd(H_3) = 1+4i, \quad HHD(H_3) = \frac{1}{17}(-15+8i). \end{cases} \tag{3.30}$$

Hence the Lusho magic hyperbola has the Pythagorean triple $(17, 144, 145)$ while the magic hyperbola $H_3$ has the Pythagorean triple $(15, 8, 17)$. A canonical hyperbola $H(a,b) : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ has:

$$HHD(H(a,b)) = \left[\frac{b^2-a^2+2abi}{a^2+b^2}\right]^2 = \left[\frac{b+ai}{\sqrt{b^2+a^2}}\right]^4. \tag{3.31}$$

The last relation makes possible the extension of the $HHD$ constant to arbitrary families of hyperbolas $H(e)$ with constant eccentricity $e$ as:

$$HHD(H(e)) = \left[\frac{\sqrt{e^2-1}+i}{e}\right]^4. \tag{3.32}$$

For example, in [10, p. 87] a Barning hyperbola is introduced with the eccentricity $e = \frac{2}{\sqrt{3}}$ and then its HHD constant is:

$$HHD(Barning) = \left[\frac{1+\sqrt{3}i}{2}\right]^4 = (e^{i\frac{\pi}{3}})^4 = e^{i\frac{4\pi}{3}}$$

which is a square root of unity of order 3.

In the Theorem of [15, p. 141] the eccentricity of a given conic $\Gamma$ is obtained as the tangent of a remarkable angle. The formula above suggest to consider two cases:
i) if $\Gamma$ is a hyperbola then we suppose $e = \frac{1}{\sin\varphi}$ with $\varphi \in (0, \frac{\pi}{2})$. Then the angle between the asymptotics of $\Gamma$ is $\pi - 2\varphi$ and the canonical function is:

$$C_{hyperbola}(e = \frac{1}{\sin\varphi}) = \frac{\sin^3\varphi}{4\cos^3\varphi}\left[\frac{\cos\varphi}{\sin^2\varphi} - \ln\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}\right]. \tag{3.33}$$

For the Barning hyperbola the angle is $\varphi = \frac{\pi}{3} = \pi - 2\varphi$. Concerning the equilateral hyperbolas we have $\varphi = \frac{\pi}{4}$ and:

$$C_{hyperbola}(\sqrt{2}) = \frac{1}{4}[\sqrt{2} - \ln(\sqrt{2}+1)] = 0.13320.....$$

which is a transcendental number.
ii) if $\Gamma$ is an ellipse then we suppose $e = \sin\varphi$ again with $\varphi \in (0, \frac{\pi}{2})$. The canonical function is:

$$C_{ellipse}(e = \sin\varphi) = \frac{\pi - 2\varphi - \sin 2\varphi}{8\cos^3\varphi}. \tag{3.34}$$

The ellipse $E_6$ has $\varphi = \frac{\pi}{4}$.

Returning to the formula (3.32) we remark that the constant $HHH(H(e))$ is the fourth power of the complex number $z(e) := \frac{\sqrt{e^2-1}+i}{e}$. This complex number belongs to the upper half-plane model of hyperbolic geometry $\mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$ endowed with the Riemannian metric given by Poincaré:

$$g(x,y) = \frac{dx^2 + dy^2}{y^2} = \left(\frac{|dz|}{y}\right)^2 \tag{3.35}$$

The geodesic polar coordinates $(\rho, \varphi) \in (0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on the Poincaré upper half-plane $(\mathbb{C}_+, g)$ are given by $\rho := dist_g(i, z)$, $\varphi := \arctan\frac{x^2+y^2-1}{2x}$ and then:

$$x = \frac{\sinh\rho\cos\varphi}{\cosh\rho - \sinh\rho\sin\varphi}, \quad y = \frac{1}{\cosh\rho - \sinh\rho\sin\varphi}. \tag{3.36}$$

The angle $\varphi(e)$ of our $z(e)$ is zero and then $\cosh\rho = e$, $\sinh\rho = \sqrt{e^2-1}$. It follows a new formula for the canonical conic function in terms of $\rho$:

$$C_{hyperbola}(e = \cosh\rho) = \frac{\cosh\rho\sinh\rho - \ln(e^{\rho} + e^{-\rho})}{4\sinh^3\rho}. \tag{3.37}$$

