An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVIII, 2022, f. 1
https://doi.org/10.47743/anstim.2022.00010

A characterization of projective special linear groups $PSL(5,2)$ and $PSL(4,5)$

Behnam Ebrahimzadeh · Behnam Azizi

Abstract In this paper, we prove that projective special linear groups $PSL(5,2)$ and $PSL(4,5)$ can be uniquely determined by their order and the largest order of elements.

Keywords the largest order of elements · prime graph · Frobenius group · projective special linear group

Mathematics Subject Classification (2010) 20D06 · 20D60

1 Introduction

For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the set of orders of elements of G is denoted by $\pi_e(G)$. The set $\pi_e(G)$ is closed and partially ordered by the divisiblity relation. Hence, it is uniquely determined by the subset of its elements, which are maximal under the divisibility relation. Suppose n is an integer number. In this case, $\pi(n)$ is the set of all prime numbers that divide n. We say the largest order of elements of G is denote by $k(G)$. Also G is called a simple K_n-group if G is simple with $|\pi(G)| = n$. Moreover, we denote a Sylow p-subgroup of G by G_p. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we always assume that $2 \in \pi_1$.
One of the important problems in group theory is the characterization by given properties, that is, there exists only one group with given properties(up to isomorphism). There is different kinds of characterization, for example, the characterization by the set of orders of elements, prime graph, the set of the numbers of elements with same order. Recently, He and Chen studied the characterization of groups by the largest order of elements. For example, in this way the authors proved that the groups $PSL(2,q)$ with

$q = p^n < 125$ ([17]), K_4-group with type $PSL(2,p)$ ([11]), the simple K_n-groups where $n = 3, 4$ ([12,13]), sporadic simple groups ([10]), $PSL(3,q)$ and $PSU(3,q)$, for some q ([14]), $PGL(2,q)$([19]) and Suzuki group $Sz(q)$ where $q = 2^{2m+1}$ and $q-1$ or $q \pm \sqrt{2q}+1$ is a prime number, projective special unitary groups $PSU(3,3^n)$, projective symplectic groups $PSP(8,q)$ and symplectic groups $C_4(q)$ ([3–8]), are characterizable by using their order together with the largest, the second largest and the third largest orders of elements. In this article, we prove that the projective special linear groups $PSL(5,2)$ and $PSL(4,5)$ are characterizable by their order and the largest orders of elements. In fact, we prove the following main theorems:
Main Theorem A. Let G be a group with $|G| = |PSL(5,2)|$ and $k(G) = k(PSL(5,2))$. Then $G \cong PSL(5,2)$.

Main Theorem B. Let G be a group with $|G| = |PSL(4,5)|$ and $k(G) = k(PSL(4,5))$. Then $G \cong PSL(4,5)$.

2 Notation and preliminaries

In this section, we give some useful lemmas which will be used in the proof of the main theorems.

Lemma 2.1 *[9, Theorem1] Let G be a Frobenius group of even order with kernel K and complement H. Then*

(i) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(ii) *$|H|$ divides $|K| - 1$;*
(iii) *K is nilpotent.*

Definition 2.2 *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.3 *[2, Theorem 2] Let G be a 2-Frobenius group of even order. Then*

(i) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(ii) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.4 *[21, Theorem A] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(i) *G is a Frobenius group;*
(ii) *G is a 2-Frobenius group;*
(iii) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Lemma 2.5 *[15] Let G be a simple K_3-group. Then G is isomorphic to one of the following groups:*
A_5, A_6, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,17)$, $PSL(3,3)$, $PSU(3,3)$ *or* $PSU(4,2)$.

Lemma 2.6 *[1] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(i) $PSL(2,q)$ *where q is a prime power satisfying $q(q^2-1) = (2,q-1)2^a \cdot 3^b \cdot p^c \cdot r^d$ with p, $r > 3$ distinct primes and a,b,c,d natural numbers;*
(ii) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $PSL(2,16)$, $PSL(2,25)$, $PSL(2,49)$, $PSL(2,81)$, $PSL(3,4)$, $PSL(3,5)$, $PSL(3,8)$, $PSL(3,17)$, $PSL(4,3)$, $O_5(4), O_5(5), O_5(7), O_5(9), O_7(2), O_8^+(2), G_2(3), PSU(3,4), PSU(3,5)$, $PSU(3,7), PSU(3,8), PSU(3,9), PSU(4,3), PSU(5,2), Sz(8), Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Lemma 2.7 *[16] Let G be a simple K_5-group. Then G is isomorphic to one of the following groups:*

