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(s_1, s_2)-Padovan matrix sequence and the case of generalization

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Abstract In this work we present the (s, t)-Padovan matrix sequence generalization by means of (s_1, s_2, s_3)-Tridovan matrix sequence definition. The first one studied primarily by [5] as (s, t)-Padovan matrix sequence, where *s* and *t* are arbitraries integers coefficients, in this work we present a new matrix sequence definition for a generic number of integer coefficients, denominated from now on as (s_1, s_2, $\ldots$, s_z)-Z-dovan. In addition the (s_1, s_2, s_3)-Tridovan matrix sequence is presented. In general, one can note that the *Padovan or Cordonnier sequence* is a particular case of this generalized matrix sequence. A study along this linear, recurrent and recursive sequence is carried out in order to support new definitions, theorems and other studies with a consolidated theoretical foundation.

Keywords generalization · matrix · Cordonnier sequence · matrix sequence · Padovan sequence

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1 Introduction

The Padovan or Cordonnier sequence is named after the Italian Richard Padovan and the Frenchman Gérard Cordonnier, where the latter contributed presenting relations and others definition. This sequence is a linear third-order recurrent sequence defined as the sum of two previous sequence terms ignoring the immediately one, the recurrence relation is given by:

Definition 1.1 *For all $n \geq 3$ integers, the recurrence relation is given by:*

$$P_n = P_{n-2} + P_{n-3}, n \geq 3$$

With the initial terms $P_0 = P_1 = P_2 = 1$ and P_n the nth term.

One can obtain the polynomial form from the recurrence relation, thereby:

$$\begin{aligned} P_n &= P_{n-2} + P_{n-3} \\ \frac{P_n}{P_{n-2}} &= 1 + \frac{P_{n-3}}{P_{n-2}} \\ \frac{P_n}{P_{n-1}}\frac{P_{n-1}}{P_{n-2}} &= 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}} \end{aligned}$$

The presented sequence convergence relation and the roots characteristic polynomial was firstly defined in 1924 by Gérard Cordonnier [17]. Assuming the following convergence relation limit ψ exists:

$$\lim_{n\to\infty}\frac{P_n}{P_{n-1}} = \lim_{n\to\infty}\psi_n = \psi \therefore \frac{P_n}{P_{n-1}} = \psi_n, n \geqslant 1$$

So we have

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{P_n}{P_{n-1}}\frac{P_{n-1}}{P_{n-2}}\right) &= \lim_{n\to\infty}\left(1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}}\right) \\ \lim_{n\to\infty}(\psi_n\psi_{n-1}) \\ \psi\psi &= 1 + \frac{1}{\psi} \\ \psi^3 - \psi - 1 &= 0 \end{aligned}$$

The aforementioned equations have three roots, two complex conjugates and one real. The real one is shown below.

$$\psi = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1,324717957244746025960908 5\ldots$$

The convergence relation ψ, between the subsequent terms of its sequence. Be Padovan sequence P_n and ψ the plastic constant ψ is also defined as:

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n} = \psi \approx 1,32$$

In the next sections, the studies related to the (s_1, s_2) Padovan matrix sequence (s, t) will be performed through the works of [5], [6], [10], [1]. So that one can obtain the generalization of matrix sequence ($(s_1, s_2, \ldots, s_z)$-Z-dovan matrix sequence)

2 $(s_1,\, s_2,\ldots,\, s_z)$-Z-dovan matrix sequence

The (s_1, s_2)-Padovan matrix sequence was defined primarily by [5], based on [6], [10] and [15] works. Cerda propose an generalizaton approach for sequence coefficients, its matrix sequence generator and other definitions.

The terms $s_1, s_2, s_3, \ldots$ vary according to the sequence length, in the case of Padovan numbers, there will be only s_1 and s_2. Thus, the generalization of the Padovan sequence, Z-dovan, will occur from the z terms sum, varying according to the number of terms that will be summed, counted from an immediately previous element. The order of these numbers is defined as $z+1$, so this new matrix sequence generalization is studied and defined in this work as $(s_1, s_2, \ldots, s_z)$ Z-dovan sequence.

For (s_1, s_2) Padovan matrix sequences, their extensions, and their generalization will be defined.

Definition 2.1 *For any integer s and t the (s,t)-Padovan sequence, represented by $P_n(s,t)$ with $n \geqslant 0$ one can define the following recurrence formula:*

$$P_{(n+3)(s,t)} = sP_{(n+1)(s,t)} + tP_{(n)(s,t)}.$$

With initial values: $P_{(0)(s,t)} = 0$, $P_{(1)(s,t)} = 1$ and $P_{(2)(s,t)} = 0$. And for the notation purpose, we will use $P_{(n)(s,t)} = P_n$.

Definition 2.2 *For any integer the (s,t)-Padovan matrix sequence, represented by $Q^{(n)}_{(s,t)}$ with $n \geqslant 0$ is defined as:*

$$Q^{(n+3)}_{(s,t)} = sQ^{(n+1)}_{(s,t)} + tQ^{(n)}_{(s,t)}.$$

With $Q^0_{(s,t)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q^1_{(s,t)} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *and* $Q^2_{(s,t)} = \begin{bmatrix} s & t & 0 \\ 0 & s & t \\ 1 & 0 & 0 \end{bmatrix}$.

