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Approximate solutions of a generalized functional equation on restricted domains and their asymptotic behaviors

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Abstract In this paper, we prove the Hyers-Ulam stability on restricted domains of the following generalized Cauchy and quadratic functional equation

$$\sum_{k=0}^{m-1} f(x+b_k y) = mf(x) + mf(y),\ x, y \in E,$$

where $b_k = \exp(\frac{2i\pi k}{m})$ for $0 \leq k \leq m-1$. These results are applied to study of an asymptotic behavior of this functional equation.

Keywords Cauchy functional equation · quadratic functional equation · Hyers-Ulam stability

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1 Introduction

The concept of stability for functional equations arises when we replace a functional equation by an inequality which acts as a perturbation of the equation. The starting point of studying stability of functional equations seems to be the famous talk of S.M. Ulam of 1940 [11], in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:
Given a group G_1, a metric group G_2 with metric $d(\cdot,\cdot)$ and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d\left(f(xy), f(x)f(y)\right) \leq \delta$ for all $x, y \in G_1$, then a homomorphism $\phi : G_1 \to G_2$ exists with $d\left(f(x), \phi(x)\right) \leq \varepsilon$ for all $x \in G_1$?
These kinds of questions form the material of the stability theory. The case of approximately additive mappings was solved by D. H. Hyers [4] under the assumption that G_1 and G_2 are Banach spaces.

In 1950, T. Aoki [1] provided a generalization of the Hyers' theorem for additive mappings and in 1978 Th. M. Rassias [10] generalized the Hyers' theorem for linear mappings by considering an unbounded Cauchy difference.

In 1998, S. M. Jung [6] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In 2002, J. M. Rassias [9] improved the bounds and thus the stability results obtained by Jung [6]. Besides, he established the Ulam stability for more general equations of two types on a restricted domain then he applied his recent results to the asymptotic behavior of functional equations of different types.

Throughout this paper, let E be a $\mathbb{C}$-vector space, F be a $\mathbb{C}$-Banach space and $2 \leq m \in \mathbb{N}$. Our aim is to prove the Hyers-Ulam stability on restricted domains of the following generalized Cauchy and quadratic functional equation

$$\sum_{k=0}^{m-1} f(x + b_k y) = mf(x) + mf(y), \; x, y \in E, \tag{1.1}$$

where $b_k = \exp(\frac{2i\pi k}{m})$ for $0 \leq k \leq m-1$. Indeed, if $m = 1$ in (1.1), we get the Cauchy functional equation

$$f(x + y) = f(x) + f(y).$$

When we put $m = 2$ in (1.1), then we obtain the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

These results are applied to study of an asymptotic behavior of this functional equation.

2 Notations and preliminary results

In this section, we need to introduce some notions and notations.

Definition 2.1 *[2] A function $A : E \to F$ between two vector spaces E and F is said to be additive provided if $A(x + y) = A(x) + A(y)$ for all $x, y \in E$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.*

Definition 2.2 *[2] Let $k \in \mathbb{N}$ and $A : E^k \to F$ be a function, then we say that A is k-additive provided if it is additive in each variable. In addition, we say that A is symmetric provided if*

$$A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k)$$

whenever $x_1, x_2, \ldots, x_k \in E$ and σ is a permutation of $\{1, 2, \ldots, k\}$.

Definition 2.3 *[2] Let $k \in \mathbb{N}$ and $A : E^k \to F$ be symmetric and k-additive and let $A_k(x) = A(x, x, \dots, x)$ for $x \in E$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x \in E$ and $r \in \mathbb{Q}$.*

In this way, a function $A_k : E \longrightarrow F$ which satisfies $A_k(rx) = r^k A_k(x)$ for all $r \in \mathbb{Q}$ and $x \in E$, will be called a rational-homogeneous form of degree k (assuming $A_k \neq 0$).

Definition 2.4 *[2] A function $p : E \to F$ is called a generalized polynomial (GP) function of degree $m \in \mathbb{N}$ if there exist $a_0 \in E$ and a rational-homogeneous form $A_k : E \to F$ (for $1 \leq k \leq m$) of degree k, such that*

$$p(x) = a_0 + \sum_{k=1}^{m} A_k(x)$$

for all $x \in E$.