(i) $O_5(q)$ *where q satisfies $|\pi(q^4-1)| = 4$;*
(ii) $PSL(2,q)$ *where q satisfies $|\pi(q^2-1)| = 4$;*
(iii) $PSL(3,q)$ *where q satisfies $|\pi((q^2-1)(q^3-1))| = 4$;*
(iv) $PSU(3,q)$ *where q satisfies $|\pi((q^2-1)(q^3+1))| = 4$;*
(v) $Sz(q)$ *where $q = 2^{2m+1}$ satisfies $|\pi((q-1)(q^2+1))| = 4$;*
(vi) $R(q)$ *where q is an odd power of 3 and $|\pi(q^2-1)| = 3$ and $|\pi(q^2-q+1)| = 1$;*
(vii) A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $PSL(4,4)$, $PSL(4,5)$, $PSL(4,7)$, $PSL(5,2)$, $PSL(5,3)$, $PSL(6,2)$, $O_7(3)$, $O_9(2), PSP(6,3), PSP(8,2)$, $PSU(4,4)$, $PSU(4,5)$, $PSU(4,7)$, $PSU(4,9)$, $PSU(5,3)$, $PSU(6,2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.

3 Proof of the main theorem A

In this section, we prove the main theorem A. From now on, we denote the group $PSL(5,2)$ by L. We recall that G is a group with $|G| = |L|$ and $k(G) = k(L)$.

Since $k(L) = 31$, we conclude that $k(G) = 31$. Thus 31 is an isolated vertex of $\Gamma(G)$ and hence $t(G) \geq 2$. So Lemma (2.4) implies that G satisfies one of the following cases:

Case 1. Let G be a Frobenius group with kernel K and complement H. Then by Lemma (2.1), $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since 31 is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/31$ and $|K| = 31$, or (ii) $|H| = 31$ and $|K| = |G|/31$. Assume that $|H| = |G|/31$ and $|K| = 31$. Then Lemma (2.1) implies that $|G|/31$ divides $31-1$ and hence $322560 \mid 30$ which is impossible. So the case

$|H| = 31$ and $|K| = |G|/31$ is the only possibility. Lemma (2.1) implies that 31 divides $|G|/31 - 1$. Therefore, $31 \mid 322559$ which is a contradiction. As a result we deduce that G is not a Frobenius group.

Case 2. Let G be a 2-Frobenius group. Then by Lemma (2.3), there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|\mathsf{Aut}(K/H)|$. Since 31 is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{31\}$ and $|K/H| = 31$. Hence $|\mathsf{Aut}(K/H)| = 30$. As $|G/K|$ divides $|\mathsf{Aut}(K/H)|$, we deduce that $|G/K|$ divides 30. Hence $|G/K|$ belongs $\{1, 2, 3, 5, 6, 10, 15, 30\}$. For this purpose, we consider only the case 30 (with lost generality), so $|H| \mid \frac{|G|}{|G/K||K/H|}$. It follows that $|H| \mid \frac{|PSL(5,2)|}{30 \cdot 31}$. Hence, $|H| \mid 2^9 \cdot 3 \cdot 7$. Since that H is nilpotent group, so $H \cong H_2 \times H_3 \times H_7$. As a result, G must be have the element of order $2 \cdot 3 \cdot 7$, which is a contradiction, because $k(G) = 31$. So, we conclude that G is not 2-Frobenius group.