Theorem 2.3 *According to [5], the generating (s,t)-Padovan matrix sequence, for $n \geq 0$ is given by:*

$$Q_{(s,t)} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Q^n_{(s,t)} = \begin{bmatrix} P_{n+1} & P_{n+2} & s_2P_n \\ P_n & P_{n+1} & s_2P_{n-1} \\ P_{n-1} & P_n & s_2P_{n-2} \end{bmatrix}.$$

Proof. Using the second principle of finite induction and the definition 2.2, one can demonstrate the theorem.

□

We can also rewrite this theorem proposed by [5] dismembering some terms, in order to generalize this matrix sequence later.

Theorem 2.4 *(s,t)-Padovan matrix sequence, for $n \geq 0$ is given by [5]:*

$$Q_{(s,t)} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Q^n_{(s,t)} = \begin{bmatrix} P_{n+1} & sP_n + tP_{n-1} & t_n \\ P_n & sP_{n-1} + tP_{n-2} & tP_{n-1} \\ P_{n-1} & sP_{n-2} + tP_{n-3} & tP_{n-2} \end{bmatrix}.$$

Proof. Using the same procedure performed in the proof of Theorem 2.3, it is possible to prove this theorem.

□

In order to generalize this matrix sequence, we will modify the notation s, t, as studied by [5], to s_1, s_2, resulting in (s_1, s_2)-Padovan.

The (s_1, s_2, s_3)-Tridovan matrix sequence of fourth order and which occurs through the sum of three previous sequence terms, counted from a jump. Introduced primarily in this work and with support in [5], and is then defined according to the conjecture presented below.

Definition 2.5 *For any integer the (s_1, s_2, s_3)-Tridovan sequence, represented as $T_{(n)}(s_1, s_2, s_3)$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:*

$$T_{(n+3)(s_1,s_2,s_3)} = s_1 T_{(n+1)(s_1,s_2,s_3)} + s_2 T_{(n)(s_1,s_2,s_3)} + s_3 T_{(n-1)(s_1,s_2,s_3)}.$$

With initial values: $T_{(0)(s_1,s_2,s_3)} = 0$, $T_{(1)(s_1,s_2,s_3)} = 1$, $T_{(2)(s_1,s_2,s_3)} = 0$ and $T_{(3)(s_1,s_2,s_3)} = s_1$ with $T_{(n)}(s_1, s_2, s_3) = T_{(n)}$.

Definition 2.6 *For any integer the (s_1, s_2, s_3)-Tridovan sequence, represented by $\Gamma_{(n)}(s_1, s_2, s_3)$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:*

$$\Gamma^{(n+3)}_{(s_1,s_2,s_3)} = s_1 \Gamma^{(n+1)}_{(s_1,s_2,s_3)} + s_2 \Gamma^{(n)}_{(s_1,s_2,s_3)} + s_3 \Gamma^{(n-1)}_{(s_1,s_2,s_3)}.$$

With: $\Gamma^0_{(s_1,s_2,s_3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\Gamma^1_{(s_1,s_2,s_3)} = \begin{bmatrix} 0 & s_1 & s_2 & s_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

$$\Gamma^2_{(s_1,s_2,s_3)} = \begin{bmatrix} s_1 & s_2 & s_3 & 0 \\ 0 & s_1 & s_2 & s_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \Gamma^3_{(s_1,s_2,s_3)} = \begin{bmatrix} s_2 & (s_1)^2 + s_3 & s_1.s_2 & s_1.s_3 \\ s_1 & s_2 & s_3 & 0 \\ 0 & s_1 & s_2 & s_3 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 2.7 *The generating matrix of (s_1, s_2, s_3)-Tridovan, represented by $\Gamma^{(n)}_{(s_1,s_2,s_3)}$ for $n \geq 0$, is expressed by:*

$$\Gamma^1_{(s_1,s_2,s_3)} = \begin{bmatrix} 0 & s_1 & s_2 & s_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Gamma^n_{(s_1,s_2,s_3)} = \begin{bmatrix} T_{(n+1)} & s_1T_{(n)} + s_2T_{(n-1)} + s_3T_{(n-2)} & s_2T_{(n)} + s_3T_{(n-1)} & s_3T_{(n)} \\ T_{(n)} & s_1T_{(n-1)} + s_2T_{(n-2)} + s_3T_{(n-3)} & s_2T_{(n-1)} + s_3T_{(n-2)} & s_3T_{(n-1)} \\ T_{(n-1)} & s_1T_{(n-2)} + s_2T_{(n-3)} + s_3T_{(n-4)} & s_2T_{(n-2)} + s_3T_{(n-3)} & s_3T_{(n-2)} \\ T_{(n-2)} & s_1T_{(n-3)} + s_2T_{(n-4)} + s_3T_{(n-5)} & s_2T_{(n-3)} + s_3T_{(n-4)} & s_3T_{(n-3)} \end{bmatrix}.$$

