An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXVIII, 2022, f. 1
https://doi.org/10.47743/anstim.2022.00002

Steinhaus type theorems over complete ultrametric fields

P.N. Natarajan

Abstract Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Entries of infinite matrices, sequences and infinite series are in K. In this paper, following ([4], [5]), we prove some more Steinhaus type theorems over K.

Keywords ultrametric (or non-archimedean) field · Steinhaus type theorem · unconditional convergence · Banach algebra · Mercerian theorem

Mathematics Subject Classification (2010) 40C05 · 40G05 · 46S10

1 Introduction

If X, Y are sequence spaces and $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$ is an infinite matrix, we write

$$A \in (X, Y)$$

if $A(x) = \{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$, $x_k \in K$, $k = 0, 1, 2, \ldots$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots,$$

supposing that the series on the right converge. If there is a notion of limit or sum in X and Y and if $A \in (X, Y)$ and A preserves limits or sums in X and Y, we write

$$A \in (X, Y; P).$$

If $A \in (X, Y)$ and A preserves the limit in X and sum in Y or vice versa, we write

$$A \in (X, Y; P').$$

In the sequel, we need the following sequence spaces

$$\ell_p = \left\{ x = \{x_k\} \Big/ \sum_{k=0}^{\infty} |x_k|^p < \infty \right\}, p \geq 1;$$

$$c_0 = \left\{ x = \{x_k\} \Big/ \lim_{k\to\infty} x_k = 0 \right\};$$

$$c = \left\{ x = \{x_k\} \Big/ \lim_{k\to\infty} x_k \text{ exists} \right\};$$

$$\ell_\infty = \left\{ x = \{x_k\} \Big/ \sup_{k\geq 0} |x_k| < \infty \right\};$$

$$bv = \left\{ x = \{x_k\} \Big/ \sum_{k=0}^{\infty} |x_k - x_{k+1}| < \infty \right\};$$

and

$$bv_0 = bv \cap c_0.$$

Note that $\ell_p \subset c_0 \subset c \subset \ell_\infty$, $p \geq 1$ and $\ell_1 \subset bv \subset c$.

In the classical case, the famous Steinhaus theorem can be written in the form:

$$(c, c; P) \cap (\ell_\infty, c) = \phi.$$

We can call any result of the form:

$$(X, Y; P) \cap (Z, Y) = \phi, X \subset Z$$

or

$$(X, Y; P') \cap (Z, Y) = \phi, X \subset Z$$

a "Steinhaus type theorem". Natarajan [2] proved that the classical Steinhaus theorem continues to hold when K is a complete, non-trivially valued, ultrametric (or non-archimedean) field.

A general reference for analysis in ultrametric fields is [1].

2 Some Steinhaus type theorems over K

In [3], Natarajan proved that

$$(\ell_\alpha, \ell_\alpha; P)' \cap (\ell_\beta, \ell_\alpha) = \phi, \beta > \alpha > 0,$$

where, $(\ell_\alpha, \ell_\alpha; P)'$ denotes the subclass of $(\ell_\alpha, \ell_\alpha; P)$ such that

$$a_{nk} \to 0, k \to \infty, n = 0, 1, 2, \ldots. \tag{2.1}$$

Noting that $\ell_\beta \subset c_0$, $(\ell_\beta, \ell_\alpha) \supset (c_0, \ell_\alpha)$, $\beta > \alpha > 0$, it follows that

$$(\ell_\alpha, \ell_\alpha; P)' \cap (c_0, \ell_\alpha) = \phi.$$

Thus, we have, the following result, which we call a Steinhaus type theorem again.

Theorem 2.1

$$(\ell_\alpha, \ell_\alpha; P)' \cap (X, \ell_\alpha) = \phi,$$

when $X = \ell_\beta, c_0, \beta > \alpha > 0$.

Now, we have the following Steinhaus type theorem.