Case 3. Let G have a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$. Since 31 is an isolated vertex of $\Gamma(G)$, we deduce that $31 \mid |K/H|$. On the other hand $|\pi(G)| = 5$, thus K/H is a simple K_n-group where $n < 6$. It is clear that K/H is not a simple K_3-group, since $31 \notin \pi(K/H)$. Suppose now that K/H is a simple K_4-group. As $31 \in \pi(K/H)$ and $k(K/H) = 31$, we conclude that K/H is isomorphic to $PSL_3(5)$. and $Sz(32)$. Since $|PSL(3,5)|$ does not divide $|G|$, we have a contradiction. Now if $K/H \cong Sz(32)$, then $|Sz(32)| \nmid |G|$, which is a contradiction. Now, we consider the groups in Lemma 2.6(i). For this purpose, we have $K/H \cong PSL_2(q)$, then by [18], $k(K/H) = q + 1$ and q. Since $31|k(K/H)$, so we consider $q + 1 = 31$ and $q = 31$. It follows that $q = 30$ and also $q = 31$. Now, if $q = 31$, then we have $K/H \cong PSL(2, 31)$. On the other hand $|G/K| \mid |Out(K/H)|$, so $|G/K| \mid |\mathsf{Out}(PSL(2, 31)|2$. As a result $|H| \mid \frac{|G|}{2}$, so $|H| \mid 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Since that H is nilpotent, so $H \cong H_2 \times H_3 \times H_5 \times \times H_7 \times H_{31}$, which is a contradiction. Now, assume $q = 30$ where it is impossible, because q can be prime number or power of prime number. Therefore, it remains to consider the case when K/H is a simple K_5-group, which we now assume.

Since $31 \in \pi(K/H)$, we deduce that K/H is isomorphic with one of the groups $PSL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, $O_5(q)$, $R(q)$, $Sz(q)$, $PSL(4, 5)$, $G_2(5)$, $PSL(5, 2)$, where the restrictions on q are given as in Lemma(2.7). Now, we have the following subcases:

(i) If $K/H \cong PSL(4, 5)$, then since $|PSL(4, 5)| \nmid |G|$ which is a contradiction, also if $K/H \cong G_2(5)$, then since $|G_2(5)| \nmid |G|$ which is a contradiction.

(ii) If $K/H \cong PSL(2, q)$, then by [18], $k(K/H) = q + 1$ and q. Since $31|k(K/H)$, so we consider $q + 1 = 31$ and $q = 31$. It follows that $q = 30$ and also $q = 31$. Now, if $q = 31$, then we have $K/H \cong PSL(2, 31)$. On the other hand $|G/K| \mid |\mathsf{Out}(K/H)|$, so $|G/K| \mid |\mathsf{Out}(PSL(2, 31)|2$. As a

result $|H| \mid \frac{|G|}{2}$, so $|H| \mid 2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Since that H is nilpotent, so $H \cong H_2 \times H_3 \times H_5 \times \times H_7 \times H_{31}$, which is a contradiction. Now, $q = 30$ is impossible, because q can be prime number or power of prime number.
(iii) If $K/H \cong PSL(3,q)$, then by by [18], $k(K/H) = (q^2 + q + 1)$ or $(q^2 + q + 1)/3$. Since $31|k(K/H)$, we consider$(q^2 + q + 1) = 31$, so we deduce $q = 5$. But $|PSL(3,5)| \nmid |G|$ which is a contradiction. Now, if $(q^2 + q + 1)/3 = 31$ so we have the equation $q^2 + q - 92 = 0$. As a result this equation has not any solution, so we have a contradiction.
(iv) If $K/H \cong PSU(3,q)$, then by [18], $k(PSU(3,q)) = p(q+1)/3$ or $p(q+1)$ where $p \leq q$. Since $31|k(K/H)$, we consider $q(q+1)/3 = 31$ and $q^2+q = 31$. Now, if $q(q+1)/3 = 31$, then we deduce $q^2 + q - 93 = 0$. But, we can see easily this equation has not any solution, which is a contradiction. Also the case $q^2 + q = 31$ is impossible.
(v) If $K/H \cong Sz(q)$ where $q = 2^{2m+1} > 2$, then by [18], $k(Sz(q)) = (q + \sqrt{2q} + 1)$. Since $31|k(K/H)$, we consider $(q + \sqrt{2q} + 1) = 31$, so $2^{2m+1}+2^{m+1}+1 = 31$. Now, we consider $2^m = x$ as a result $2x^2+2x-30 = 0$. We can see easily this equation has not any solution, which is a contradiction.
(vi) If $K/H \cong O_5(q)$, then by [18], [20], $k(O_5(q)) = (q^2 + 1)/2$. Since $31|k(K/H)$, so we consider $(q^2 + 1)/2 = 31$, it follows that $q^2 = 61$, which it is impossible.
(vii) If $K/H \cong R(q)$ where $q = 3^{2m+1}$, $m \geq 1$, then by [18], $k(R(q)) = ((q + \sqrt{3q} + 1)$. Since $31|k(K/H)$, so we consider$(q + \sqrt{3q} + 1) = 31$, so we have $3^{2m+1} + 3^{m+1} + 1 = 31$. Now, we consider $3^m = x$ so we have the equation $3x^2 + 3x - 30 = 0$. It follows that this equation has not any solution. Hence $K/H \cong PSL(5,2)$. Now, since $|K/H| = |G| = |L|$, this implies that $G = K/H \cong PSL(5,2)$.