Proof. Using the Definition 2.6 and the second principle of finite induction, we have:

$$\Gamma^{(n+1)}_{(s_1,s_2,s_3)} = s_1.\Gamma^{(n-1)}_{(s_1,s_2,s_3)} + s_2.\Gamma^{(n-2)}_{(s_1,s_2,s_3)} + s_3.\Gamma^{(n-3)}_{(s_1,s_2,s_3)}.$$

$$\Gamma^{(n+1)}_{(s_1,s_2,s_3)} = s_1$$

$$\begin{bmatrix} T_{(n)} & s_1T_{(n-1)} + s_2T_{(n-2)} + s_3T_{(n-3)} & s_2T_{(n-1)} + s_3T_{(n-2)} & s_3.T_{(n-1)} \\ T_{(n-1)} & s_1T_{(n-2)} + s_2T_{(n-3)} + s_3T_{(n-4)} & s_2T_{(n-2)} + s_3T_{(n-3)} & s_3T_{(n-2)} \\ T_{(n-2)} & s_1T_{(n-3)} + s_2T_{(n-4)} + s_3T_{(n-5)} & s_2T_{(n-3)} + s_3T_{(n-4)} & s_3T_{(n-3)} \\ T_{(n-3)} & s_1T_{(n-4)} + s_2T_{(n-5)} + s_3T_{(n-6)} & s_2T_{(n-4)} + s_3T_{(n-5)} & s_3T_{(n-4)} \end{bmatrix} +$$

$$s_2 \begin{bmatrix} T_{(n-1)} & s_1T_{(n-2)} + s_2T_{(n-3)} + s_3T_{(n-4)} & s_2T_{(n-2)} + s_3T_{(n-3)} & s_3T_{(n-2)} \\ T_{(n-2)} & s_1T_{(n-3)} + s_2T_{(n-4)} + s_3T_{(n-5)} & s_2T_{(n-3)} + s_3T_{(n-4)} & s_3T_{(n-3)} \\ T_{(n-3)} & s_1T_{(n-4)} + s_2T_{(n-5)} + s_3T_{(n-6)} & s_2T_{(n-4)} + s_3T_{(n-5)} & s_3T_{(n-4)} \\ T_{(n-4)} & s_1T_{(n-5)} + s_2T_{(n-6)} + s_3T_{(n-7)} & s_2T_{(n-5)} + s_3T_{(n-6)} & s_3T_{(n-5)} \end{bmatrix}$$

$$+s_3 \begin{bmatrix} T_{(n-2)} & s_1T_{(n-3)} + s_2T_{(n-4)} + s_3T_{(n-5)} & s_2T_{(n-3)} + s_3T_{(n-4)} & s_3T_{(n-3)} \\ T_{(n-3)} & s_1T_{(n-4)} + s_2T_{(n-5)} + s_3T_{(n-6)} & s_2T_{(n-4)} + s_3T_{(n-5)} & s_3T_{(n-4)} \\ T_{(n-4)} & s_1T_{(n-5)} + s_2T_{(n-6)} + s_3T_{(n-7)} & s_2T_{(n-5)} + s_3T_{(n-6)} & s_3T_{(n-5)} \\ T_{(n-5)} & s_1T_{(n-6)} + s_2T_{(n-7)} + s_3T_{(n-8)} & s_2T_{(n-6)} + s_3T_{(n-7)} & s_3T_{(n-6)} \end{bmatrix}$$

$$\Gamma^{(n+1)}_{(s_1,s_2,s_3)} = \begin{bmatrix} T_{(n+2)} & s_1T_{(n+1)} + s_2T_{(n)} + s_3T_{(n-1)} & s_2T_{(n+1)} + s_3T_{(n)} & s_3T_{(n+1)} \\ T_{(n+1)} & s_1T_{(n)} + s_2T_{(n-1)} + s_3T_{(n-2)} & s_2T_{(n)} + s_3T_{(n-1)} & s_3T_{(n)} \\ T_{(n)} & s_1T_{(n-1)} + s_2T_{(n-2)} + s_3T_{(n-3)} & s_2T_{(n-3)} + s_3T_{(n-2)} & s_3T_{(n-1)} \\ T_{(n-1)} & s_1T_{(n-2)} + s_2T_{(n-3)} + s_3T_{(n-4)} & s_2T_{(n-2)} + s_3T_{(n-3)} & s_3T_{(n-2)} \end{bmatrix}$$

□

In order to perform the matrix sequence generalization, we can consider the relations used for the previous sequences, and thus reach the generalized form of this matrix sequence, named $(s_1, s_2, \ldots, s_z)$-Z-dovan. Based on the work of [1], where the Fibonacci sequence generalization is performed, one can define the generalization of this matrix sequence.

Definition 2.8 *For any integer the $(s_1, s_2, \ldots, s_z)$-Z-dovan sequence, named as $Z_{(n)}(s_1, s_2, \ldots, s_z)$ with $n \geqslant 0$, is defined by:*

$$\begin{aligned} Z_{(n+3)(s_1,s_2,\ldots,s_z)} &= s_1.Z_{(n+1)(s_1,s_2,\ldots,s_z)} + s_2Z_{(n)(s_1,s_2,\ldots,s_z)} \\ &\quad + s_3Z_{(n-1)(s_1,s_2,\ldots,s_z)} + \ldots + s_zZ_{(n-z+2)(s_1,s_2,\ldots,s_z)} \\ Z_{(n+1)(s_1,s_2,\ldots,s_z)} &= \sum_{i=1}^{z} s_iZ_{(n-i)(s_1,s_2,\ldots,s_z)}. \end{aligned}$$

Where z represents the number of terms to be summed counted from a jump, and the sequence as being of order $z+1$.