Definition 2.5 *[2] Let F^E denote the vector space, over a field K, consisting of all maps from E into F. For each $h \in E$ define the linear difference operator Δ_h on F^E by*

$$\Delta_h f(x) = f(x+h) - f(x)$$

for all $f \in F^E$ and all $x \in E$.

Notice that these difference operators commute ($\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ for all $h_1, h_2 \in E$) and if $h \in E$ and $n \in \mathbb{N}$, then Δ_h^n the n-th iterate of Δ_h satisfies

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh)$$

for all $f \in F^E$ and all $x, h \in E$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The following theorem was proved by Mazur and Orlicz [7], [8] then in greater generality by Djoković [3].

Theorem 2.6 *Let $n \in \mathbb{N}$ and $f : E \to F$ be a function between two vector spaces E and F, then the following assertions are equivalent.*

1. *$\Delta_h^n f(x) = 0$ for all $x, h \in E$.*
2. *$\Delta_{h_n} \dots \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_n \in E$.*
3. *f is a GP function of degree at most $n-1$.*

3 Main results

Lemma 3.1 *Suppose that $2 \leq m \in \mathbb{N}$, $b_k \in K$ for $0 \leq k \leq m$, E is a $\mathbb{C}$-vector space, F is a $\mathbb{C}$-Banach space, $\delta \geq 0$ and $f_k : E \to F$ for $0 \leq k \leq m$, be mappings fulfilling*

$$\left\| \sum_{k=0}^{m} \Big[f_k(x + b_k y) - f_k(b_k y) \Big] \right\| \leq \delta \tag{3.1}$$

for all $x, y \in E$ and assume that $b_k - b_j \neq 0$ and $b_0 \neq 0$ whenever $0 \leq j < k \leq m$. Then,

$$\left\| \Delta_{h_{m+1}} \dots \Delta_{h_1} f_0(x) \right\| \leq 3^m \delta \tag{3.2}$$

for all $x, h_1, \dots, h_{m+1} \in E$.

Proof. Letting $d_{jk} = b_k - b_j$ for all $0 \leq k \leq m$, so we get that $d_{jk} \neq 0$ where $j < k$ and $d_{kk} = 0$. We get

$$(x + b_m h_1) + b_k(y - h_1) = x + b_k y + d_{km} h_1$$

for all $0 \leq k \leq m$ and all $x, y, h_1 \in E$. By using these equalities and (3.1), we find that

$$\begin{aligned} &\left\| \sum_{k=0}^{m} \big[f_k(x + b_k y + d_{km} h_1) - f_k(x + b_k y) \big] - \sum_{k=0}^{m} \big[f_k(b_k y + d_{km} h_1) - f_k(b_k y) \big] \right\| \\ &\leq \left\| \sum_{k=0}^{m} \big[f_k(x + b_k y + d_{km} h_1) - f_k\big(b_k(y - h_1)\big) \big] \right\| \\ &\quad + \left\| \sum_{k=0}^{m} \big[f_k(b_k y + d_{km} h_1) - f_k\big(b_k(y - h_1)\big) \big] \right\| \\ &\quad + \left\| \sum_{k=0}^{m} \big[f_k(x + b_k y) - f_k(b_k y) \big] \right\| \leq 3\delta \end{aligned}$$

and, since $d_{mm} = 0$,

$$\left\| \sum_{k=0}^{m-1} \big[\Delta_{d_{km} h_1} f_k(x + b_k y) - \Delta_{d_{km} h_1} f_k(b_k y) \big] \right\| \leq 3\delta \tag{3.3}$$

for all $x, y, h_1 \in E$. By repeating the argument that leads from (3.1) to (3.3), we find that

$$\left\| \sum_{k=0}^{m-2} \Big[\Delta_{d_{k,m-1} h_2} \Delta_{d_{km} h_1} f_k(x + b_k y) - \Delta_{d_{k,m-1} h_2} \Delta_{d_{km} h_1} f_k(b_k y) \Big] \right\| \leq 3^2 \delta \tag{3.4}$$

for all $x, y, h_1, h_2 \in E$. By applying this reasoning $m-2$ more times, we are inclined to admit that

$$\left\|\Delta_{d_{01}h_m} \dots \Delta_{d_{0m}h_1}\left[f_0(x+b_0y) - f_0(b_0y)\right]\right\| \leq 3^m\delta \tag{3.5}$$

for all $x, y, h_1, \dots h_m \in E$. Consequently,

$$\|\Delta_{b_0y} \dots \Delta_{h_1} f_0(x))\| \leq 3^m\delta \tag{3.6}$$

for all $x, y, h_1, ..., h_m \in E$. Since $d_{0k} \neq 0$ for $1 \leq k \leq m$, and $b_0 \neq 0$ the last inequalities assert that

$$\left\|\Delta_{h_{m+1}} \dots \Delta_{h_1} f_0(z))\right\| \leq 3^m\delta \tag{3.7}$$

for all $z, h_1, \dots, h_{m+1} \in E$.