4 Proof of the main theorem B

Here we prove the main theorem B. First, we denote the group $PSL(4,5)$ by L. We recall that G is a group with $|G| = |L|$ and $k(G) = k(L)$.
Since $k(L) = 39$, we conclude that $k(G) = 39$. Thus 31 is an isolated vertex of $\Gamma(G)$ and hence $t(G) \geq 2$. So, Lemma (2.4) implies that G satisfies in item (iii).
First, we prove that G is not a Frobenius and 2-Frobenius group. Hence, Let G be a Frobenius group with kernel K and complement H. As seen before, we have either $|H| = |G|/31$ and $|K| = 31$, or $|H| = 31$ and $|K| = |G|/31$. Then Lemma (2.1) implies that $|G|/31$ divides $31 - 1$ and hence $234000000 \mid 30$ which is impossible. So the case $|H| = 31$ and $|K| = |G|/31$ can be occured. Lemma (2.1) implies that 31 divides $|G|/31 - 1$. Therefore $31|233999999$ that is a contradiction. Therefore, G is not a Frobenius group. Now, we show the group G is not a 2-Frobenius group. For this purpose as-

sume G be a 2-Frobenius group. Then, by Lemma (2.3), there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|\mathsf{Aut}(K/H)|$. Since 31 is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{31\}$ and $|K/H| = 31$. Hence $|\mathsf{Aut}(K/H)| = 30$. As $|G/K|$ divides $|\mathsf{Aut}(K/H)|$, we deduce that $|G/K||30$. Hence, $|G/K|$ belongs $\{1, 2, 3, 5, 6, 10, 15, 30\}$. For this purpose, we consider only the case 30 (with lost generality) so $|H| \mid \frac{|G|}{|G/K||K/H|}$. It follows that $|H| \mid \frac{|PSL(4,5)|}{30.31}$. Hence, $|H| \mid 2^8 \cdot 3 \cdot 5^5 \cdot 13$. Since that H is nilpotent group, so $H \cong H_2 \times H_3 \times H_5 \times H_{13}$. As a result, G must be have the element of order $2 \cdot 3 \cdot 5 \cdot 13$, which is a contradiction. So G is not 2-Frobenius group.
Now by Lemma (2.4) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$. Since 31 is an isolated vertex of $\Gamma(G)$, we deduce that $31 \mid |K/H|$. On the other hand $|\pi(G)| = 5$, thus K/H is a simple K_n-group where $n < 6$. It is clear that K/H is not a simple K_3-group, since $31 \notin \pi(K/H)$. Suppose now that K/H is a simple K_4-group. As $31 \in \pi(K/H)$ and $k(K/H) = 31$, we conclude that $31 \leq k(K/H) \leq 39$. Thus K/H is isomorphic with the groups $PSL(3, 5)$ or $PSL(2, q)$. If $K/H \cong PSL(3, 5)$, then since $|K/H| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, we conclude that $|G/K|$ divides $|\mathsf{Out}(K/H)| = 2$, hence $|G/K| \mid 2$. So we have $|H| = 2^i \cdot 3 \cdot 5^3 \cdot 13$ where $i \mid 2$. Since H is a nilpotent group, H is the direct product of Sylow subgroups. Thus we have $H \cong H_2 \times H_3 \times H_5 \times H_{13}$. So G must has an element of order $2 \cdot 3 \cdot 5 \cdot 13$ which is a contradiction because $k(K/H) \leq 39$. If $K/H \cong PSL(2, q)$, where $q(q^2-1) = 2^a \cdot 3^b \cdot p^c \cdot r^d$ where a, b, c, $d \geq 1$ and p, r is a prime number larger than 3, then $k(K/H) = q+1$ or q. Since $31 \in \pi(K/H)$ and $k(K/H) \leq 39$, we deduce that $31 \leq k(K/H) \leq 39$. Hence $q = 31$ and $|\mathsf{Out}(K/H)| = 2$. Thus $|G/K||2$ and $|H| = 2^i \cdot 3 \cdot 5^5 \cdot 13$ where $i \mid 2$. Now since H is nilpotent group, similary to part (i) G must has an element of order $2 \cdot 3 \cdot 5 \cdot 13$ which is a contradiction because $k(K/H) \leq 39$. Therefore, it remains to consider the case when K/H is a simple K_5-group, which we now assume.
Since $31 \in \pi(K/H)$, we deduce that K/H is isomorphic with one of the groups $PSL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, $O_5(q)$, $R(q)$, $Sz(q)$, $PSL(4, 5)$, $G_2(5)$, $PSL(5, 2)$, where the restrictions on q are given as in Lemma(2.7). Now, we have the following subcases:
(i) If $K/H \cong PSL(2, q)$, then by [18] $k(PSL(2, q)) = q + 1$ or q. Since $31 \leq k(K/H) \leq 39$, we conclude that $q = 31$, $q = 32$ or 37. Hence, if $q = 31$, then $K/H \cong PSL(2, 31)$. On the other hand, $|G/K| \mid |\mathsf{Out}(PSL(2, 31))|2$, so $|H| \mid \frac{|G|}{2.2^5.3.5.31}$. As a result $|H| \mid 2.3.5^5.13$ so $H \cong H_2 \times H_3 \times H_5 \times H_{13}$. As a result, G must be have the element of order $2 \cdot 3 \cdot 5 \cdot 13$, which is a contradiction.
(ii) If $K/H \cong PSL(3, q)$, then by [18] $k(PSL(3, q)) = (q^2 + q + 1)$ or $(q^2 + q + 1)/3$. Since $31 \leq k(K/H) \leq 39$, we conclude that $q = 5$. if $q = 31$, then $K/H \cong PSL(3, 5)$. On the other hand, $|G/K| \mid |\mathsf{Out}(PSL(3, 5))|2$, so