This case $e = \cosh\rho$ is connected to the case i) above through the Lobachevsky's *angle of parallelism function* $\Pi$ defined by: $\sin\Pi(x) = \frac{1}{\cosh x}$. Hence, the angle $\varphi$ of i) is exactly the angle of parallelism $\Pi(\rho)$ and (2.37) is exactly (3.33). The formulae (3.36) becomes:

$$x = \frac{\cos \Pi(\rho) \cos \varphi}{1 - \cos \Pi(\rho) \sin \varphi}, \quad y = \frac{\sin \Pi(\rho)}{1 - \cos \Pi(\rho) \sin \varphi}. \tag{3.38}$$

For an equilateral hyperbola we have $\cosh \rho = e = \sqrt{2}$, $\sinh \rho = 1$, $\Pi(\rho) = \frac{\pi}{4}$ and then:

$$x = \frac{\cos \varphi}{\sqrt{2} - \sin \varphi}, \quad y = \frac{1}{\sqrt{2} - \sin \varphi}. \tag{3.39}$$

## 4 Complementary ellipses

For two real numbers $a > b > 0$ let $E(a, b)$ be the standard ellipse:

$$E(a, b) : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

and recall its eccentricity:

$$e = e(E) = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

An important class of ellipses is introduced in [2, p. 11]:

**Definition 4.1** *The ellipses $E(e)$, $\tilde{E}(\tilde{e})$ are called complementary if $(e, \tilde{e}) \in S^1$ i.e.:*

$$e^2 + \tilde{e}^2 = 1. \tag{4.1}$$

*Hence $E$ is self-complementary if its eccentricity is $e = \frac{1}{\sqrt{2}}$.*

**Remark 4.1** i) A picture of a confocal self-complementary ellipse and lemniscate is Figure 2 of [1, p. 1098].
ii) To the given ellipse $E(a, b)$ we can associate the hyperbola:

$$H(a, b) : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

with eccentricity:

$$e = e(H) = \frac{\sqrt{a^2 + b^2}}{a}.$$

Hence $e^2(E(a, b)) + e^2(H(a, b)) = 2$ and then we can define the complementarity of two hyperbolas $H(e)$, $\tilde{H}(\tilde{e})$ (associated to ellipses with horizontal major axis) through:

$$e^2 + \tilde{e}^2 = 3.$$

Moreover, we can generalize the notion of complementarity for two nondegenerate conics of the same type through:

$$e^2 + \tilde{e}^2 = 2 + sign(\Delta).$$

For example, the complementary hyperbola $\tilde{H}_3$ of our $H_3$ has the eccentricity:

$$\tilde{e}^2 = 3 - \frac{17 - \sqrt{17}}{8} = \frac{7 + \sqrt{17}}{8}.$$

The canonical conic function and the $HHD$ constant of a self-complementary hyperbola $H(e = \sqrt{\frac{3}{2}} = 1.2247...)$ are:

$$\begin{cases} C_{hyperbola}\left(\sqrt{\frac{3}{2}}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \ln \frac{\sqrt{3}+1}{\sqrt{2}}, \\ HHD(H\left(\sqrt{\frac{3}{2}}\right)) = \left[\frac{1+i\sqrt{2}}{\sqrt{3}}\right]^4 = -\frac{7+4\sqrt{2}i}{9}. \end{cases} \tag{4.2}$$

The minimal polynomial of the complex number from the squares of $HHD$ is $P(x) = 3x^4 + 2x^2 + 3$ and the corresponding angle of parallelism is $\varphi = 54.74°$.

All integer solutions of the Pell equation provided by the generic self-complemented hyperbola:

$$H_2 : x^2 - 2y^2 = 1$$

are given by $x_n + y_n\sqrt{2} = \pm(3 + 2\sqrt{2})^n$ i.e., conform the equations (3.2.6) of [4, p. 36]:

$$x_n = \pm\frac{(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n}{2}, y_n = \pm\frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{2\sqrt{2}}, n \geq 0$$

with $(x_0, y_0) = \pm(1, 0)$ and $(x_1, y_1) = \pm(3, 2)$. The signs correspond to the two branches of the hyperbola $H_2$.

