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Result on an Integral function of an Integral function represented by Dirichlet series

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Abstract This paper deals in exhibiting a property for which type (R) of an integral function of an integral function represented by Dirichlet series for a finite order (R) is finite.

Keywords Dirichlet series · entire function

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1 Introduction

Consider a sequence $\{a_n\}$ of complex numbers and a strictly increasing sequence $\{\lambda_n\}$ of real numbers such that $\lambda_n \to \infty$ as n $\to \infty$ with $\lambda_1 > 0$. Define a function $g : \mathbb{C} \to \mathbb{C}$ such that

$$g(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}\,;\ s = \sigma + it\ (\sigma, t \in \mathbb{R}) \tag{1.1}$$

satisfying

$$\limsup_{n\to\infty} \frac{\log \lambda_n}{n} < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

We further define

$$f(s) = \sum_{n=1}^{\infty} b_n e^{ns}$$

satisfying

$$\limsup_{n\to\infty} \frac{\log |b_n|}{n} = -\infty.$$

From [1], f(s) and g(s) denote entire functions represented by Dirichlet series in one complex variable 's'.
Clearly domain of absolute convergence and convergence of $f(s)$ coincide with each other.
The maximum modulus of the entire functions f(s) and g(s) are defined as

$$F(\sigma) = \sup_{|t|<\infty} |f(\sigma + it)|$$

and

$$G(\sigma) = \sup_{|t|<\infty} |g(\sigma + it)|$$

respectively.
Ritt [2] defined the order (R) of an entire function g(s) as

$$\rho_g = \limsup_{\sigma\to\infty} \frac{\log\log G(\sigma)}{\sigma} \quad (0 \le \rho_g \le \infty)$$

For $0 < \rho_g < \infty$, type (R) 'τ_g' of g(s) is defined as

$$\tau_g = \limsup_{\sigma\to\infty} \frac{\log G(\sigma)}{e^{\sigma\rho_g}} \quad (0 \le \tau_g \le \infty)$$

We put

$$m(\sigma, g) = max\{|a_n|e^{\lambda_n\sigma} : n\}$$

as the maximum term of g(s).

K.N. Srivastava in [3] established the conditions under which order (R) of an integral function of an integral function represented by Dirichlet series is finite. In this paper we establish the result for which type (R) of an integral function of an integral function represented by Dirichlet series for a finite order (R) is finite.
We take

$$\begin{aligned} h(s) &= f(\log g(s)) \\ &= \sum_{n=1}^{\infty} b_n(g(s))^n. \end{aligned}$$

2 Basic results

K. Sugimura [4] proved the following two lemmas.

Lemma 2.1 *If g(s) is an entire function of form* (1.1) *and* $m(\sigma, g)$ *is its maximum term, then*

$$|a_m|e^{\lambda_m \sigma} \leq m(\sigma, g) \leq G(\sigma).$$

Lemma 2.2 *Let* $g(-\infty) = 0$ *and* $H(\sigma) = \sup_{|t|<\infty} |h(\sigma + it)|$.
Then there exists a definite number C not dependent on f(s), g(s) and σ*, such that*

$$H(\sigma) \geq F\{\log(C.G(\sigma - \theta))\}$$

where θ *is a fixed constant.*

The following result has been proven by Srivastava [3]:

Theorem 2.3 *If h(s) is of finite order (R) then one of the following holds*
a) Either, g(s) is a Dirichlet polynomial and f(s) is of finite order (R); or
b) g(s) is of finite order (R) and f(s) is of order (R) zero.

3 Main result

Theorem 3.1 *Let* $h(s)$ *be an entire function of order (R) '*ρ*'* $(0 < \rho < \infty)$ *then h(s) is of infinite type (R) unless*
a) g(s) is a Dirichlet polynomial and f(s) is of finite type (R); or
b) g(s) is of finite type (R) and f(s) is of same type (R) as that of h(s).