$|H| \mid \frac{|G|}{2.2^5.3.5^3.31}$. As a result $|H| \mid 2.3.5^3.13$ so $H \cong H_2 \times H_3 \times H_5 \times H_{13}$. As a result, G must be have the element of order $2 \cdot 3 \cdot 5 \cdot 13$, which is a contradiction.Thus $|K/H| \nmid |G|$ which is a contradiction.
(iii) If $K/H \cong PSU(3,q)$, then by [18] $k(PSU(3,q)) = (q^2+q+1)$ or $(q^2+q+1)/3$. Since $31 \leq k(K/H) \leq 39$, we deduce that $q = 5$ which is a contradiction.
(iv) If $K/H \cong Sz(q)$ where $q = 2^{2m+1} > 2$, then by [18] $k(Sz(q)) = (q+\sqrt{2q}+1)$. Since that $31 \leq k(K/H) \leq 39$, so we consider $31 \leq 2^{2m+1} + 2^{m+1} + 1 \leq 39$, it follows that$m = 1$ so $q = 8$. But $|Sz(8)| \nmid |G|$, which this is a contradiction.
(v) If $K/H \cong O_5(q)$, then by [18], [20], $k(O_5(q)) = (q^2+1)/2$. Since $31 \leq k(K/H) \leq 39$, so $31 \leq (q^2+1)/2 \leq 39$, we deduce that $q = 8$. But $|S_4(8)| \nmid |G|$ which is a contradiction.
(vi) If $K/H \cong R(q)$ where $q = 3^{2m+1}$, $m \geq 1$, then by [18] $k(R(q)) = (q+\sqrt{3q}+1)$. Since $31 \leq k(K/H) \leq 39$, we deduce that $q = 27$. Thus $|R(27)| \nmid |G|$ which is a contradiction.
So, we conclude that $K/H \cong PSL(4,5)$. Now since $|K/H| = |G| = |L|$, this implies that $G = K/H \cong PSL(4,5)$.