Definition 2.9 *For any integer the $(s_1, s_2, \dots, s_z)$-Z-dovan matrix sequence, named as $\zeta^{(n)}_{(s_1,s_2,\dots,s_z)}$ with $n \geqslant 0$, is defined by:*

$$\begin{aligned}\zeta^{(n+3)}_{(s_1,s_2,\dots,s_z)} &= s_1.\zeta^{(n+1)}_{(s_1,s_2,\dots,s_z)} + s_2\zeta^{(n)}_{(s_1,s_2,\dots,s_z)} + s_3\zeta^{(n-1)}_{(s_1,s_2,\dots,s_z)} \\ &\quad + \dots + s_z\zeta^{(n-z+2)}_{(s_1,s_2,\dots,s_z)} \\ \zeta^{(n+1)}_{(s_1,s_2,\dots,s_z)} &= \sum_{i=1}^{z} s_i\zeta^{(n-i)}_{(s_1,s_2,\dots,s_z)}.\end{aligned}$$

Definition 2.10 *$(s_1, s_2, \dots, s_z)$-Z-dovan has a matrix that generates its sequence, represented by $\zeta^{(n)}_{(s_1,s_2,\dots,s_z)}$ for $n \geq 0$, thus defined as $\zeta^{(n)}_{(s_1,s_2,\dots,s_z)}$ for a better understanding at where:*

$$\zeta^{1}_{(s_1,s_2,\dots,s_z)} = \begin{bmatrix} 0 \; s_1 \; s_2 \; \dots \; s_z \\ I^z \qquad\quad 0^z \end{bmatrix}$$

and

$$\zeta^{(n)}_{(s_1,s_2,s_3,\dots,s_z)} = \begin{bmatrix} Z_{(n+1)} & \dots & \sum_{k=j}^{z+1} s_{(k-1)}Z_{(n+j-k)} \\ Z_{(n)} & \dots & \sum_{k=j}^{z+1} s_{(k-1)}Z_{(n+1-i+j-k)} \\ \vdots & \vdots & \vdots \\ Z_{(n-z+1)} & \dots & \sum_{k=j}^{z+1} s_{(k-1)}Z_{(n-z+j-k)} \end{bmatrix}.$$

where i and j represent the row and column, with $j \neq 1$, $i > 0$, $j > 1$, i, $j \in \mathbb{N}$, [8].

We can observe that the first row of the matrix represents the operator or the recurrence of the sequence. Subsequently, there is an identity matrix just below, being expressed according to the number of summed terms of the sequence and finally, a column of zeros, varying according to the number of summed terms of the sequence (z).

From the definition 2.10, it is concluded that the matrix sequence generalization can easily be calculated according to the number of summed terms of the sequence. That is, we express a generalization model starting from the Padovan sequence (two terms summed), then the tridovan and finally one can reach a more general perspective (z terms add).

3 Binet's formula (s_1, s_2,…,s_z)-Z-dovan matrix sequence

Binet's formula is an alternative to find the sequence's weights without depending on the recurrence and its necessary to know first their respective characteristic equations.

Theorem 3.1 *Be r_1,r_2 the equation roots $r^2 + pr + q = 0$, with $r_1 \neq r_2$, thereby all recurrence solutions $x_{n+2} + px_{n+1} + qx_n = 0$ are expressed in the form of:*

$$a_n = C_1 r_1^n + = C_2 r_2^n.$$

With C_1, C_2 the coefficients of formula.

[5] defined the characteristic $(s_1,\ s_2)$-Padovan matrix sequence equation as being $x^3 - sx - t = 0$, however, in this article we will define using other variables s, t.

Definition 3.2 *The characteristic equation of (s, t)-Padovan matrix sequence is defined from the recurrence 2.1 as:*

$$x^3 - sx - t = 0.$$

Resulting in a real root and two complex conjugates, r_1, r_2, r_3. Where:
$r_1 = A + B$,
$r_2 = -\frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B)$,
$r_3 = -\frac{1}{2}(A + B) - \frac{i\sqrt{3}}{2}(A - B)$.
And:
$A = (\frac{1}{2} + \sqrt{\Delta})^{(\frac{1}{3})}$,
$B = (\frac{1}{2} - \sqrt{\Delta})^{(\frac{1}{3})}$,
$\Delta = (\frac{t}{2})^2 - (\frac{s}{3})^3$.

With this definition and theorem 3.1 one can easily reach the Binet's formula for $(s,\ t)$-Padovan matrix sequence.

Theorem 3.3 *For $n \in \mathbb{N}$, we have the Binet's formula for (s, t)-Padovan matrix sequence defined as:*

$$Q_{(s,t)}^{(n)} = C_1 r_1^n + = C_2 r_2^n + C_3 r_3^n.$$

where C_1, C_2, C_3 are the coefficients, r_1, r_2, r_3 the characteristic roots equation and $27t^2 - 4s^3 \neq 0$.