Proof. Let h(s) be of finite type (R) 'τ' for the finite order (R) ρ then, for $\epsilon > 0$

$$H(\sigma) < exp\{(\tau + \epsilon)e^{\sigma\rho}\}$$

where $H(\sigma) = \sup_{|t|<\infty} |h(\sigma + it)|$.

From Lemma 2.1, we have

$$|a_m|e^{\lambda_m \sigma} \leq m(\sigma, g) \leq G(\sigma).$$

and from Lemma 2.2, we have

$$F[\log(C.G(\sigma - \theta))] \leq H(\sigma)$$

Here,

$$\begin{aligned} F[\log\{C.|a_m|e^{-\lambda_m\theta}\}+\lambda_m\sigma] &= F[\log\{C.|a_m|e^{\lambda_m(\sigma-\theta)}\}] \\ &\leq F[\log\{C.G(\sigma-\theta)\}] \\ &\leq H(\sigma) \\ &< exp\{(\tau+\epsilon)e^{\sigma\rho}\} \end{aligned}$$

$$\Rightarrow F[\log\{C.|a_m|e^{-\lambda_m\theta}\}+\sigma] < exp\{(\tau+\epsilon)e^{\frac{\sigma\rho}{\lambda_m}}\}$$

Let ρ_f be the order (R) of function f(s).
Type (R) of function f(s),

$$\begin{aligned} \tau_f &= \limsup_{\sigma\to\infty}\frac{\log F(\sigma)}{e^{\sigma\rho_f}} \\ &< \limsup_{\sigma\to\infty}\frac{e^{\frac{\sigma\rho}{\lambda_m}}(\tau+\epsilon)}{e^{\sigma\rho_f}} \\ &= \limsup_{\sigma\to\infty} e^{\sigma(\frac{\rho}{\lambda_m}-\rho_f)}(\tau+\epsilon) \end{aligned}$$

As h(s) is of finite order (R) therefore by Theorem 2.3, we have

Case 1: g(s) is a Dirichlet polynomial and f(s) is of finite order (R).

As g(s) is a Dirichlet polynomial, therefore λ_m holds a finite value.

If $\frac{\rho}{\lambda_m}-\rho_f>0$ then τ_f is finite.

If $\frac{\rho}{\lambda_m}-\rho_f=0$ then $\tau_f<\tau+\epsilon$ which holds for all $\epsilon>0$ then $\tau_f=\tau$.

If $\frac{\rho}{\lambda_m}-\rho_f<0$ then $\tau_f=0$.

Case 2: g(s) is of finite order (R) and f(s) is of order (R) zero.

If g(s) is not a Dirichlet polynomial, then λ_m can be chosen large enough, thus

$$\tau_f < \limsup_{\sigma\to\infty} e^{-\sigma\rho_f}(\tau+\epsilon)$$

As $\rho_f=0$, therefore

$$\tau_f<\tau+\epsilon$$

which holds for all $\epsilon>0$. Hence $\tau(f)=\tau$.

From Lemma 2.1, $|b_n|e^{n\sigma} \leq F(\sigma)$.
We have,

$$\begin{aligned}|b_n|e^{n\log\{C.G(\sigma-\theta)\}} &\leq F(\log\{C.G(\sigma-\theta)\}) \\ &\leq H(\sigma) \\ &< exp\{(\tau+\epsilon)e^{\sigma\rho}\}\end{aligned}$$

Thus type of g(s) is finite.
Hence the theorem.

□

References

1. Mandelbrojt, S. – *Dirichlet series. Principles and methods*, D. Reidel Publishing Co., Dordrecht (1972).
2. Ritt, J.F. – *On certain points in the theory of Dirichlet series*, Amer. J. Math, **50** (1928), no. 1, 73-86.
3. Srivastava, K.N. – *On an entire function of an entire function defined by Dirichlet series*, Pacific J. Math., **18** (1966), 379-383.
4. Sugimura, K. – *Übertragung einiger Sätze and der Theorie der ganzen Funktionen auf Dirichletsche Reihen*, Mathematische Zeitschrift, **29** (1929), 264-277.

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