Theorem 2.2

$$(\ell_\alpha, \ell_\alpha; P) \cap (c, \ell_\alpha) = \phi, \alpha > 0.$$

Proof. Let $A = (a_{nk}) \in (\ell_\alpha, \ell_\alpha; P) \cap (c, \ell_\alpha)$, $\alpha > 0$. For the sequence $x = \{x_k\} = \{1, 1, 1, \dots\} \in c$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}$$

is defined, $n = 0, 1, 2, \dots$ so that (2.1) holds. Thus

$$A \in (\ell_\alpha, \ell_\alpha; P)' \cap (c, \ell_\alpha).$$

Now, $\ell_\beta \subset c$, $\beta > \alpha$ and so $(\ell_\beta, \ell_\alpha) \supset (c, \ell_\alpha)$, $\beta > \alpha$.
Consequently

$$A \in (\ell_\alpha, \ell_\alpha; P)' \cap (\ell_\beta, \ell_\alpha), \beta > \alpha$$

which is a contradiction of Theorem 2.1, completing the proof.

□

Corollary 2.3 *Since* $c \subset \ell_\infty$, $(c, \ell_\alpha) \supset (\ell_\infty, \ell_\alpha)$ *so that*

$$(\ell_\alpha, \ell_\alpha; P) \cap (\ell_\infty, \ell_\alpha) = \phi, \alpha > 0.$$

Consequently, we have

$$(\ell_\alpha, \ell_\alpha; P) \cap (X, \ell_\alpha) = \phi, \alpha > 0,$$

when $X = c, \ell_\infty$.

We shall now consider the class $(c_0, \ell_1; P)$. We recall that $(c_0, \ell_1; P)$ is the subclass of (c_0, ℓ_1) such that

$$\sum_{k=0}^{\infty} x_k = \sum_{n=0}^{\infty} (Ax)_n, x = \{x_k\} \in c_0.$$

We now have the following Steinhaus type result.

Theorem 2.4

$$(c_0, \ell_1; P) \cap (c, \ell_1) = \phi.$$

Proof. Let $A = (a_{nk}) \in (c_0, \ell_1; P) \cap (c, \ell_1)$. Now, $A \in (c_0, \ell_1; P)$. For $k = 0, 1, 2, \ldots,$ the sequence $x = \{0, 0, \ldots, 0, 1, 0, \ldots\}$, 1 occurring in the kth place is in c_0 and $(Ax)_n = a_{nk}$. Hence,

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots. \tag{2.2}$$

Also, $A \in (c, \ell_1)$. For $x = \{1, 1, 1, \ldots\} \in c$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}$$

and so

$$\sum_{n=0}^{\infty} |(Ax)_n| \text{ converges,}$$

$$i.e., \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} \right| \text{ converges.}$$

Consequently

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} \text{ converges.}$$

Since convergence in K is equivalent to unconditional convergence (see [6]),

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{nk} \text{ converges,}$$

which is a contradiction, in view of (2.2), proving the theorem.

□

Corollary 2.5 *Since $c \subset \ell_\infty$, $(c, \ell_1) \supset (\ell_\infty, \ell_1)$ so that*

$$(c_0, \ell_1; P) \cap (\ell_\infty, \ell_1) = \phi.$$

Thus we have

$$(c_0, \ell_1; P) \cap (X, \ell_1) = \phi,$$

when $X = c, \ell_\infty$.

We next claim that

$$(bv, \ell_1; P') = \phi,$$

where, $(bv, \ell_1; P')$ is the subclass of (bv, ℓ_1) such that

$$\lim_{k\to\infty} x_k = \sum_{n=0}^{\infty}(Ax)_n, x = \{x_k\} \in bv.$$

Let now, $A = (a_{nk}) \in (bv, \ell_1; P')$. The sequence $x = \{1, 1, 1, \dots\} \in bv$, for which,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}, n = 0, 1, 2, \dots.$$

$\{(Ax)_n\} \in \ell_1$ so that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{nk} \text{ converges.}$$

Also

$$\sum_{n=0}^{\infty}(Ax)_n = \lim_{k\to\infty} x_k = 1,$$

$$i.e., \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{nk} = 1. \tag{2.3}$$

Now, $x = \{0, 0, \dots, 0, 1, 0, \dots\}$, 1 occurring in the kth place is in bv, for which,

$$(Ax)_n = a_{nk}, k = 0, 1, 2, \dots$$

so that

$$\sum_{n=0}^{\infty} a_{nk} = \lim_{k\to\infty} x_k = 0. \tag{2.4}$$

Since convergence in K is equivalent to unconditional convergence (see [6]), using (2.3) and (2.4),

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{nk} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{nk} = 0, \end{aligned}$$

a contradiction, proving our claim.