The discriminant $\Delta = \frac{(-t)^2}{4} + \frac{(-s)^3}{27}$, relative to the 3rd degree equation, determines how the equation roots will be. Therefore, when $\Delta \neq 0$ all roots will be different, if $\Delta \neq 0$ the equation $27t^2 - 4s^3 \neq 0$ must fit. In fact, if the condition does not fit, there will not be the coefficients stated above because at least two roots are equal, so there will be no Binet formula. Note also that $r_1 r_2 r_3 = t$ and $r_1 + r_2 + r_3 = 0$, this implies that $t \neq 0$.

Further, we will explore the characteristic equation and Binet's formula the $(s_1,\ s_2,\ s_3)$-Tridovan matrix sequence, based on $(s,\ t)$-Padovan matrix sequence studies.

Definition 3.4 *The characteristic (s_1, s_2, s_3)-Tridovan matrix sequence equation is defined based on recurrence 2.5, being:*

$$x^4 - s_1x^2 - s_2x - s_3 = 0.$$

At where r_1, r_2, r_3, r_4 are the roots, and:
$r_1 = -S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$,
$r_2 = -S - \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$,
$r_3 = S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$,
$r_4 = S - \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$,
$p = -8s_1$, $q = -s_2$,
$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3}(Q + \frac{\Delta_0}{Q})}$,
$Q = \sqrt[3]{\frac{\Delta_1 + (\sqrt{\Delta_1^2 - 4\Delta_0^3})}{2}}$,
$\Delta_0 = s_1^2 - 12s_3$,
$\Delta_1 = -2s_1^3 + 27s_2^2 - 72s_1s_3$.

Theorem 3.5 *For $n \epsilon \mathbb{N}$, we have (s_1, s_2, s_3)-Tridovan matrix sequence, its Binet's formula is:*

$$\Gamma^{(n)}_{(s_1,s_2,s_3)} = C_1r_1^n + C_2r_2^n + C_3r_3^n + C_4r_4^n.$$

Where C_1, C_2, C_3, C_4 are the coefficients, r_1, r_2, r_3, r_4 the characteristic roots equation and $s_3 \neq 0$ and $\Delta < 0$ or ($\Delta > 0$ and $P < 0$ and $D < 0$) where $\Delta = s_2^2(4s_1^3 - 27s_2^2) - 16s_3(16s_3^2 + 8s_1^2s_3 + 9s_1s_2^2 + s_1^4)$, $P = -8s_1$ and $D = -64s_3 - 16s_1$.

This condition is obtained from the principle that all roots must be distinct, thus $\Delta < 0$ or $\Delta > 0$ with $P < 0$ and $D < 0$. Starting from the general 4th form degree equation represented by $ax^4+bx^3+cx^2+dx+e = 0$, on what a, b, c, d, e are the coefficients of this equation, we have that the discriminant Δ is obtained by the formula [14]:

$$\begin{gathered}\Delta = 256a^3e^3 - 192a^2bd^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 \\ -6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde \\ -4b^3d^3 - 4b^2c^3e + b^2c^2d^2.\end{gathered}$$

Therefore, for Δ we have to:

$$s_2^2(4s_1^3 - 27s_2^2) - 16s_3(16s_3^2 + 8s_1^2s_3 + 9s_1s_2^2 + s_1^4)$$

The parameters P and D are found through the formula:

$$\begin{aligned}&P = 8ac - 3b^2 = -8s_1,\\ &D = 64a^3e - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4 = -64s_3 - 16s_1.\end{aligned}$$

Thereby: $-8.s_1 > 0$ and $-64.s_3 - 16.s_1 > 0$

The roots equation product, $r_1 r_2 r_3 r_4 = s_3$, and the roots sum, $r_1 + r_2 + r_3 + r_4 = 0$, implying that the term $s_3 \neq 0$, so that the Binet's formula existence is possible.

Definition 3.6 *The characteristic (s_1, s_2, ..., s_z)-Z-dovan matrix sequence equation is defined based on recurrence 2.8, so:*

$$x^{z+1} = s_1 x^{(z-1)} + s_2 x^{(z-2)} + \ldots + s_{(z)} x^{(z-z)} \Longrightarrow$$
$$x^{z+1} - s_1 x^{(z-1)} - s_2 x^{(z-2)} - \ldots - s_{(z)} x^{(z-z)} = 0$$
$$x^{z+1} - (\sum_{j=1}^{z} s_j . x^{z-j}) = 0.$$

Where $r_1, r_2, r_3, \ldots, r_z$ are the roots, and to perform the characteristic roots calculation equation from the 5th, it is necessary to use some computational resource.

Theorem 3.7 *For $n \in \mathbb{N}$, we have (s_1, s_2, ..., s_z)-Z-dovan matrix sequence Binet's formula defined as follow:*

$$\zeta^{(n)}_{(s_1, s_2, \ldots, s_z)} = C_1 r_1^n + C_2 r_2^n + \ldots + C_z r_z^n.$$

Where $C_1, C_2, \ldots, C_z$ are the coefficients, $r_1, r_2, \ldots, r_z$ the characteristic roots equation. The discriminant and the equation roots must be different of zero. This can be easy verified calculating the Binet's formula coefficients with Vandermonde's determinant [2], [11], as presented below:

$$Det = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_z \\ r_1^2 & r_2^2 & \ldots & r_z^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^z & r_2^z & \ldots & r_z^z \end{bmatrix} = \prod_{z \geq i > j \geq 2} (r_i - r_j).$$

The discriminant must be performed according to the equation order and if one root is equal to another, the determinant will be equal to 0. Thus there will be no Binet formula for the matrix sequence extension.