□

Theorem 3.2 *Suppose that E is a $\mathbb{C}$-vector space and F is a $\mathbb{C}$-Banach space. If $f : E \to F$ satisfies*

$$\left\|\sum_{k=0}^{m-1} f(x+b_ky) - mf(x) - mf(y)\right\| \leq \delta \tag{3.8}$$

for all $x, y \in E$, where $2 \leq m \in \mathbb{N}$, $b_k = \exp(\frac{2ik\pi}{m})$ for $0 \leq k \leq m-1$ and $\delta \geq 0$. Then there exists a unique GP function $p : E \to F$ of degree at most m such that $p(0) = 0$ and

$$\|f(x) - f(0) - p(x)\| \leq 2 \cdot 3^m\delta \ for\ all\ x \in E.$$

Moreover

$$\sum_{k=1}^{m} p(x+b_ky) - mp(x) - mp(y) = 0 \ for\ all\ x, y \in E.$$

Proof. Letting $x = 0$ in (3.8), we get

$$\left\|\sum_{k=0}^{m-1} f(b_ky) - mf(0) - mf(y)\right\| \leq \delta \ for\ all\ x, y \in E. \tag{3.9}$$

By (3.8) and (3.9), we have

$$\left\|\sum_{k=0}^{m-1} g(x+b_ky) - mg(x) - \sum_{k=0}^{m-1} g(b_ky)\right\| \leq 2\delta,\ x, y \in E, \tag{3.10}$$

where $g = f - f(0)$. If we put $f_k = g$ for each $0 \leq k \leq m-1$, $f_m = -mg$ and $b_m = 0$ in (3.10), then we get

$$\left\|\sum_{k=0}^{m}\left[f_k(x+b_ky) - f_k(b_ky)\right]\right\| \leq 2\delta. \tag{3.11}$$

Thus, by Lemma 3.1

$$\left\|\Delta_{h_{m+1}}\ldots\Delta_{h_1}g(z)\right\|\leq 2\cdot 3^m\delta$$

for all $z,h_1,\ldots,h_{m+1}\in E$.
Hence, according to [Theorem II [5]], there exists a GP function $p:E\to F$, of degree at most m such that

$$\|g(x)-p(x)\|\leq 2\cdot 3^m\delta \tag{3.12}$$

and

$$p(x)=g(0)+\sum_{j=1}^{m}A_j(x), \tag{3.13}$$

where $A_j:E^j\to F$ are symmetric, j-additive mappings.

By (3.12), we obtain

$$\left\|\sum_{k=0}^{m-1}\big(p(x+b_ky)\big)-mp(x)-mp(y)\right\|\leq\left\|\sum_{k=0}^{m-1}\big[p(x+b_ky)-g(x+b_ky)\big]\right\| \\ +\left\|mg(x)-mp(x)\right\|+\left\|mg(y)-mp(y)\right\|,$$

so

$$\left\|\sum_{k=0}^{m-1}\big(p(x+b_ky)\big)-mp(x)-mp(y)\right\|\leq 6m\cdot 3^m\delta \tag{3.14}$$

for all $x,y\in E$.
In right of (3.13), the inequality (3.14) says that

$$\left\|\sum_{j=1}^{m}\left(\sum_{k=0}^{m-1}\Big(A_j(x+b_ky)\Big)-mA_j(x)-mA_j(y)\right)\right\|\leq 6m\cdot 3^m\delta \tag{3.15}$$

for all $x,y\in E$. Replacing x by rx and y by ry ($r\in\mathbb{Q}$) in (3.15), we conclude that