Note that $bv_0 = bv \cap c_0 \subset bv$. We now prove the following Steinhaus type result.

Theorem 2.6

$$(bv_0, \ell_1; P) \cap (bv, \ell_1) = \phi.$$

Proof. We recall that $(bv_0, \ell_1; P)$ is the subclass of (bv_0, ℓ_1) such that

$$\sum_{k=0}^{\infty} x_k = \sum_{n=0}^{\infty} (Ax)_n, x = \{x_k\} \in bv_0,$$

noting that $bv_0 \subset c_0$. Let $A = (a_{nk}) \in (bv_0, \ell_1; P) \cap (bv, \ell_1)$. For $k = 0, 1, 2, \ldots$, the sequence $x = \{0, 0, \ldots, 0, 1, 0, \ldots\}$, 1 occurring in the kth place is in bv_0, for which

$$(Ax)_n = a_{nk}, k = 0, 1, 2, \ldots.$$

Hence

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots,$$

i.e., (2.2) holds.

Again, the sequence $x = \{1, 1, 1, \ldots\}$ is in bv. For this sequence, $\{(Ax)_n\} \in \ell_1$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}, n = 0, 1, 2, \ldots.$$

So $\sum_{n=0}^{\infty} (Ax)_n$ converges, i.e.,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} \text{ converges.}$$

In view of the fact that convergence in K is equivalent to unconditional convergence (see [6]),

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{nk} \text{ converges,}$$

which leads to a contradiction, because of (2.2). The proof of the theorem is now complete.

□

Corollary 2.7 *Noting that* $bv \subset c \subset \ell_\infty$,

$$(bv, \ell_1) \supset (c, \ell_1) \supset (\ell_\infty, \ell_1).$$

Consequently

$$(bv_0, \ell_1; P) \cap (c, \ell_1) = \phi$$

and

$$(bv_0, \ell_1; P) \cap (\ell_\infty, \ell_1) = \phi.$$

Thus we have:

$$(bv_0, \ell_1; P) \cap (X, \ell_1) = \phi,$$

when $X = bv, c, \ell_\infty$.

In the above context, one can note that

$$(bv_0, \ell_1; P) \cap (c_0, \ell_1) \neq \phi.$$

We note that $\ell_1 \subset bv$. We have the following result.

Theorem 2.8

$$(\ell_1, \ell_1; P) \cap (bv, \ell_1) = \phi.$$

Proof. Let $A = (a_{nk}) \in (\ell_1, \ell_1; P) \cap (bv, \ell_1)$. Since $A \in (\ell_1, \ell_1; P)$, (2.2) holds. The sequence $x = \{1, 1, 1, \dots\}$ is in bv and so $\{(Ax)_n\} \in \ell_1$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}, n = 0, 1, 2, \dots.$$

Consequently

$$\sum_{n=0}^{\infty} (Ax)_n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} \text{ converges.}$$

So $\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{nk}$ converges, again using the fact that convergence and unconditional convergence are equivalent in K (see [6] again). This leads to a contradiction, in view of (2.2), completing the proof.

□

In ([5], p. 153), the following Steinhaus type result was proved.

Theorem 2.9

$$(c_0, c_0; P) \cap (\ell_\infty, c_0) = \phi,$$

where, $(c_0, c_0; P)$ *denotes the subclass of* (c_0, c_0) *with*

$$\sum_{k=0}^{\infty} x_k = \sum_{n=0}^{\infty} (Ax)_n, x = \{x_k\} \in c_0.$$

In the context of Theorem 2.9, it is worth noting that

$$(c_0, c_0; P) \cap (c, c_0) \neq \phi,$$

as the following example illustrates:

$$A \equiv (a_{nk}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & -3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 4 & -4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is easy to check that

$$\sup_{n,k} |a_{nk}| \leq 1 < \infty;$$
$$\lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots;$$
$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots;$$

and

$$\sum_{k=0}^{\infty} a_{nk} = 0, n = 0, 1, 2, \ldots$$

so that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0.$$

Consequently

$$A \in (c_0, c_0; P) \cap (c, c_0).$$

It is also worthwhile to note that (c_0, c_0) is an ultrametric (or non-archimedean) Banach algebra under the norm

$$\|A\| = \sup_{n,k} |a_{nk}|, A = (a_{nk}) \in (c_0, c_0).$$

In the context of the ultrametric Banach algebra $((c_0, c_0)$, we prove a Mercerian theorem, similar to the one proved in [3].