For (s_1, $s_2, \ldots, s_z$)-Z-dovan matrix sequence one can define the roots product of equation, $r_1 r_2 \ldots r_z = s_z$, and the $r_1 + r_2 + \ldots + r_z = 0$ sum roots. With this, it is possible to conclude that $s_z \neq 0$, because if this condition isn't true, Binet's formula will not exist.

The Binet's formula can be applied to (s_1, $s_2, \ldots, s_z$)-Z-dovan matrix sequence as long as the conditions proposed in this section are met.

4 The generating $(s_1,\ s_2, \ldots, s_z)$-Z-dovan matrix sequence function

A generative function allows to find any sequence terms in constant coefficients linear recurrences sequences without having to know each term before [12].

Definition 4.1 *The Padovan sequence generating function is defined as:*

$$G(P_n, x) = \frac{x}{(1 - x^2 - x^3)}.$$

With $P_0 = 0, P_1 = 1, P_2 = 0$

These initialization values can be changed, according to the sequence as performed in [16] and [9].

Theorem 4.2 *The generating* $(s_1,\ s_2)$*-Padovan sequence function for* $n \in \mathbb{N}$ *is given by:*

$$G(P_{(n)(s,t)}, x) = \frac{x}{1 - sx^2 - tx^3}.$$

Proof. For Padovan numbers this function is multiplied by sx^2, tx^3 in the equations below due to their recurrence relation.

Thus, for $(s,\ t)$-Padovan sequence, whereas $P_0 = 0; P_1 = 1; P_2 = 0$, and $P_{(n)(s,t)}$, it has to be:

$$\begin{aligned}
G(P_{(n)(s,t)}, x) &= P_0 + P_1x + P_2x^2 + P_3x^3 + P_4x^4 + \ldots \\
sx^2G(P_{(n)(s,t)}, x) &= sP_0x^2 + sP_1x^3 + sP_2x^4 + P_3x^5 + P_4x^6 + \ldots \\
tx^3G(P_{(n)(s,t)}, x) &= tP_0x^3 + tP_1x^4 + tP_2x^5 + tP_3x^6 + tP_4x^7 + \ldots
\end{aligned}$$

Based on the equation

$$G(P_{(n)(s,t)}, x) - sx^2G(P_{(n)(s,t)}, x) - tx^3G(P_{(n)(s,t)}, x)$$

We have to:

$$G(P_{(n)(s,t)}, x)(1 - sx^2 - tx^3) = P_0 + P_1x + (P_2 - P_0)x^2$$

$$G(P_{(n)(s,t)}, x)(1 - sx^2 - tx^3) = x$$

$$G(P_{(n)(s,t)}, x) = \frac{x}{1 - sx^2 - tx^3}$$

□

Through the Taylor series, we perform the direct generating function division, where the variable exponent x is the sequence position term, and its coefficient represents the term of that sequence [13]:

$$x + sx^3 + tx^4 + (s)^2x^5 + 2stx^6 + (s^3t^2)x^7)$$
$$+3s^2tx^8 + (s^4 + 3st^2)x^9 + (4s^3t + t^3)x^{10}$$

Theorem 4.3 *The generating (s, t)-Padovan matrix sequence function for* $n \in \mathbb{N}$ *and for the purpose of notation we will use* $Q_{(n)(s,t)} = Q_n$ *is given by:*

$$G(Q^{(n)}_{(s,t)}, x) = \frac{Q^0 + Q^1x + (Q^2 - sQ^0)x^2}{1 - sx^2 - tx^3}.$$

Proof. For these numbers the function is multiplied by sx^2, tx^3 in the equations below due to their recurrence relation 2.2.

Thus, for the (s, t)-Padovan matrix sequence, we have:
$Q^0_{(s,t)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q^1_{(s,t)} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $Q^2_{(s,t)} = \begin{bmatrix} s & t & 0 \\ 0 & s & t \\ 1 & 0 & 0 \end{bmatrix}$, we have to:

$$G(Q^{(n)}_{(s,t)}, x) = Q^0 + Q^1x + Q^2x^2 + Q^3x^3 + Q^4x^4 + \dots$$
$$sx^2G(Q^{(n)}_{(s,t)}, x) = sQ^0x^2 + sQ^1x^3 + sQ^2x^4 + sQ^3x^5 + sQ^4x^6 + \dots$$
$$tx^3G(Q^{(n)}_{(s,t)}, x) = tQ^0x^3 + tQ^1x^4 + tQ^2x^5 + tQ^3x^6 + tQ^4x^7 + \dots$$

Based on the equation $G(Q^{(n)}_{(s,t)}, x)$-$sx^2G(Q^{(n)}_{(s,t)}, x)$-$tx^3G(Q^{(n)}_{(s,t)}, x)$, we have to:

$$G(Q^{(n)}_{(s,t)}, x)(1 - sx^2 - tx^3) = Q^0 + Q^1x + (Q^2 - sQ^0)x^2$$
$$G(Q^{(n)}_{(s,t)}, x)(1 - sx^2 - tx^3) = Q^0 + Q^1x + (Q^2 - sQ^0)x^2$$
$$G(Q^{(n)}_{(s,t)}, x) = \frac{Q^0 + Q^1x + (Q^2 - sQ^0)x^2}{1 - sx^2 - tx^3}$$

□

In a similar way, one can investigate another theorem to be used for (s_1, s_2, s_3)-Tridovan sequence.