iii) Suppose $E$ and $\hat{E}(\hat{a}, \hat{b})$ are confocal ellipses; hence:

$$\hat{a}^2 = a^2 - \lambda, \quad \hat{b}^2 = b^2 - \lambda$$

for a real $\lambda$. Imposing $\hat{E}$ be the complementary ellipse to $E$ i.e. $\hat{e} = \tilde{e}$ gives the relations:

$$\lambda = 2a^2 - \frac{a^4}{b^2} = a^2\left[1 - \left(\frac{e}{\tilde{e}}\right)^2\right], \quad \hat{a} = a\frac{e}{\tilde{e}}, \quad \hat{b} = a\frac{e^2}{\tilde{e}}. \tag{4.3}$$

The area bounded by the complementary confocal ellipses is:

$$\mathcal{A} = \mathcal{A}(E) - \mathcal{A}(\hat{E}) = \pi ab\left[1 - \left(\frac{e}{\tilde{e}}\right)^3\right]. \tag{4.4}$$

iv) The expression of the complementary condition in terms of Hermitian coefficients is:

$$\frac{4|A|}{2|A|+B}+\frac{4|\tilde{A}|}{2|\tilde{A}|+\tilde{B}}=1. \tag{4.5}$$

v) The $\odot_c$-product of the complementary eccentricities of ellipses is:

$$e\odot_c\tilde{e}=\sqrt{1-(e\tilde{e})^2}, \quad e^2_{\odot_c}=\frac{\tilde{e}}{e}\sqrt{\frac{1+e^2}{2}}. \tag{4.6}$$

In the paper [20] it is proved that *the regular refraction interval* for a conic is $\left[0, \frac{\sqrt{1+e^2}}{e}\right]$ in terms of its eccentricity. Hence, if we denote by $L(rri;e)=\frac{\sqrt{1+e^2}}{e}$ the upper bound of this interval it follows that:

$$e^2_{\odot_c}=\frac{\tilde{e}}{\sqrt{2}}L(rri;e)$$

and that for self-complementary ellipses we have $L(rri;1/\sqrt{2})=\sqrt{3}$.
vi) Recall after [9, p. 16] that the motion in Newton's gravitational potential is governed by the effective potential:

$$V_{eff}(r):=V(r)+\frac{L^2}{2r^2}=-\frac{M}{r}+\frac{L^2}{2r^2}$$

and that the bounded motions are ellipses with the eccentricity $e=\frac{kL^2}{M}$ for a constant $k\geq 0$. For $L>0$ and $k>0$ let us call *Kepler ellipses* these bounded trajectories. Hence for a self-complementary Kepler ellipse we have $M=\sqrt{2}kL^2$ and the effective potential is:

$$V_{eff}(r)=\frac{L^2}{2r^2}(1-2\sqrt{2}kr).$$

vii) In [5] is presented a method to obtain ellipses by using the Joukowski map $J:\mathbb{C}^*\to\mathbb{C}$, $J(z):=z+\frac{1}{z}$. Namely, the circle of radius $r>1$ is transformed into the ellipse $E(r)$ with $a=r+\frac{1}{r}$ and $b=r-\frac{1}{r}$. Since its eccentricity is $e(r)=\frac{2r}{r^2+1}$ we get that $E(r)$ is self-complementary only for $r=\sqrt{2}+1$ which gives $a=2\sqrt{2}$ and $b=2$.
viii) At the page 76 of the book [22] the perimeter of the ellipse $E(a,b)$ is computed by means of an elliptic integral involving the following functions of $\lambda:=1-e^2=\tilde{e}^2$:

$$g_2(\lambda):=\frac{\sqrt[3]{4}}{3}(\lambda^2+\lambda-1), \quad g_3(\lambda):=-\frac{1}{27}(2\lambda^3+3\lambda^2-3\lambda-2).$$

Hence, this Weierstrass-type invariants can be expressed directly in terms of $e$ as:

$$g_2(e)=\frac{\sqrt[3]{4}}{3}(e^4-3e^2+1), \quad g_3(e)=\frac{1}{27}(2e^6-9e^4+3e^2)$$

and therefore, a self-complementary ellipse has:

$$g_2(\frac{1}{\sqrt{2}}) = -\frac{\sqrt[3]{4}}{12}, \quad g_3(\frac{1}{\sqrt{2}}) = \frac{5}{54}.$$