$$\left\|\sum_{j=1}^{m}\left(r^j\sum_{k=0}^{m-1}\Big(A_j(x+b_ky)\Big)-mA_j(x)-mA_j(y)\right)\right\|\leq 6m\cdot 3^m\delta \tag{3.16}$$

for all $x,y\in E$ and all $r\in\mathbb{Q}$. By continuity, (3.16) holds for all real r, and all $x,y\in E$. Now suppose that $\phi:F\to\mathbb{R}$ is a continuous linear functional. Then

$$\left\|\sum_{j=1}^{m}r^j\phi\left(\sum_{k=0}^{m-1}\Big(A_j(x+b_ky)\Big)-mA_j(x)-mA_j(y)\right)\right\|\leq ||\phi||6m\cdot 3^m\delta \tag{3.17}$$

for all $x, y \in E$ and all $r \in \mathbb{R}$. Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that

$$\phi\left(\sum_{k=0}^{m-1}\Big(A_j(x+b_ky)\Big) - mA_j(x) - mA_j(y)\right) = 0 \tag{3.18}$$

for all $x, y \in E$ where $1 \le j \le m-1$. Since this is so for every continuous linear functional $\phi : F \to \mathbb{R}$, by the Hahn-Banach theorem we get that

$$\sum_{k=0}^{m-1}\Big(A_j(x+b_ky)\Big) - mA_j(x) - mA_j(y) = 0, \tag{3.19}$$

for all $x, y \in E$ and $1 \le j \le m$. Then, each p is a (GP) function of degree at most m and from (3.19) we find that

$$\sum_{k=0}^{m-1} p(x+b_ky) - mp(x) - mp(y) = 0, \tag{3.20}$$

for all $x, y \in E$. Finally, let p' be another generalized polynomial (GP) of degree at most m such that $p^{'}(0) = 0$, $p^{'}$ is a solution of (1.1) and

$$||f(x) - p^{'}(x)|| \le 2 \cdot 3^m\delta, \ x \in E.$$

Then, from (3.20) and (3.12) we get that $p - p^{'}$ is generalized polynomial (GP) of degree at most m such that $||p(x) - p'(x)|| \le 4 \cdot 3^m\delta, \ x \in E$. Thus, necessarily $p = p' + p(0) - p^{'}(0)$. Since $p(0) = p^{'}(0) = 0$ we get that $p = p^{'}$ witch ends the proof.

□

Theorem 3.3 *Suppose that E is a $\mathbb{C}$-vector space and F is a $\mathbb{C}$-Banach space. Let $d > 0$ and $\delta \ge 0$ be fixed. If a mapping $f : E \to F$ satisfies the generalized quadratic inequality (3.8) for all $x, y \in E$, with $||x|| + ||y|| \ge d$, then there exists a unique GP function $p : E \to F$ such that $p(0) = 0$ and*

$$\|f(x) - f(0) - p(x)\| \le 2 \cdot 3^{m+1}\delta \ for \ all \ x \in E.$$

Moreover

$$\sum_{k=0}^{m-1} p(x+b_ky) - mp(x) - mp(y) = 0 \ for \ all \ x, y \in E.$$

Proof. Assume that $\|x\| + \|y\| < d$ and let $t \in E$ such that $\|t\| = d$ where $x = 0$ and

$$t = \left(1 + \frac{d}{\|x\|}\right)x,$$

where $x \neq 0$. We note that

$$\|t\| = \|x\| + d \geq d,$$

$$\|x\| + \|t\| \geq d$$

and

$$\|t\| + \|x + b_k y\| \geq d.$$

We have b_j for all $0 \leq j \leq m-1$ is a root of unity, if we put $b_j = e^{i\theta_j}$, we obtain

$$\|x + b_j t\|^2 = \left(2\left(1 + \frac{d}{\|x\|}\right)\left(1 + cos\theta_j\right) + \frac{d^2}{\|x\|^2}\right)\|x\|^2 \geq d^2$$