Theorem 4.4 *The* (s_1, s_2, s_3)*-Tridovan sequence generating function for* $n \in \mathbb{N}$ *is given by:*

$$G(T_{(n)}, x) = \frac{x}{1 - s_1x^2 - s_2x^3 - s_3x^4}.$$

Proof. The function $G(T_{(n)}, x)$ is multiplied by s_1x^2, s_2x^3 and s_3x^4, according to the recurrence formula (Definition 2.5). Let $T_{(0)} = 0; T_{(1)} = 1; T_{(2)} = 0; T_{(3)} = s_1$, and $T_{(n)(s_1,s_2,s_3)}$, we have to:

$$\begin{aligned}
G(T_{(n)(s_1,s_2,s_3)}, x) &= T_{(0)} + T_{(1)}x + T_{(2)}x^2 + T_{(3)}x^3 + T_{(4)}x^4 + \dots \\
s_1x^2G(T_{(n)(s_1,s_2,s_3)}, x) &= s_1T_{(0)}x^2 + s_1T_{(1)}x^3 + s_1T_{(2)}x^4 + s_1T_{(3)}x^5 \\
&\quad + s_1T_{(4)}x^6 + \dots \\
s_2x^3G(T_{(n)(s_1,s_2,s_3)}, x) &= s_2T_{(0)}x^3 + s_2T_{(1)}x^4 + s_2T_{(2)}x^5 + s_2T_{(3)}x^6 \\
&\quad + s_2T_{(4)}x^7 + \dots \\
s_3x^4G(T_{(n)(s_1,s_2,s_3)}, x) &= s_3T_{(0)}x^4 + s_3T_{(1)}x^5 + s_3T_{(2)}x^6 + s_3T_{(3)}x^7 \\
&\quad + s_3T_{(4)}x^8 + \dots
\end{aligned}$$

According to equation

$$\begin{aligned}
G(T_{(n)(s_1,s_2,s_3)}, x) - s_1x^2G(T_{(n)(s_1,s_2,s_3)}, x) - s_2x^3G(T_{(n)(s_1,s_2,s_3)}, x) \\
- s_3x^4G(T_{(n)(s_1,s_2,s_3)}, x)
\end{aligned}$$

We have to:

$$\begin{aligned}
G(T_{(n)(s_1,s_2,s_3)}, x)(1 - s_1x^2 - s_2x^3 - s_3x^4) &= T_{(0)} + T_{(1)}x + (T_{(2)} \\
&\quad - T_{(0)})x^2 + (T_{(3)} - s_1T_{(1)} \\
&\quad - s_2T_{(0)})x^3 \\
G(T_{(n)(s_1,s_2,s_3)}, x)(1 - s_1x^2 - s_2x^3 - s_3x^4) &= x \\
G(T_{(n)(s_1,s_2,s_3)}, x) &= \frac{x}{1 - s_1x^2 - s_2x^3 - s_3x^4}
\end{aligned}$$

□

Below we show a generating exploration (s_1, s_2, s_3)-Tridovan sequence function through the series of powers of Tylor, in the same way as was done in the previous sequence.

$$\begin{aligned}
x + s_1x^3 + s_2x^4 + (s_1^2 + s_3)x^5 + 2s_1s_2x^6 \\
+(s_1^3 + 2s_1s_3 + s_2^2)x^7 + (3s_1^2s_2 + 2s_2s_3)x^8 \\
+(s_1^4 + 3s_1^2s_3 + 3s_1s_2^2 + s_3^2)x^9 + (4s_1^3s_2 + 6s_1s_2s_3 + s_2^3)x^{10}
\end{aligned}$$

It's worth noting that these terms can be calculated using the function generating calculations.

Theorem 4.5 *The generating function of (s_1, s_2, s_3)-Tridovan matrix sequence for $n \epsilon \mathbb{N}$ and for the purpose of notation we will use $\Gamma_{(n)(s_1,s_2,s_3)} = \Gamma_n$ is given by:*

$$G(\Gamma^{(n)}_{(s_1,s_2,s_3))}, x) = \frac{\Gamma^0 + \Gamma^1 x + (\Gamma^2 - s_1\Gamma^0)x^2 + (\Gamma^3 - s_1\Gamma^1 - s_2\Gamma^0)x^3}{1 - s_1x^2 - s_2x^3 - s_3x^4}.$$