The associated depressed cubic equation $4y^3 - g_2(\frac{1}{\sqrt{2}}) - g_3(\frac{1}{\sqrt{2}}) = 0$ has the form:

$$y^3 + \frac{\sqrt[3]{4}}{48}y - \frac{5}{216} = 0.$$

The only possible zero of the function $g_2$ is $e = \frac{\sqrt{5}-1}{2} = \tilde{e}^2$ and the equation:

$$e^4 - 3e^2 + (1 - \frac{3g_2}{\sqrt[3]{4}}) = 0$$

considered as a quadratic equation in $e^2$ has the discriminant: $\varDelta = 5 + \frac{12g_2}{\sqrt[3]{4}}$ and hence we suppose $g_2 \geq -\frac{5\sqrt[3]{4}}{12}$. In fact, we consider now:

$$e^2 = \frac{3 - \sqrt{\varDelta}}{2}$$

but the condition $e^2 \in (0,1)$ gives that $g_2 > -\frac{\sqrt[3]{4}}{3} > -\frac{5\sqrt[3]{4}}{12}$ and finally:

$$e = \sqrt{\frac{3 - \sqrt{\varDelta}}{2}} \in (0,1).$$

□

The motivation for the present section consists in the fact that both [2] and [1] do not provide examples of (remarkable) self-complementary ellipses although their general form is $E(a, b = \frac{a}{\sqrt{2}})$. But our magic $E_6$ is such a desired example. So, we include here some direct consequences of Definition 3.1. For example, the canonical function (2.25) becomes:

$$C_{ellipse}(e) = \frac{\pi - 2e\tilde{e} - 2\arcsin e}{(2\tilde{e})^3} \tag{4.7}$$

and then:

$$(2\tilde{e})^3 C_{ellipse}(e) + 2\arcsin e = (2e)^3 C_{ellipse}(\tilde{e}) + 2\arcsin \tilde{e}. \tag{4.8}$$

Following a construction of [2, p. 11] for a given ellipse $E(a, b; e)$ we can associate at least other three ellipses:
i) the arithmetic-geometric $E_{a,g}(m_a, m_g; e_{a,g})$,
ii) the arithmetic-harmonic $E_{a,h}(m_a, m_h; e_{a,h})$,
iii) the geometric-harmonic $E(g, h)(m_g, m_h; e_{g,h})$,
where, as usual, $m_a = m_a(a, b) = \frac{a+b}{2}$ is the arithmetic mean of $a$ and $b$,

$m_g = m_g(a,b) = \sqrt{ab}$ is the geometric mean of $a$ and $b$ an respectively $m_h = m_h(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$ is the harmonic mean of $a$ and $b$. In the cited book only the first new ellipse is given with:

$$e_{a,g} = \frac{a-b}{a+b} = \frac{1-\tilde{e}}{1+\tilde{e}} \tag{4.9}$$

which means that $e_{a,g} = f_{a,g}(e)$ for $f_{a,g} : [0,1) \to [0,1)$:

$$f_{a,g}(x) = \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \left[\frac{1-\sqrt{1-x^2}}{x}\right]^2 = \left[\tan\left(\frac{\arcsin x}{2}\right)\right]^2,$$

$$f_{a,g}\left(\frac{1}{\sqrt{2}}\right) = (\sqrt{2}-1)^2. \tag{4.10}$$

We have the identity $E_{a,g} = \tilde{E}$ if and only if $\tilde{e} = \sqrt{2}-1$; equivalently $e = \sqrt{2(\sqrt{2}-1)} = 0.9101$ which has the minimal polynomial $P(x) = x^4 + 4x^2 - 4$. In a similar manner we have:

$$e_{a,h} = \frac{(1-\tilde{e})\sqrt{1+5\tilde{e}}}{(1+\tilde{e})^{\frac{3}{2}}}, \quad e_{g,h} = e_{a,g}. \tag{4.11}$$

With WolframAlpha we obtain the identity $E_{a,h} = \tilde{E}$ only for $\tilde{e} = 0.5042....$ The canonical function for the eccentricity $e_{a,g}$ is:

$$C_{ellipse}\left(\frac{1-\tilde{e}}{1+\tilde{e}}\right) = \frac{(1+\tilde{e})^3}{64\tilde{e}^{\frac{3}{2}}}\left[\pi - \frac{4\sqrt{\tilde{e}}(1-\tilde{e})}{(1+\tilde{e})^2} - 2\arcsin\frac{1-\tilde{e}}{1+\tilde{e}}\right]. \tag{4.12}$$