Theorem 2.10 *Let $K = \mathbb{Q}_p$, the p-adic field for a prime p.*
If

$$y_n = x_n + \lambda p^n (x_0 + x_1 + \cdots + x_n) \tag{2.5}$$

and $\{y_n\} \in c_0$, then $\{x_n\} \in c_0$, provided

$$|\lambda|_p < 1,$$

$|\cdot|_p$ being the p-adic valuation.

Proof. Since (c_0, c_0) is an ultrametric Banach algebra, if $\lambda \in \mathbb{Q}_p$ is such that $|\lambda|_p < \frac{1}{\|A\|}$, $A \in (c_0, c_0)$, then $I + \lambda A$, where I is the identity matrix, has an inverse in (c_0, c_0). The matrix of transformation in the present context is $I + \lambda A$, where

$$A \equiv (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ p & p & 0 & 0 & 0 & \cdots \\ p^2 & p^2 & p^2 & 0 & 0 & \cdots \\ p^3 & p^3 & p^3 & p^3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We note that

$$\sup_{n,k} |a_{nk}| = 1 < \infty;$$

and

$$\lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots,$$

so that

$$A \in (c_0, c_0).$$

Also

$$\|A\| = \sup_{n,k} |a_{nk}| = 1.$$

So, if $|\lambda|_p < 1$, $I + \lambda A$ has an inverse in (c_0, c_0). We can now write (2.5) in the form

$$y = (I + \lambda A)x, \quad \text{where } x = \{x_k\}, y = \{y_k\},$$

from which it follows that

$$(I + \lambda A)^{-1} y = x.$$

Since $y \in c_0$ and $(I + \lambda A)^{-1} \in (c_0, c_0)$, it follows that $x \in c_0$, completing the proof of the theorem.

□

For the sake of completeness, we conclude the paper with a few observations recorded earlier (see [5]). With the usual meaning,

(i) $(c_0, c; P') = (\ell_1, c; P')$; ([5], p. 176)
(ii) $(c_0, c; P') \cap (c, c) \neq \phi$; ([5], p. 185)
(iii) $(c_0, c; P') \cap (\ell_\infty, c) = \phi$ (a Steinhaus type result); ([5], p. 185-186)
(iv) $(c, c_0; P') = \phi$; ([5], p. 187).

References

1. Bachman, G. – *Introduction to p-adic numbers and valuation theory*, Academic Press, New York-London, (1964).
2. Natarajan, P.N. – *The Steinhaus theorem for Toeplitz matrices in non-Archimedean fields*, Comment. Math. Prace Mat., **20** (1977/78), no. 2, 417-422.
3. Natarajan, P.N. – *Characterization of a class of infinite matrices with applications*, Bull. Austral. Math. Soc., **34** (1986), no. 2, 161-175.
4. Natarajan, P.N. – *Steinhaus type theorems over valued fields: a survey, Advanced topics in mathematical analysis*, CRC Press, Boca Raton, FL, (2019), 529–562.
5. Natarajan, P.N. – *Sequence spaces and summability over valued fields*, CRC Press, Boca Raton, FL, (2020).
6. Van Rooij, A.C.M.; Schikhof, W.H. – *Non-Archimedean analysis*, Nieuw. Arch. Wisk. **19** (1971), no. 3, 120-160.

Received: 25.XII.2020 / Revised: 23.IV.2021 / Accepted: 26.IV.2021

Author

P.N. Natarajan,
Old no. 2/3, New no. 3/3,
Second Main Road, R.A. Puram,
Chennai-600028, India,
E-mail: pinnangudinatarajan@gmail.com