Proof. The function $G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)$ is multiplied by s_1x^2, s_2x^3 and s_3x^4, according to the recurrence formula (Definition 2.6). So we have: $\Gamma^0_{(s_1,s_2,s_3)} = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}$, $\Gamma^1_{(s_1,s_2,s_3)} = \begin{bmatrix} 0&s_1&s_2&s_3\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}$, $\Gamma^2_{(s_1,s_2,s_3)} = \begin{bmatrix} s_1&s_2&s_3&0\\ 0&s_1&s_2&s_3\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix}$

and $\Gamma^3_{(s_1,s_2,s_3)} = \begin{bmatrix} s_2 & (s_1)^2+s_3 & s_1s_2 & s_1s_3\\ s_1 & s_2 & s_3 & 0\\ 0 & s_1 & s_2 & s_3\\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned}
G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x) &= \Gamma^0 + \Gamma^1x + \Gamma^2x^2 + \Gamma^3x^3 + \Gamma^4x^4 + \dots\\
s_1x^2G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x) &= s_1\Gamma^0x^2 + s_1\Gamma^1x^3 + s_1.\Gamma^2x^4 + s_1\Gamma^3x^5\\
&\quad + s_1\Gamma^4x^6 + \dots\\
s_2x^3G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x) &= s_2\Gamma^0x^3 + s_2\Gamma^1x^4 + s_2\Gamma^2x^5 + s_2\Gamma^3x^6 + s_2\Gamma^4x^7 + \dots\\
s_3x^4G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x) &= s_3\Gamma^0x^4 + s_3\Gamma^1x^5 + s_3\Gamma^2x^6 + s_3\Gamma^3x^7 + s_3\Gamma^4x^8 + \dots
\end{aligned}$$

According to equation:
$G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)$ - $s_1x^2G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)$
-$s_2x^3G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)$-$s_3x^4G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)$, we have to:

$$\begin{aligned}
G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)(1 - s_1x^2 - s_2x^3 - s_3x^4) &= \Gamma^0 + \Gamma^1x + (\Gamma^2 - s_1\Gamma^0)x^2\\
&\quad + (\Gamma^3 - s_1\Gamma^1 - s_2\Gamma^0)x^3\\
G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x)(1 - s_1.x^2 - s_2x^3 - s_3x^4) &= \Gamma^0 + \Gamma^1x + (\Gamma^2 - s_1\Gamma^0)x^2\\
&\quad + (\Gamma^3 - s_1\Gamma^1 - s_2\Gamma^0)x^3
\end{aligned}$$

$$G(\Gamma^{(n)}_{(s_1,s_2,s_3)}, x) = \frac{\Gamma^0 + \Gamma^1x + (\Gamma^2 - s_1\Gamma^0)x^2 + (\Gamma^3 - s_1\Gamma^1 - s_2\Gamma^0)x^3}{1 - s_1x^2 - s_2x^3 - s_3x^4}$$

□

Given the results regarding the functions generating the previous matrix sequences, we can obtain the generating $(s_1, s_2, \dots s_z)$-Z-dovan and $(s_1, s_2, \dots s_z)$-Z-dovan matrix sequence function.

Theorem 4.6 *The generating function of (s_1, s_2, ..., s_z)-Z-dovan sequence for $n \in \mathbb{N}$ where $Z_{(n)(s_1,s_2,\ldots,s_z)} = Z_{(n)}$ is given by:*

$$G(Z_{(n)}, x) = \frac{x}{1 - (\sum_{i=1}^{z} s_i x^{i+1})}.$$

Where z represents the number of terms that will be added to the sequence counted from the jump.

Theorem 4.7 *The generating function of (s_1, s_2,..., s_z)-Z-dovan matrix sequence for $n \in \mathbb{N}$ where $\zeta^{(n)}_{(s_1,s_2,\ldots,s_z)}$ and for the purpose of notation we will use $\zeta^{(n)}_{(s_1,s_2,\ldots,s_z)} = \zeta^n$ is given by:*

$$G(\zeta^{(n)}_{(s_1,s_2,\ldots,s_z)}, x) = \frac{\sum_{i=0}^{z} f(i) x^i}{1 - (\sum_{i=1}^{z} s_i x^{i+1})}.$$

Where: $s_0 = 0$ e $f(i) = \zeta_i - (\sum_{j=0}^{i-1} s_j \zeta_{i-1-j})$.

5 Conclusion

This work presents new Padovan Matrix Sequence extensions and generalization based on (s,t)-Padovan. The (s_1, s_2)-Padovan its an alternative visualization of [5] (s,t)-Padovan matrix sequence which its used as background to the generalization definition. It's also presented the (s_1, s_2, s_3)-Tridovan matrix sequence, which is defined as the recurrence sum of three previous terms after ignore the immediatly next one.

With the previous definitions we conjecture the existence of other matrix sequences resulting from the increase in the number of summed antecedent terms, thus its proposed the models generalization $(Z_{(n)(s_1,s_2,\ldots,s_z)})$ and $(\zeta_{(n)(s_1,s_2,\ldots,s_z)})$. The Binet's formula, alternative ways to get the sequence terms without using recurrence are addressed as well the generating function. It is worth emphasizing that this generalization is presented in a primordial way in this work, contributing to the area of pure mathematics and to the teaching of the Padovan matrix sequence.

It can be concluded that the Padovan Sequence is a particular case of $(s_1, s_2, \ldots, s_z)$-Z-dovan matrix sequence, where it's obtained after considering the last two terms after ignoring the immediately next on and assigns values equal to 1 to the variables defined by s_1, s_2 .

For future work, it is expected that new generalizations of this sequence will be made, inserting definitions of other sequences, so that they become a mixed sequence [3, 4, 7].

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