Since we know the parametrization of unit circle $S^1$ a class of complementary ellipses is given by:

**Definition 4.2** *For $m, n \in \mathbb{N}^*$ with $m > n$ we define the Pythagorean complementary ellipses $E(m^2+n^2, 2mn; e = \frac{m^2-n^2}{m^2+n^2})$ and $\tilde{E}(m^2+n^2, m^2-n^2; \tilde{e} = \frac{2mn}{m^2+n^2})$.*

The ellipse confocal and complementary to $E$ is
$\hat{E}(\hat{a} = \frac{m^4-n^4}{2mn}, \hat{b} = \frac{(m^2-n^2)^2}{2mn}; \tilde{e})$ and the area bounded by $E$ and $\hat{E}$ is:

$$\mathcal{A}(m,n) = 2mn(m^2+n^2)\pi\left[1-\left(\frac{m^2-n^2}{2mn}\right)^3\right]. \tag{4.13}$$

For the associated arithmetic-geometric ellipse the eccentricity is:

$$f_{a,g}(e) = \left(\frac{m-n}{m+n}\right)^2, \quad f_{a,g}(\tilde{e}) = \left(\frac{n}{m}\right)^2. \tag{4.14}$$

The first Hermitian coefficient for the ellipse $E(a,b;e)$ is real and satisfies:

$$A(E(a,b;e)) = \frac{1}{2}B(H(a,b)) = \frac{1}{4}\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = -\left(\frac{e}{2b}\right)^2 < 0 \qquad (4.15)$$

and hence:

$$A(E)\cdot A(\tilde{E}) = \frac{1}{16a^4}, \quad < f_E, f_{\tilde{E}} >= \frac{1}{2(bc)^2}. \qquad (4.16)$$

The right triangle $\Delta ABC$ with $a$ as hypothenuse and $b,c$ as legs has the area $S(ABC)=\frac{bc}{2}$ and hence the above inner product is:

$$< f_E, f_{\tilde{E}} >= \frac{1}{8[S(ABC)]^2}. \qquad (4.17)$$

The Pythagorean complementary ellipses have $< f_E, f_{\tilde{E}} >= \frac{1}{8[mn(m^2-n^2)]^2}$ and:

$$A(E) = -\left(\frac{m^2-n^2}{4mn(m^2+n^2)}\right)^2, A(\tilde{E}) = -\left(\frac{mn}{m^4-n^4}\right)^2. \qquad (4.18)$$

*Example 4.1* For $m=2>n=1$ we have:

$$\begin{cases} E = E\left(5,4;e=\frac{3}{5}\right), \quad \tilde{E} = \tilde{E}\left(5,3;\tilde{e}=\frac{4}{5}\right), \\ \hat{E}\left(\frac{15}{4},\frac{9}{4}\right), \quad E_{a,g} = E_{a,g}\left(\frac{9}{2},2\sqrt{5};e_{a,g}=\frac{1}{9}\right), \\ A(E) = -\left(\frac{3}{40}\right)^2, \quad A(\tilde{E}) = -\left(\frac{2}{15}\right)^2, \quad \mathcal{A}(2,1) = \frac{185}{16}\pi. \end{cases} \qquad (4.19)$$

Also $\frac{3}{5}\odot_c\frac{4}{5} = \frac{\sqrt{37\cdot 13}}{25}$, $< f_E, f_{\tilde{E}} >= \frac{1}{288}$ and:

$$C_{ellipse}\left(\frac{3}{5}\right) = \frac{5}{8^3}[25\pi - 24 - 50\arcsin\frac{3}{5}],$$

$$C_{ellipse}\left(\frac{4}{5}\right) = \frac{5}{6^3}[25\pi - 24 - 50\arcsin\frac{4}{5}].$$

A final example of self-complementary (and symmetric) ellipse appears in [19, p. 214]:

$$E: 3x^2+2xy+3y^2+6x-2y-5=0 \qquad (4.20)$$

having two integer points: $(-3,2)$ and $(0,-1)$.

□

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