Thus, from (3.6) and the new functional identity we obtain that

$$m\left[\sum_{k=0}^{m-1} f(x + b_k y) - mf(y) - mf(x)\right]$$

$$= \left[\sum_{k=0}^{m-1}\Big(mf(x + b_k y) + mf(t) - \sum_{n=0}^{m-1} f(x + b_k y + b_n t)\Big)\right]$$

$$+ \left[\sum_{n=0}^{m-1}\Big(-mf(x + b_n t) - mf(y) + \sum_{k=0}^{m-1} f(x + b_n t + b_k y)\Big)\right]$$

$$+m\left[\sum_{n=0}^{m-1} f(x + b_n t) - mf(x) - mf(t)\right],$$

then

$$m\left\|\sum_{k=0}^{m-1} f(x + b_k y) - mf(x) - mf(y)\right\| \leq 3m\delta.$$

Therefore,

$$\left\|\sum_{k=0}^{m-1} f(x + b_k y) - mf(x) - mf(y)\right\| \leq 3\delta.$$

for all $x, y \in E$. Finally, we get the desired result by applying Theorem 3.2. □

Corollary 3.4 *Suppose that E is a $\mathbb{C}$-vector space and F is a $\mathbb{C}$-Banach space. A mapping $f : E \to F$ such that $f(0) = 0$ satisfies the functional equation (1.1) if and only if the asymptotic condition*

$$\left\|\sum_{k=0}^{m-1} f(x + b_k y) - mf(x) - mf(y)\right\| \to 0, \text{ as } \|x\| + \|y\| \to \infty. \tag{3.21}$$

holds true.

Proof. By the asymptotic condition (3.21), there exists a sequence δ_n monotonically decreasing to 0 such that

$$\left\| \sum_{k=0}^{m-1} f(x+b_k y) - mf(x) - mf(y) \right\| \leq \delta_n \tag{3.22}$$

for all $x, y \in E$ with $\|x\| + \|y\| \geq n$. Hence, it follows from Theorem 3.3 that there exists a unique generalized polynomial $p_n : E \to F$ such that $p_n(0) = 0$ and

$$\|f(x) - p_n(x)\| \leq 2 \cdot 3^{m+1} \delta_n \; for \; all \; x \in E. \tag{3.23}$$

Since δ_n is a monotonically decreasing sequence, the generalized polynomial p_n satisfies (1.1) for all $n \geq N$. The uniqueness of p_n implies for all $n \geq N$, $p_N = p_n$. Hence, by letting $n \to \infty$, we conclude that f is generalized polynomial at most m, and it satisfies the functional equation (1.1).

□

If we put $m = 1$ then $m = 2$ in Theorem 3.2, we find the following particular cases.

Corollary 3.5 *Suppose that E is a vector space, F is a real (or complex) Banach space and $\delta \geq 0$. If $f : E \to F$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ such that $\|x\| + \|y\| \geq d$. Then there exists an additive function $A : E \to F$ such that

$$\|f(x) - f(0) - A(x)\| \leq 6\delta \; for \; all \; x \in E.$$

Corollary 3.6 *Let $d > 0$ and $\delta \geq 0$ be fixed. If a mapping $f : E \to F$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta$$

for all $x, y \in E$ such that $\|x\| + \|y\| \geq d$. Then there exists a quadratic function $Q : E \to F$ such that

$$\|f(x) - f(0) - Q(x)\| \leq 18\delta \; for \; all \; x \in E.$$

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References

1. Aoki, T. – *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64-66.
2. Baker, J. A. – *A general functional equation and its stability*, Proc. Amer. Math. Soc., **133** (2005), no. 6, 1657-1664.
3. Djoković, D. Ž – *A representation theorem for* $(X_1 - 1)(X_2 - 1) \cdots (X_n - 1)$ *and its applications*, Ann. Polon. Math., **22** (1969/1970), 189-198.
4. Hyers, D. H. – *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222-224.
5. Hyers, D. H. – *Transformations with bounded m-th differences*, Pacific J. Math., **11** (1961), 591-602.
6. Jung, S. M. – *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., **222** (1998), no. 1, 126-137.
7. Mazur, S. ; Orlicz, W. – *Grundlegende Eigenschaften der Polynomischen Operationen, Erste Mitteilung*, Studia Mathematica, **5** (1934), no. 1, 50-68.
8. Mazur, S. ; Orlicz, W. – *Grundlegende Eigenschaften der Polynomischen Operationen, Zweite Mitteilung*, Studia Mathematica, **5** (1934), no. 1, 179-189.
9. Rassias, J. M. – *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl., **276** (2002), no. 2, 747-762.
10. Rassias, Th. M. – *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), no. 2, 297-300.
11. Ulam, S. M. – *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960, xiii+150 pp.

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