References

1. Bugeaud, Y.; Cao, Z.; Mignotte, M. – *On simple K_4-groups*, J. Algebra, **241** (2001), no. 2, 658-668.
2. Chen, G.Y. – *On the structure of Frobenius groups and 2-Frobenius groups*, J. Southwest China Normal University, **20** (1995), no. 5, 485-487.
3. Ebrahimzadeh, B.; Iranmanesh, A.; Tehranian, A.; Parvizi Mosaed, H. – *A characterization of the Suzuki groups by order and the largest elements order*, J. Sci. Islam. Repub. Iran, **27** (2016), no. 4, 353-355.
4. Ebrahimzadeh, B.; Mohammadyari, R. – *A new characterization of projective special unitary groups $PSU_3(3^n)$*, Discuss. Math. Gen. Algebra Appl., **39** (2019), no. 1, 35-41.
5. Ebrahimzadeh, B.; Sadeghi, M.Y.; Iranmanesh, A.; Tehranian, A. – *A new characterization of symplectics groups $PSP(8,q)$*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **66** (2020), no. 1, 93-99.
6. Ebrahimzadeh, B.; Mohammadyari, R.; Sadeghi, M.Y. – *A new Characterization of the simple groups $C_4(q)$, by its order and the largest order of elements*, Acta Comment. Univ. Tartu. Math., **23** (2019), no. 2, 283-290.
7. Ebrahimzadeh, B.; Asboei, A.K. – *A characterization of symplectic groups related to Fermat primes*, Comment. Math. Univ. Carolin., **62** (2021), no. 1, 33-40.
8. Ebrahimzadeh, B. – *A new characterization of simple groups ${}^2D_n(3)$*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **41** (2021), no. 4, 45-50.
9. Gorenstein, D. – *Finite groups*, Second edition, Chelsea Publishing Co., New York, (1980).
10. He, L.G.; Chen, G.Y.; Xu, H.J. – *A new characterization of sporadic simple groups*, Ital. J. Pure Appl. Math., **30** (2013), 373-392.
11. He, L.G.; Chen, G.Y. – *A new characterization of simple K_4-groups with type $L_2(p)$*, Adv. Math. (China), **43** (2014), no. 5, 667–670.
12. He, L.G.; Chen, G.Y. – *A new characterization of simple K_3-groups*, Comm. Algebra, **40** (2012), no. 10, 3903-3911.
13. He, L.G.; Chen, G.Y. – *A new characterization of simple K_4-groups*, J. Math. Res. Appl., **35** (2015), no. 4, 400-406.
14. He, L.G.; Chen, G.Y. – *A new characterization of $PSL_3(q)$ $(q \leq 8)$ and $PSU_3(q)$ $(q \leq 11)$*, Journal of Southwest University (Nature Science Edition), **33** (2011), no. 10, 81-87.

15. Herzog, M. – *On finite simple groups of order divisible by three primes only*, J. Algebra, **10** (1968), 383-388.
16. Jafarzadeh, A.; Iranmanesh, A. – *On simple K_n-groups, for $n = 5, 6$*, Groups St. Andrews, **2** (2005), 517-526, London Math. Soc. Lecture Note Ser. 340, Cambridge Univ. Press, Cambridge, (2007).
17. Jiang, Q.; Shao, C. – *Characterization of some $L_2(q)$ by the largest element orders*, Math. Rep. (Bucur), **17 (67)** (2015), no. 4, 353-358.
18. Kantor, W.M.; Seress, Á. – *Large element orders and the characteristic of Lie-type simple groups*, J. Algebra, **322** (2009), no. 3, 802-832.
19. Li, J.; Shi, W.J.; Yu, D. – *A characterization of some $PGL(2, q)$ by maximum element orders*, Bull. Korean Math. Soc, **52** (2015), no. 6, 2025-2034.
20. Shi, W.J. – *Pure quantitative characterization of finite simple groups*, Front. Math. China, **2** (2007), no. 1, 123-125.
21. Williams, J.S. – *Prime graph components of finite groups*, J. Algebra, **69** (1981), no. 2, 487-513.

Received: 23.VIII.2021 / Revised: 19.XI.2021 / Accepted: 22.XII.2021

Authors

Behnam Ebrahimzadeh (Corresponding author),
Department of Mathematics,
Persian Gulf University,
Bushehr, Iran,
E-mail: behnam.ebrahimzadeh@gmail.com

Behnam Azizi,
Department of Mathematics,
Kaleybar Branch, Islamic Azad University,
Kaleybar, Iran,
E-mail: *Azizi_behnam*396@*yahoo.com*